Computer Aided Engineering Design
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Lecture - 20

Hello and welcome again, we continue with our lectures on computer aided engineering design. We continue with our discussion on B-spline curves and segments. This is lecture number 20, today we will discuss B-spline basis functions and how to construct them.

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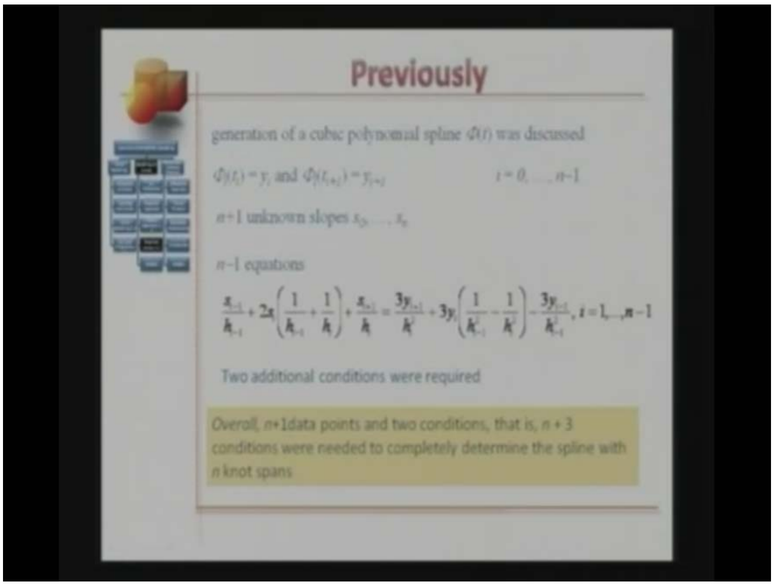

What did we do in the previous lecture, well we studied the construction or generation of a cubic polynomial spline phi t, we also talked about what knots are t i and t i plus 1 and to construct cubic polynomial splines. We had the vertical heights of the splines at each knot; that means the pair t i and y i were available to us, the index i was going from 0 to n minus 1.

Recall how we constructed the cubic polynomial spline? We did that as piecewise continuous curve, which was twice differentiable throughout the definition of domain, that is the cubic piecewise continuous spline was differentiable in the entire range of t. Anyhow, so these are individual segments of that piecewise continuous curve.

What is known to us is phi i, at t i equals phi i and phi i at t i plus 1 equals y i plus 1, once again i is going from 0 to n minus 1. We also had n plus 1 unknown slopes at each pair t i, y i those slopes were s 0, s 1, s 2 up till s n. And then we had n minus 1 equations that related these slopes; these equations were of the form s i minus 1, over h i minus 1 plus 2 times s i within parenthesis 1 over, h i minus 1 plus 1 over h i plus s i plus 1 over h i equals 3, times y i plus 1, over h i squared plus 3, times y i times within parenthesis 1 over h i minus 1, square minus 1 over h i square minus 3 times y i minus 1 over h i minus 1. The whole squared the index i went from 1 up till n minus 1. h i here was defined as the difference between the two knots, t i plus 1 and t i. This was a familiar equation that we had encountered when discussing c 2 composite (( )) curves.

If you would recall this is a triad diagonal system and that can be solved very efficiently but, anyhow to be able to find these intermediate slopes for the cubic polynomial spline to be twice differentiable everywhere, we needed 2 additional conditions overall n plus 1, data points and 2 conditions that is n plus 3, conditions were needed to completely determine the spline with n knot spans. In other words n plus 1 knots, you would have observed how intricate or how involved the mathematics was in the previous.

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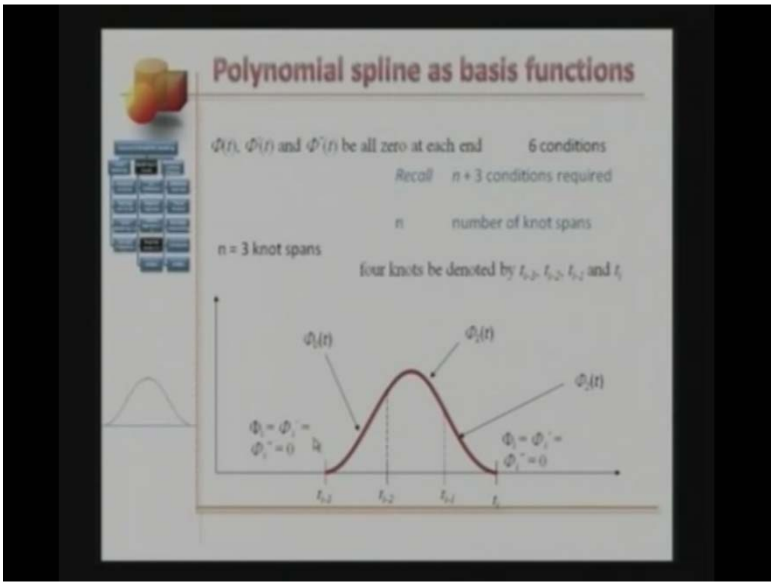

What we were trying to do is we were trying to construct a b-spline basis function via a polynomial spline essentially we will try to construct this bell shaped curve which we call a b-spline basis function. Notice the n conditions the value of the spline would be 0 at these 2 ends, the slope may also be 0 at these 2 ends, and we can also think of making the curvature also 0, at the 2 ends.

Let us start with these conditions that is phi t, phi prime t and phi double prime t. In other words the position slope and the second derivative of the spline the piecewise composite continuous line let these be all 0, at each end. How many conditions do you think there will be, well 3 plus 3 in all 6 conditions. We had just seen that we require n plus 3 conditions to completely determine a polynomial spline. Here n is the number of knot spans for which there would be n plus 1 knots in total, well lets us try to compare that we require these n plus 3 conditions and we already have 6 conditions that means n would be 3, 6 minus 3. We would require a minimum of 3 knot spans to try to construct this bell shaped curve.

Well since we have 3 knot spans will have 4 knots, let these knots be denoted by t i minus 3, t i minus 2, t i minus 1 and t i again arranged in ascending order so t i would be a knot corresponding to this point, here t i minus 3 would be a knot corresponding to this point here and t i minus 2 and t i minus 1 would be intermediate knots, remember that we would know the values of these knots beforehand before attempting to construct the b-spline basis function, so this is the parameter axis the t axis and this is the spline axis of the phi axis ,we plot t i minus 3 t i minus 2, t i minus 1 and t i as 4 knots the idea is to be able to construct this bell shaped curve we would know possibly the value at this point and the value at this point but, we will try to figure that later. We may or may not need those values for now.

So I had told u earlier that this bell shaped curve is a piecewise composite spline curve. The first segment is phi sub 0 t, the second segment is phi sub 1 t and the third segment is phi sub 2 t, each segment will be standing over a single knot span all we need to do is figure what phi 0, phi 1 and phi 2 are, we start with the conditions that are given to us the position slope and the second derivative they are all 0 here and likewise the position slope and the secondary derivative here are all 0 with these 6 conditions and 3 knot spans. We should be able to uniquely determine what phi 0, phi 1 and phi 2 would be. Now the question is what do you think would be the solution you might want to take a moment if you come up with an answer saying that the trivial solution will be phi equals 0, in other words phi 0 is 0, phi 1 is 0 and phi 2 is 0.

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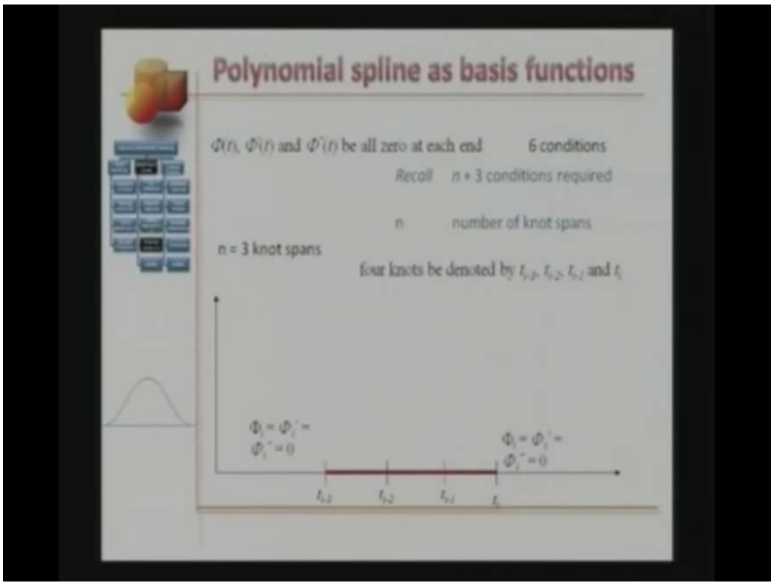

Then i would say that you are correct what do we do now we can try exercising a few options one of the options can be that we can introduce another condition and when doing that we will have to introduce other knot let us see what happens.

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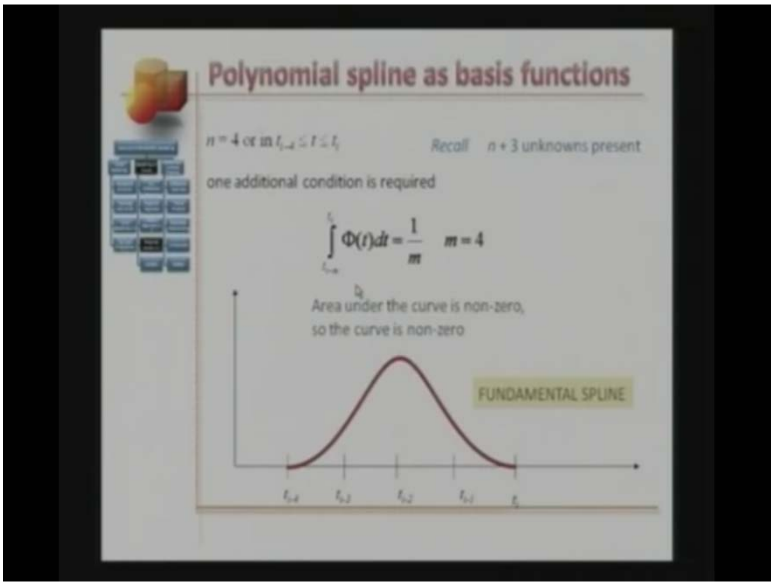

Let us introduce another knot to the left, t i minus 4 a word first about how i try to compute and construct these spline basis functions. I like to go from right to left there are many books that go from left to right but, it does not matter I am comfortable with going from right to left, anyhow coming back to our discussion we have introduced another knot and therefore, another knot span now we have 4 knot spans t i minus 4, to t i minus 1, t i minus 3, to t i minus 2 is second, t i minus 2 to t i minus 1, the third and t i minus 1 to t i the 4.

Now let us try to construct a cubic basis spline over the 4 knot, spans here n is equal to 4 and the knot span spans from t i minus 4 to t i. We have seen before that if i had introduced other knot or another knot span i would require 1 more condition in addition to the 6 conditions: that we had before why do we need 1 additional condition well recall that we have n plus 3 unknowns present overall where n would be the number of knot spans. We use something called the standardizing condition to ensure that this b-spline basis function is not 0 in between the values, t i minus 4 to t I, we ensure that the area under this line curve is non 0.

That is integral from t i minus m, to t i of phi of t, dt is equal to 1 over m here, m is the order of the curve remember n minus 1 would be the degree of the curve so in this case this is a cubic spline for which m equals 4, Now if i go leftward from t i the first knot that i, will have will be t i minus 4, if this is a curve of order m this knot over here can be generalized to t i minus m which is this value.

You have seen that the area under the curve is non 0, so that the polynomial spline is non 0 in between the values t i minus 4 and t i we call this a fundamental spline and I will tell you the reason why we call this a fundamental spline a little later.

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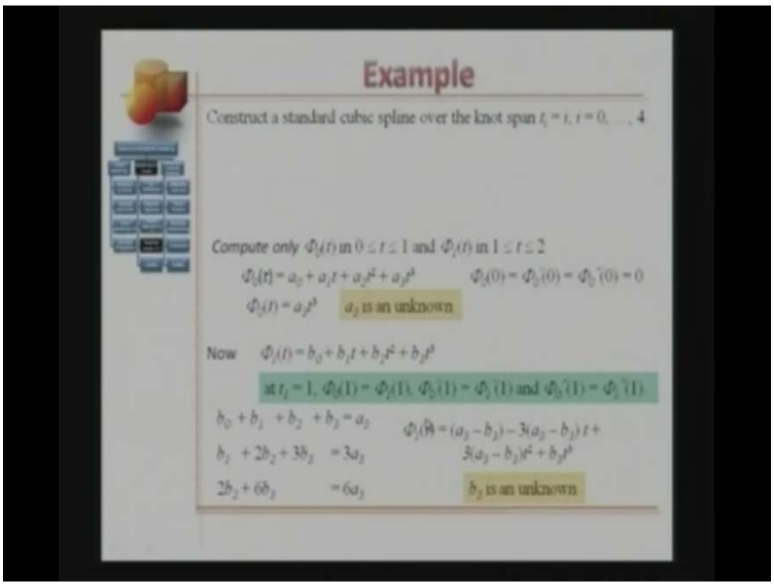

Let us take a look at an example; We try to construct a standard cubic again piecewise continuous composite spline over knot span t i equals i, where i goes from 0 to 4, we understand that the placement of these knots is uniform the boundary conditions will be symmetrical so our standardized spline will be symmetric about t equals 2. This would mean that we would need to compute only two segments of the spline phi 0 t, in the knot span 0, 1 and phi 1 t, in the knot span 1,2 since each polynomial segment is cubic. We can write phi 0 t as a 0 plus a, 1 t plus 2 t, squared plus a 3 t cube a 0, a 1. A 2 and a 3 are unknown coefficients that we need to determine.

Let us start with the boundary conditions on the left position slope and second derivative are 0 this would mean that a 0 a 1 and a 2 are all 0 and so we can write phi 0 t as simply a 3 times t cube a 3 is still an unknown now phi 1 being also cubic we can write phi 1 as b 0 plus b 1 t plus b 2 t square plus b 3 t cube where now b 0 will be 1 b 2 and b 3 are unknowns how do we find them all we need to do is ensure position slope and the continuity in secondary derivative as the junction point t equals 1, recall the basic definition of spline an n degree curve has to be n minus 1 continuous throughout the domain of definition.

So a cubic spline has to be c 2 continuous throughout. Therefore, at the junction point t 1 equals 1 we will have to have phi 0 at 1 equals phi 1 at 1 position continuity phi prime 1 at 1 equals phi prime of 1 at 1 slope continuity and phi double prime 1 at 1 equals phi double prime 1 at 1. the continuity on the secondary correspondingly we will get 3 equations b 0, plus b 1, plus b 2, plus b 3, equals a 3, b 1 plus 2 times b 2, plus 3 times b 3 equals 3 times, a 3 2, b 2 plus 6 b 3, equals 6, a 3, so far the story is that a 3 is not known to us, b 0, b 1, b 2 and b 3 are also not known to us. And then we have these 3 equations we can eliminate some of the unknowns over here. We can write b 0, b 1 and b 2 in terms of b 3 which would make phi 1 of t as a 3, minus b 3 minus 3 times, a 3 minus, b 3 times, 3 plus 3 times, a 3 minus b 3 times t square plus b 3 times t cube. now what now we have b 3 as an unknown and a 3 also as an unknown we need to figure these 2 unknowns for which we would need 2 conditions. Can you think about those two conditions let me give you two clues one symmetry of the composite spline at t equals two and two standardization of the spline.

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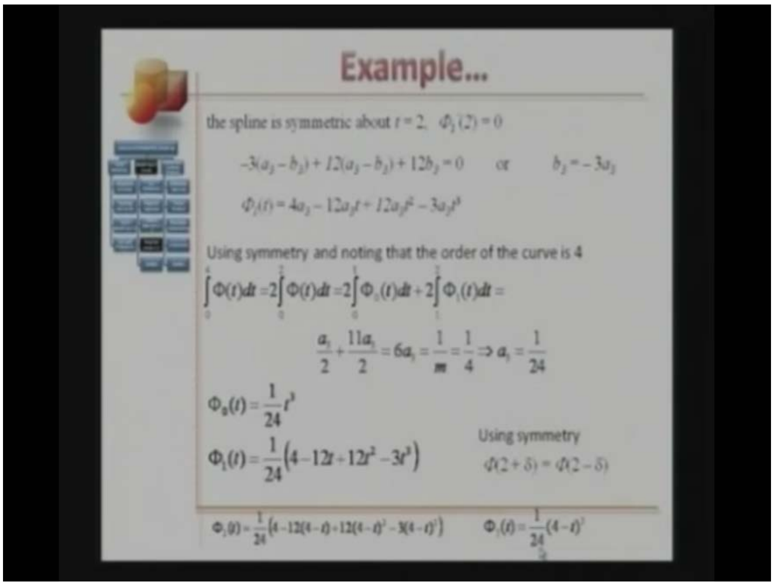

Let us use the first condition, the spline is symmetric about t equals 2, which would mean that the slope at that point will be 0, in other words: the first derivative of phi 1 at 2 will be 0, this would generate the condition minus 3 times a 3 minus b 3 plus, 12 times a 3 minus b 3 equals 0, and this would help us get a relation between a 3 and b 3.

you can express b 3 as minus two times, a 3 which would make phi 1 of 3, a 4 times a 3 minus 12 times, a 3 times, 3 plus 12 times, a 3 times, t squared minus 3 times, a 3 times t cube now how do we find a 3 using symmetry which we have used here and noting that the order of the curve is four.

We now use these standardization condition that is the area of this composite piecewise spline line in between the knots 0 and 4 is equal to 1, over 4. Now since this spline is symmetric. I can write this equation as 2 times integral 0 to 2 phi t dt. which is equal to 2 times integral 0 to 1 phi 0, t dt plus 2 times integral 1 to 2 phi 1 t dt. we have by now acquired the algebraic expressions of phi 0 and phi 1, we can use that and compute the area under the curve as a 3 over 2 plus 11 times, a 3 over 2, which is 6 times, a 3 which is equal to 1 over m, something that i have mentioned earlier and m for us is 4 cubic polynomial composite continuous is piecewise many times (( )), so a 3 will be 1 over 24, we can substitute for a 3 in phi 0 to get phi 0, of t is equal to 1 over 24 times t cube and phi 1 of t equals 1 over 24 times 4 minus 12, times t plus 12 times t square minus 3 times t cube.

Now how about phi 2 and phi 3, so phi 2 was a piece of this polynomial splice phi in the knot span 2 and 3 and phi 3 is a piece in between the knots 3 and 4. We can use symmetry and we can say that phi of 2 plus delta is the same as phi 2 minus delta for any delta. which would give us phi 2 of t equals 1 over 24 times, 4 minus 12 times, 4 minus t plus 12 times 4 minus t squared minus 3 times 4 minus t cube, all you need to do is plug in four minus t in place of t to take care of this transformation likewise phi 3 of t will be 1 over 24, 4 minus t cube phi 3 would be the symmetric part of phi 0 and phi 2 is the symmetric part of phi 1.

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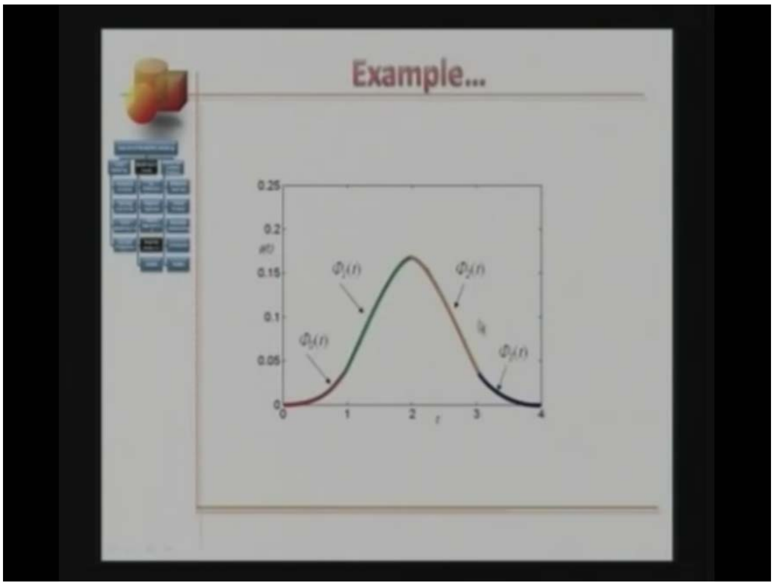

Let us try to plot these segments and understand how these segments come together and form a composite continuous basis function or spline, specifically a cubic spline in this case so this is how the entire basis function is going to look like the knot values are 0, 1, 2, 3 and 4. This red curve here is the first segment the green 1 is the second segment the orange 1 is the third segment, which is symmetric to the green 1 and this segment here is symmetric to the red segment here the segment here is phi sub 0. this one is phi sub 1 this is phi sub 2 and the last one is phi sub 3 to summarize, we have position slope and second derivatives as 0 here and the same as 0 here at this junction point we have position slope and second derivative continuum, likewise for this junction point and for this junction point position slope and continuity in the second derivative and of course,, you would notice that the slope here is 0 because this basis function is symmetric at t equals 2.

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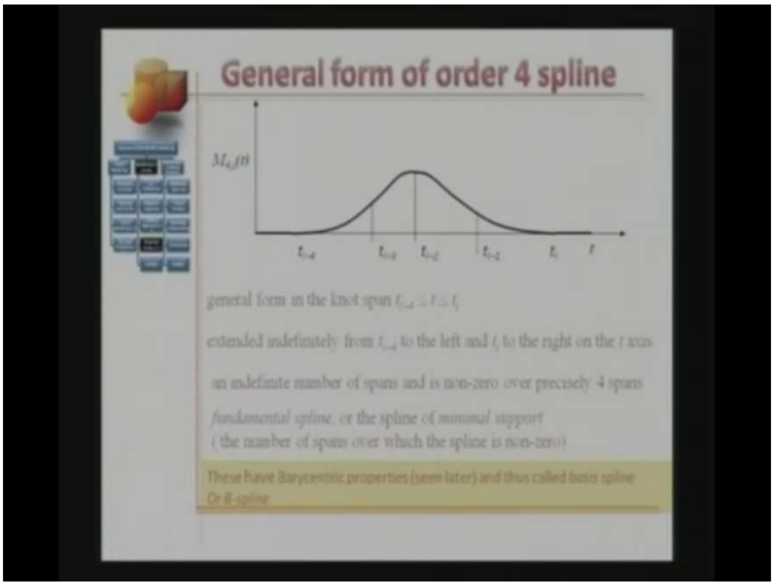

Now let us try to understand the general form for an order four spline it may or may not be symmetric the knots may or may not be uniformly placed. I start with t i minus 4 this is t i minus 3 t i minus 2, t i minus 1 and t i well although i have plotted these knots from left to right i should ideally go from right to left simply because it is convenient to me once again we have 4 knot spans over which this order 4 composite spline is standing.

Let me nomenclate this spline as m sub 4 i the first index corresponds to the order of the spline, the second index corresponds to the last knot over which this spline function stands this corresponds to t i right here further I do not need to restrict myself to the values of t, in between t i minus 4 and t i. I can use the values less than t i minus 4 and also I can use the values greater than t, I just that this line m 4 i will be 0 here and here the general form in the knot span t i minus 4 equal to smaller than t equal to smaller than t i is this this general form can be extended indefinitely from t i minus 4 to the left and t i to the right on the t axis which is something i had just mentioned.

So what will happen is, m 4, i will be standing over an indefinite number of spans only that it is going to be non 0 for values between t i minus 4, and t i which are precisely 4 spans 1, 2, 3 and 4, i had mentioned before that this is called a fundamental spline or the spline of minimal support is the minimum number of spans over which the spline can be non 0 we will come to this little later but, let me emphasize here that these bell shaped b-spline basis functions are Barycentric properties and that is the reason why they are called basis spline or b-spline polynomials or functions although i have been using the term B-spline or basis spline this is the formal reason why.

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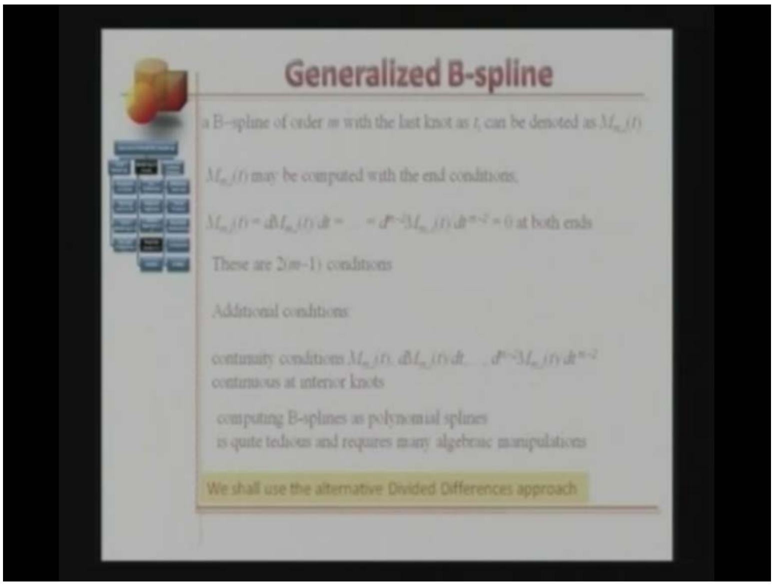

Lets us look at a a generalized b-spline or a basis spline so far we were able to construct a b-spline function of order 4 that is order of degree 3 it is very simple to construct a very similar b-spline function of order degree m with the last knot as t i and if you recall the nomenclature we have just seen this nomenclature will be capital M ,the first index is small m the second index is i capital M stands for the b-spline ,small m stands for the order of the b-spline and i stands for the index corresponding to the last knot for which the spline stands.

Try to remember how we constructed out of 4 b-spline basis functions. we can construct a generalized b-spline basis function in a similar manner that is M m i of t maybe computed with the end conditions M m i t is equal to the first derivative of the function is equal to the second derivative of the function up till the m minus (( )) derivative of order M b-spline and we set all these derivatives as 0, and we do that at both the ends the end on the left and right. How many conditions you think there will be there would be 2 times m minus 1 conditions. We will have additional conditions as well and they would correspond to those at the junction points they would be the continuity conditions the position slope. The second derivative and so on so forth up till the M minus 2, derivative it should all be continuous at the junction points over the interior knots once again go back to the definition of spline.

A spline of order m is c M minus two continuous everywhere we have seen in the previous lecture: and a bit of it today at commuting these lines as polynomial splines can be a little tedious and can require quite a few algebraic manipulations and it may not be easy for us to write a computer program to be able to handle that, we should look for better ways to construct these spline basis functions one of those ways is the divided differences approach.

We will be studying different techniques to synthesize a construct using spline basis functions later but, let us try to fix the geometry behind all these algebraic manipulations that we will be doing throughout these lectures i always emphasize on geometry because this is what makes us all understand things lot better now this is the way i remember to construct b-spline basis functions this is a general one of order m.

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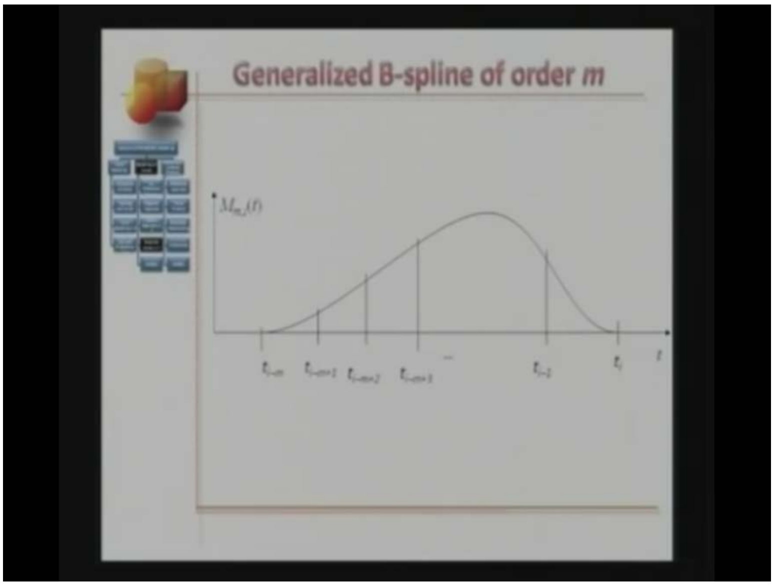

This is my vertical axis of order m this is my parameter axis now. let me ask you how you would want to construct a b-spline of order M well as i said i will go from right to left this is my M sub m sub i t now these two vertices they would have information pertaining to the entire knot span here, as i said this is the parameter axis t the last knot would correspond to t i and the first knot would correspond to t i minus m. so all you would need to do is first corresponding to this index plot the last knot and then use the index and the order to figure what the t i minus n knot is and plot it on the left so you generalized order M, square line is going to be standing on the knot span t i minus m and t i then start plotting the intermediate knots t i minus m plus one t i minus m plus 2. t i minus M plus 3 and so on so forth up till t i minus 1, and then finally, t i if you have already started.

You might want to do this exercise a few times to try to fix this geometric method of sketching a b-spline basis function of order M once you are comfortable with this it will be a lot easier for us to understand the material covered in subsequent literature.

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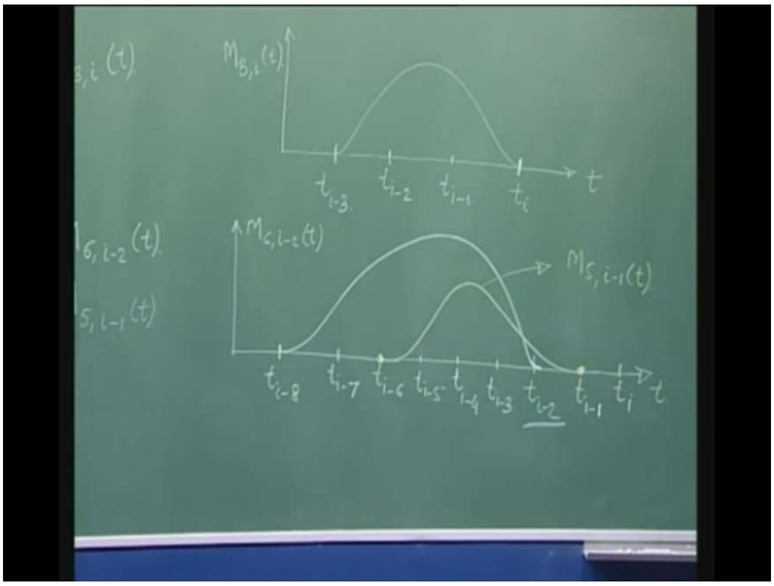

Maybe I will do a few examples: for you here on the board how about constructing a b-spline basis function of order 3, and i would represent the last spline as a function t draw the vertical axis draw the parameter axis this is M 3 i t 3, is the order of this line that means the spline would be quadratic now step one plot the last knot t i step two figure out the first knot what do you think that will be t i minus 3, how about the intermediate knots we will have t i minus 2 and then t i minus 1, and your quadratic spline will be standing over these three knot spans, so it is not very difficult to understand let us try a difficult one, M 6 say i minus 2 t, you will be constructing many of these spline functions to be able to understand their properties a lot better the vertical axis and the parameter axis this is M 6 i minus 2 t ok, can you help me out what is step one step one is to plot again the last knot this would correspond to t i minus 2, let me give you a little perspective here I will also plot t i and t i minus 1 but, this would be the knot which would be important to me at this time, how about the first knot this would correspond to t i minus 2 minus 6 which is t i minus 8, Let me plot this thing here let me not worry about these intermediated knots at this time and let me go ahead and plot M 6 i minus 2 of t i have deliberately drawn this as an unsymmetric bell shaped function, I should be careful here i have to have position slope second derivative maybe the third derivative and so on so forth of 0, so i cannot have a non 0 slope here ,so how about the intermediate knots now t i minus 7, t i minus 6, t i minus 5, t i minus 4, t i minus 3 and then we have t i minus 2 here ,if I ask you to plot over this M, let us say 5 i minus 1 of t would you be able to do that so our starting knot is, t i minus 1 which is this one here our first knot is t i minus 1, minus 5 which is t i minus 6, which is this knot, I am going to be careful at how i draw the slopes here the two end points this is M5 i minus 1 t of t 10, marks for guessing this over how many knot spans will this function span 1, 2, 3, 4 and 5 the order of the b-spline function.