Computer Aided Engineering Design
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Lecture -19

Hello and welcome, let us now start now with our introductive lecture on these lines sp curves.

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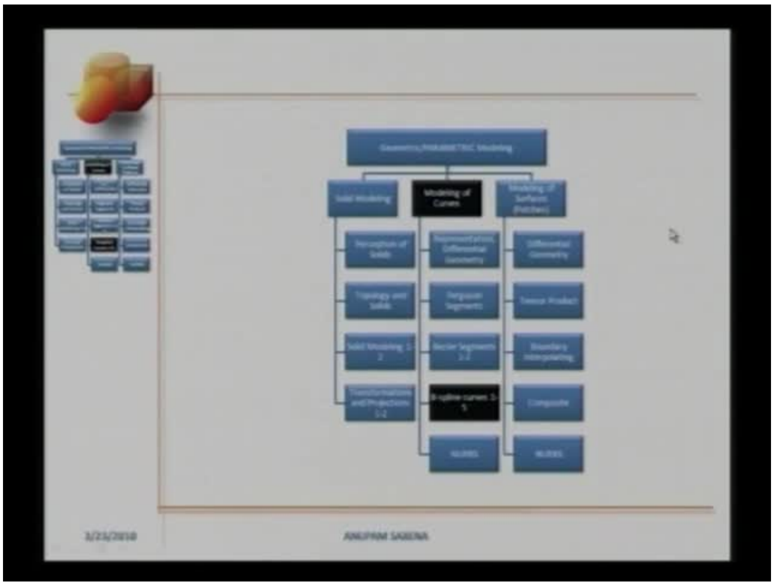

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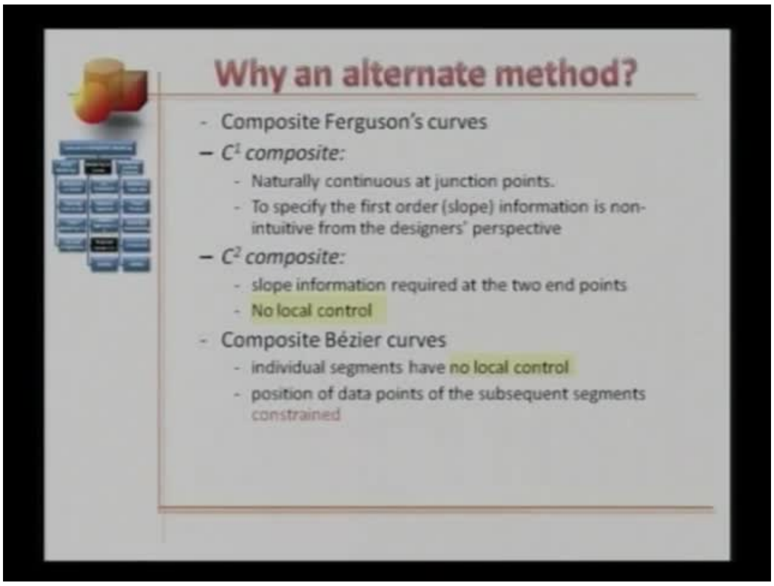

This is lecture number 19; here we will talk about polynomial lines in general. But first why these lines, rather why an alternative approach or method to design curves? Let us look some composite curve models that we have discussed previously. First composite Ferguson's curves, for C 1 composite Ferguson's curves; for C 1 composite Ferguson's curve, the model is naturally continuous at junction points. However, to specify the first order or slope information is not very intuitive from the designers perspective, and this is true especially in three dimensions. To specify individual components, the x, y, z components of slope is not very (( )) for design.

For C 2 composite of the slopes, the slope information is required at the two end points. All the intermediate slopes will get computed g 2 C 2 composite. However, there is no local shape control for the entire curve. In fact C one composite Ferguson's curve and be locally Ferguson's shapes. In case of composite Ferguson curves, that we have recently studied, once again individual segments constituting composite Bezier curves, do not have local control. Position of data points of the subsequent segments are constrained.

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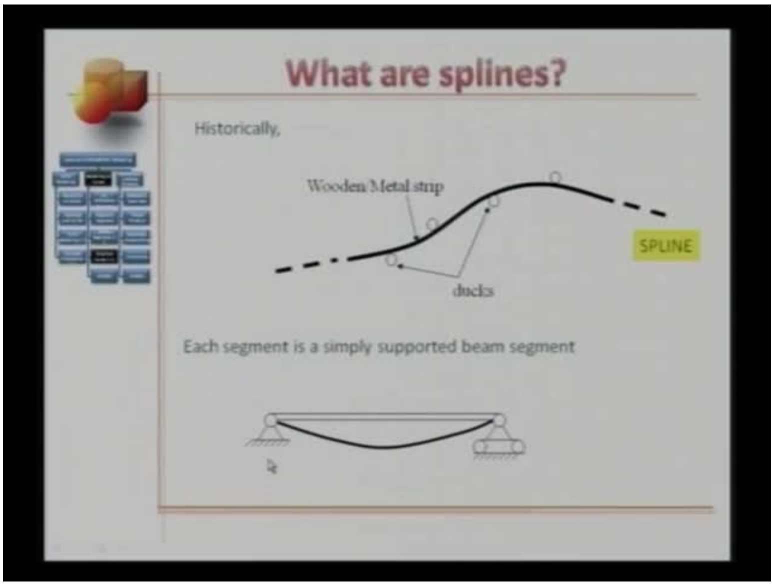

Any how this is what I emphasized and this is what going to motivate us to pursue spline segments and curves. No local control in Ferguson's and composite Bezier curves, particularly C two composite Bezier curves and Ferguson's curves that we have studied before. We would have designed like to a change local control and shape design curves be following your lectures will be devoted entirely today's discussion. After motivating us enough to study these slides first, let us figure what splines are?

How do you think British, Portuguese and Spanish would built huge ships about 3 to 400 may be 500 years ago and travel across the world. They use something called ducks and within in or in between those ducks they use to place woods strip or metallic strip and get the shape hollow of the ship. These strips use to remain within the ducks for days and days. They then possibly did not know that what they design are essentially splines curves (( )) in the third dimension.

If you look at these individual segments, each segment is a simply supported beam. You might point to revisit the discussion in your second year solid mechanics course or central materials course. Each beam therefore, will appear like this, simply supported at both ends. Let us try to analyze the structure using Euler by Euler beam theory that assumes small definition.

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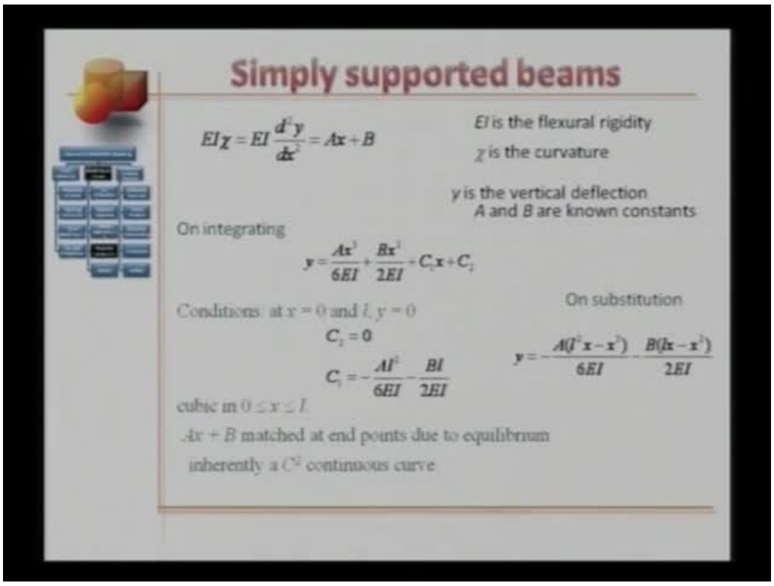

These expressions are relations would be very familiar to mechanical engineers, a virginic mechanical engineers. It says E I times chi is equal to E I times this second derivative of the vertical deflection with respect to x square, where x is along the beam length, which is equal to A times x plus B. E I is the flexural rigidity, chi is the curvature, and for small information the curvature is given by d to y over d x square. y is the vertical deflection, A and B are known constant, ten points for guessing, what this is? You are right, this is the movement at any cross-section.

If we integrate this relation twice, will have y equals A times x cube over 6 E I plus B times x square over 2 E I plus C 1 x plus C 2. C 1 and C 2 are known constants which you can compute using boundary conditions. On the conditions as say x equals 0 and x equals l the beam length are that the vertical deflections are 0. Why, because we are considering a simply curved beam? If we put x equals 0 here and know that y equals 0 this would mean C 2 equals 0 and for x equals l, we can substitute y equals 0 x equals l and find what C 1 is? C 1 will be minus of A l square over 6 E I minus B l over 2 E I.

If we substitute the value of C 1 and C 2 into this equation, we have y equals minus A times l square x minus x square over 6 E I minus B times l x minus x square over 2 E I. Once again l is the beam length, what do we see? We see the vertical deflection over here as a function of the coordinate along the beam length. We can also observe what the relation or what the nature of the relation is, it is cubic, for a values of x between 0 and l across entire simply supported beam segment.

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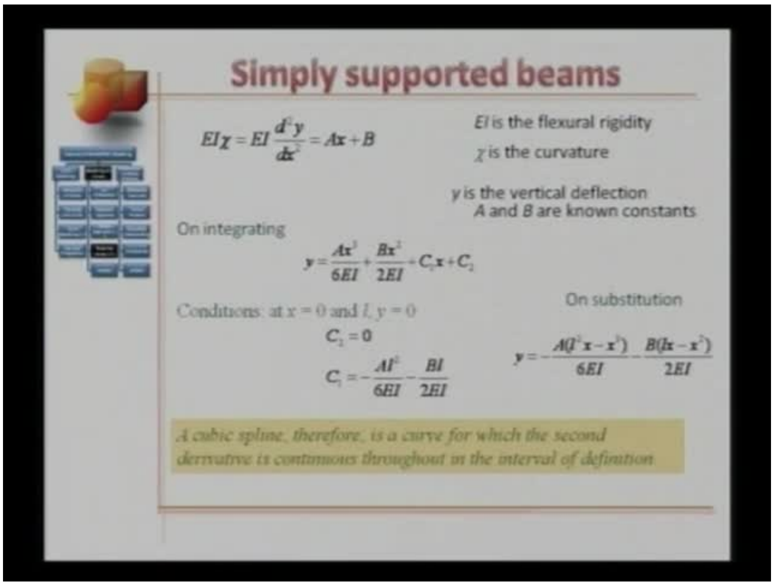

One should note, that A x plus B, which could represent the movement condition at any cross sections is matched at end points due to a cubic. If we look or if we observe this relation closely, it seems to be inherently of C 2 continuous curve because you would be able to differentiate this at least twice. If we go back and if we go back quite differentiate this thing for the third time d 3 y over d x cube equals to A and this value A will be different across different contiguous simple supported segments. What do we note now?

We can comment on what we are going to term as cubic's spline. A cubics spline is a curve for which the second derivative is continuous throughout in the interval of definition. Let us go back to this figure here. We are considering this entire strip and what we are saying in this strip is cubic spline.

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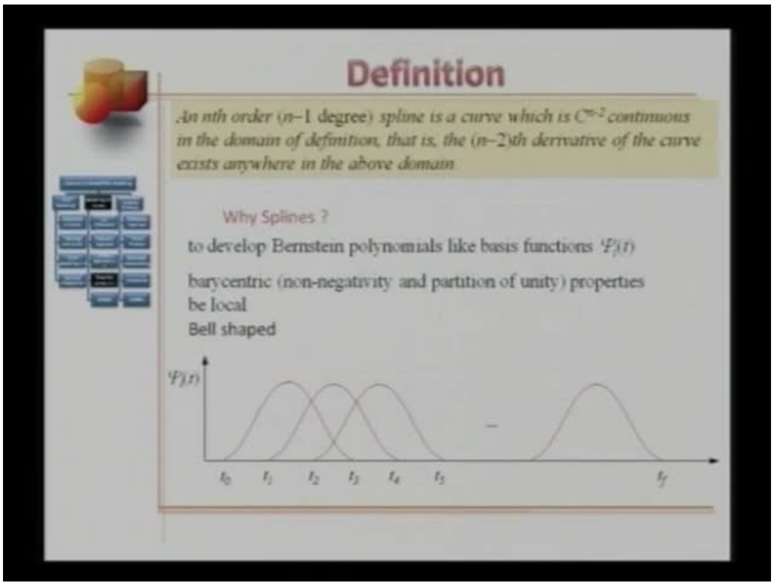

It may be possible for us to generalize and define what an n minus 1 th degree or an nth degree spline is? So, the definition of the spline, an nth order which is n minus 1 degree curve, a spline is a curve which is C n minus 2 continuous in the entire domain of definition. This means that the n minus 2 th derivative of the curve exists anywhere in the above domain. A little note about what order is, what degree is? Order is always degree plus 1; for example, nth order curve will be the degree minus 1.

Basic question, why splines recall our discussion on Bernstein polynomials, when we are discussing Bezier segments and composites. We would like to develop basic functions, which are very similar to Bernstein polynomials. Let us call those function as psi sub i as the function parameter t, t, here may here or may not give respected the values between 0 and 1. Also, recall the salient properties of Bernstein polynomials. Remember there are barycentric in nature. That all these constant polynomials are positive or non-negative in the respective parameter intervals that is 0 and 1 and that all these polynomials is up to 1. No matter, what the t value or the parametric value is?

We would want these barycentric properties to be local now. But, what do I mean? Bernstein polynomials is sum to 1 for value of t in between 0 and 1. Here we would want not all but some spline basis functions by (( )) to 1, in some sub interval of t. Of course, we would want these basic functions, psi i to be well ahaped. Say for example, you are trying to plot different apline bases functions psi I, one of those psi i look like this. Let us say for parametric value t 0 and t 3, the second one remain will look like this. Saying t 1 and t 4.

Third one look like this, saying t 2 and t 5 and we can keep on going until the final one, they look like this. Coming back to this statement local barycentricity, we do not want all these psi's sum to 1, rather we would want now only few of these basis functions goes up to 1. Irrespective of how the rest of the functions behave and it is this that would give us the local control power, we will see later how?

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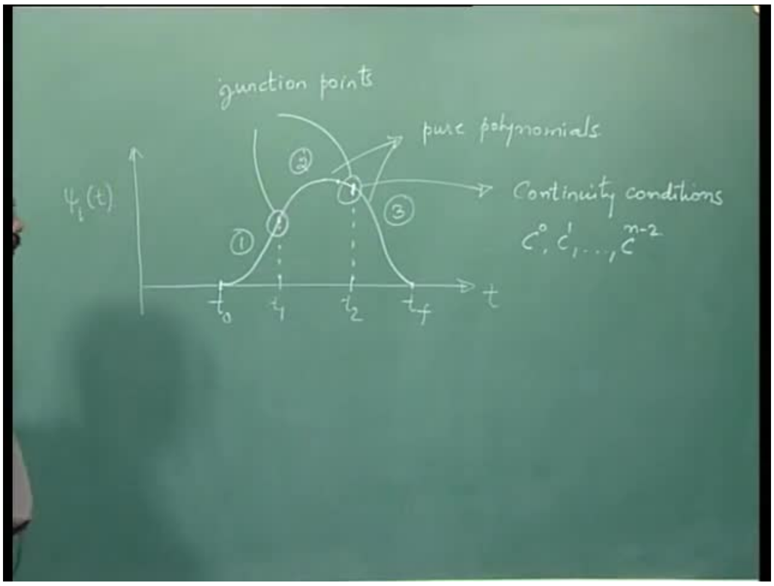

Let us continue our discussions on polynomial splines. Let us consider for this functions here and let us try to design that function. Let us say for example, that this function is a cubic polynomial, number one. Number two, that it is designed in a similar way as a piece wise continuous curve that we had seen and discussing Ferguson composite curves and Beizer composite curves. Let me explain this on the board.

I first draw the parameter axis t, I draw the vertical axis that would represent the value of psi i. So, remember I am now constructing a bell shaped basis function or a bell shape piece wise composite curve. This is how I would do this. I will start some parameter value, let us say t o and I will end at some other parameter value say t f and in between I will construct different polynomial segments. Let us say in-between t 0 and t 1, I construct this segment, in between t 1 and say t 2, I construct an another segment. And in t 2 and t f I construct the third segment segment one, segment two, segment three.

Each of these segments will be pure polynomials. As you would notice that there would be junction points, in this case will have two junction points here. Like, we had discussed in case of Ferguson Beizer curves, we will have to impose continuity conditions. C 0 C 1 after C n minus 2 in case this bell shape curve is an end order spline. Let us now continue with some mathematics. Let me work with a few of these polynomial segments.

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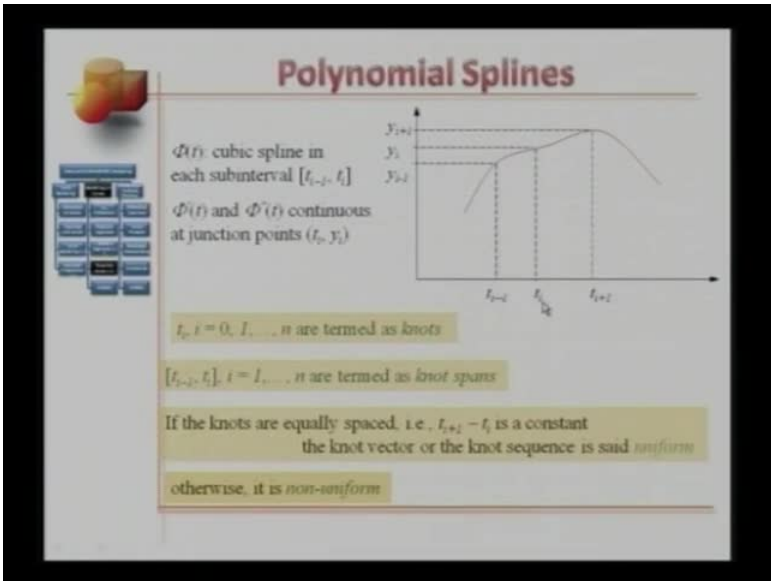

This is the polynomial segments, this is the parametric axis. This is the axis that correspond to non zero values of basis function. Let us say this is generic spline curve. We have these points at parametric value t i minus 1 and we have a value of this this generic spline curve as y minus 1 at t i. We have the value y i like wise at t i plus 1 we have the value y i plus 1 and so and so. Let us say these are the pairs given to us and we have to construct a generic spline. Let us say we are going to be constructing by t, which could be cubic spline in each sub interval t i minus 1 t i.

What would these means? At these junction points we would want that this spline is slope continues and also we would want the second derivative of this entire spline is continues. Of course position continuity is implicit. Some terminologies, these parameter values t i minus 1, t I, t i plus 1 and so and so forth. The fixed values for a given spline for i equals 0 1 and till end and these fixed values for t are called knots. You might want to keep this in mind because we are going to be frequently using the term knots later on. Going further these splines, t i minus 1 into t i, t i into t i plus 1 for different values of i are called knots splines. If the knots are equally spaced, that would mean that if these knot intervals are of constant length, the knot sequence is said to be uniform. Otherwise if these fixed parameter values are arbitrary spaced the knot spline called non uniform. We assume here that these knots are arranged in ascending order that means t i minus 1 is smaller than t I and so on and so forth.

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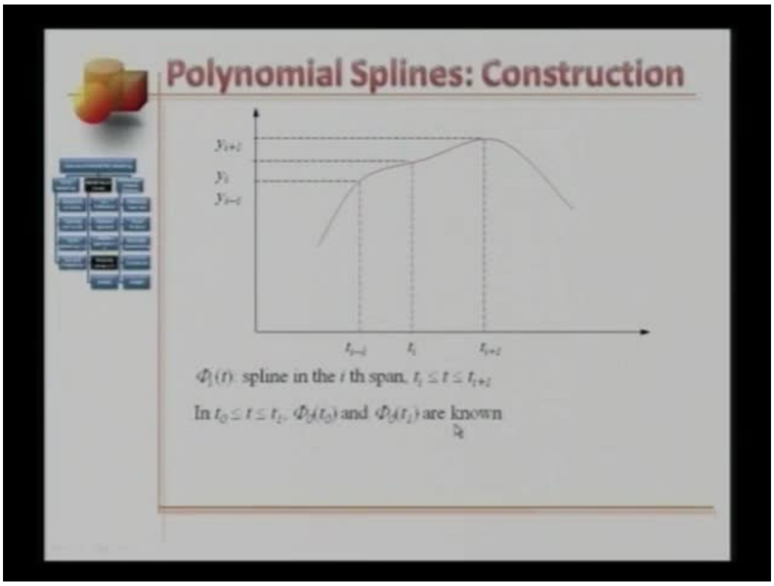

Let us enlarge this figure now. Say we have knots now and we have the values of this composite spline at each knot. Let us spline in this line in the i th span. The i th span would be corresponding to the values of t in between t i and t i plus 1. So, here this is the i th span. Let us assume that the first span for values of t in between t 0 t 1 phi 0 t 0 phi 0 t 1 are known.

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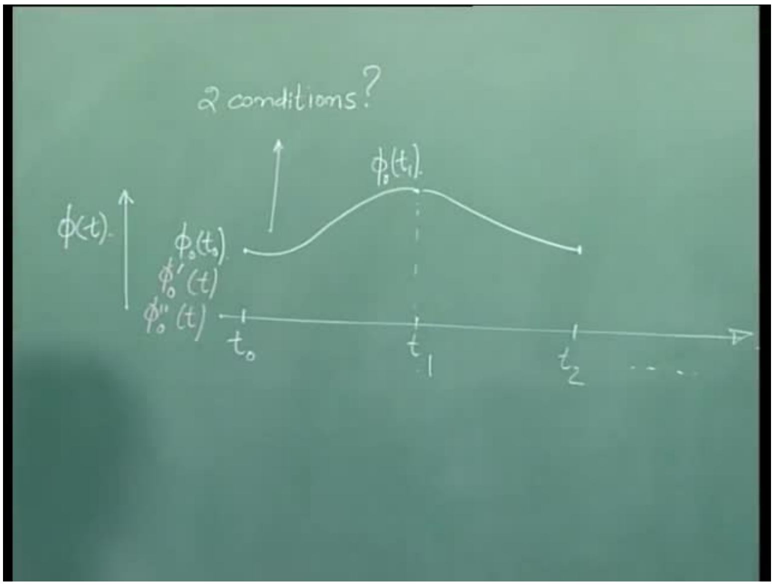

Coming back to the board, this is my parameter axis key. I start with parametric values t 0 t 1 t 2 and so and so forth. Of course, I am interested in constructing a bell shape bases function, but let me generalize this discussion here. This is my y axis, let me start constructing spline curve from here. At t 0 let me assume that I know the value of phi 0.

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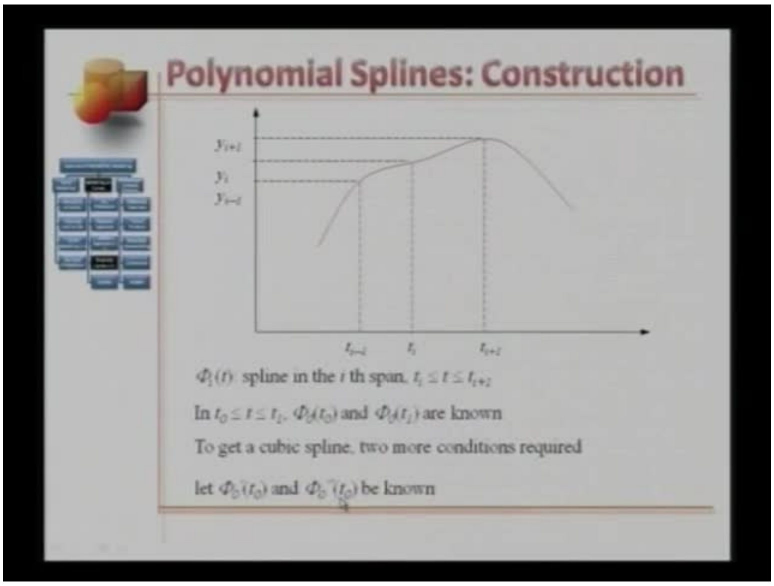

Likewise at t 1, let me assume that I know the value of phi 0 again phi 0 would be the curve in between t 0 and t 1 that is in the first span. Since, I am trying to construct cubic curve or cubic polynomial, let us say have a shape like this. I have these two conditions. I would need two more conditions. What are they? They can be a slope and this second derivative and the first parameter value. Let us assume that we have slope information given here and also the information corresponding to the second derivative given here. Three conditions, fourth condition this is a cubic polynomial I would be able to determine this polynomial. This is the first piece of the cubic spline that we have interested in constructing. How about the second one? We would need four conditions again, because the second would be a sphere cubic polynomial. Let us construct those conditions. Let me call this phi 1 t.

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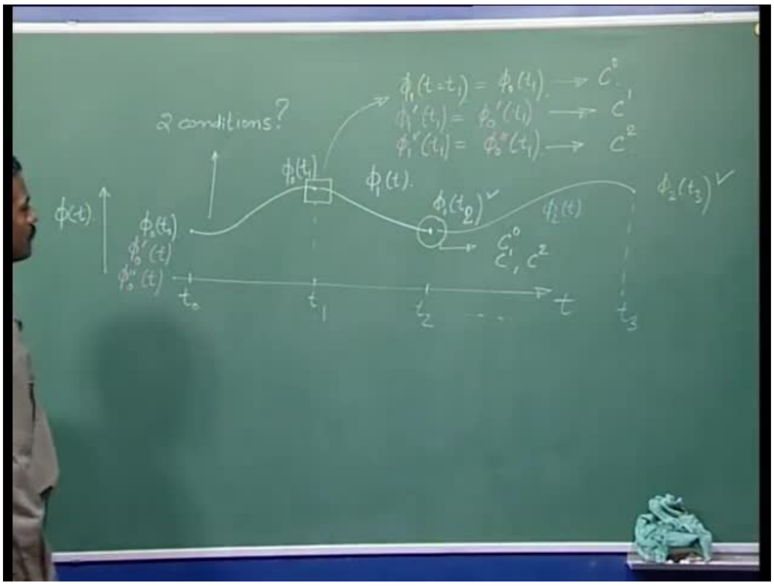

Phi 1 t at t 2, let us say it is known that is the first condition. How about other three conditions? Let us concentrate on this point here, which is the junction point. Do I know the value of phi 1 at t equals t 1? Yes and that is the same as phi 0 at t 1, which comes from here. This would give us position continuity for C 0 continuity. Let us recall now what would be a cubic spline? A cubic spline would be such that this entire curve would be C 2 continues throughout that would means that the second derivative should be uniquely available at any point on this curve.

In the sense we need to ensure that we have not only C 1 continuity or slope continuity, but also continuity in the second derivative at this junction point. What would that mean? It would means we need to able to generate two more conditions. The slope should be equal and the second derivative would also be equal. The slope and the second derivative for this segment and the slope and this second derivative for this segment. This ensures slope continuity and this condition ensures the continuity in the second derivative.

In the sense for this cubic segment phi 1 t, we now have four conditions the value at t equals t 2. The value at knot t 1 the slope continuity condition and the curvature continuity condition. Since, this again a cubic segment with four conditions should be able to the four unknowns constructing the third segment. We would construct the third segment in an exact passion, that we used to construct the second segment. Let me call this segment phi 2. Let this knot value t 3, let me assume the value right here. Phi 2 t 3, let me assume this is known at this junction point you will have C 0 continuity C 1 continuity and C 2 continuity.

Now, that you know what the second segment is you can generate these three conditions. Three conditions here, the fourth condition here, the third segment of this phi is known. Once again, let me emphasize that this is a cubic spline. By definition a cubic spline suppose to have C two continuity condition in cap thought out.

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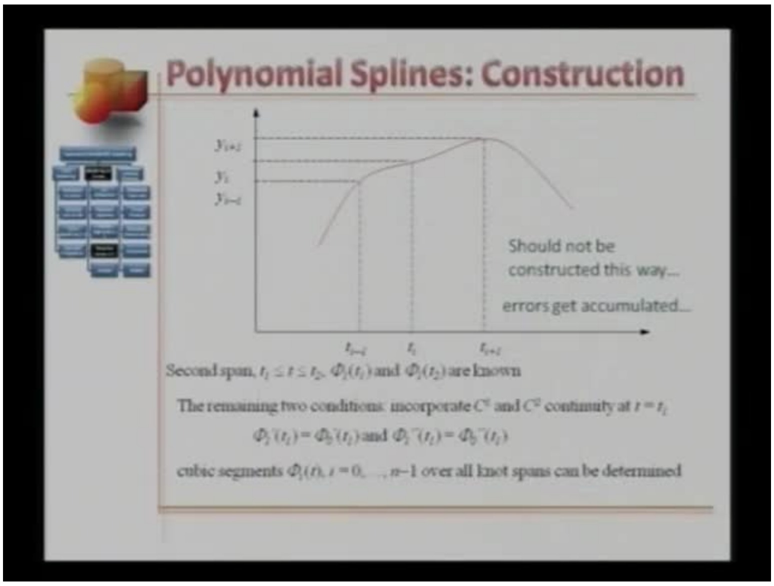

If we keep on constructing subsequent segments, we would be able to extend this curve as much as t 1. In other words to any knot value t 1. By the way any idea has to how we can make this shape of this spline to resemble to a bell shape curve? Well one way would be to start with the first value as 0 and end the last value as 0. How about the slope and curvature? Let us preserve this discussion later.

So, this is a summary of construction procedure that I have just described on board. In the sense cubic segments phi i t, i going from 0 n minus 1, over the entire knot spans can be determined. But should spline be constructed this way, possibly not. Because if there are errors starts from here will keep on getting accumulated, will keep on getting added. To avoid that we have a slightly different way of constructing along splines. Let us start from the figure again.

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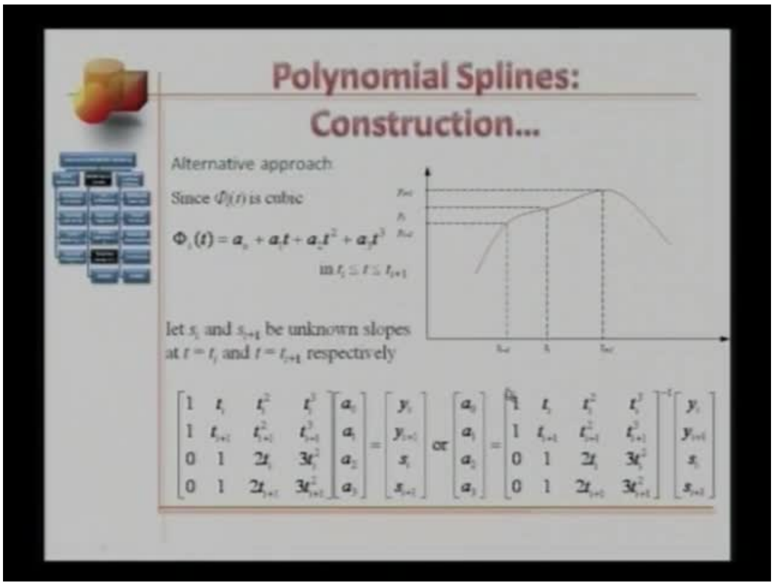

Let us say we know the values at different knots of the spline. The value would be y i minus 1, t i minus 1, y i and t i and y i plus 1 and y i and so and so forth. Let us discuss an alternate approach. Since, each segment over here represented by phi i cubic, I can write phi i in this form as a function parametric t. Phi i is equals to a sub zero plus a sub one times t plus a sub t times t square plus t times t cube. a 0, a 1, a 2, a 3 are unknown (( )). Phi would be defines within the knot span t i and t i plus 1. Now, let us say that s i and s i plus 1 are unknown slopes at values of t equals t i and t i plus 1.

Now, in terms of the two positions and two slopes, what do we have? We have a linear system in the coefficients. Will have the coefficient matrix as 1, t i, t i square, t i cube, 1 t i plus 1, t i plus 1 the whole square, t i plus 1 the whole cube, 0, 1, 2 t I, 3, t i square, 0, 1, 2 t i plus 1 and 3 t i plus 1 square. What could be these coefficients corresponds to? Well this corresponds to the first equation if we substitute phi i equals phi i for t it is t i. This row here corresponds to the second equation phi i at t i plus 1 is phi i plus 1. If we differentiate this, we have a 0 which is called, will have a 1 plus 2 a t plus 3 a t t square.

If we substitute t equals t i these two terms are 0 corresponding to the first coefficient a 0. These two terms will be 1 will have a 1 here. These to term corresponding to t i and t i plus 1 and still first differentiation of this one and these two terms here would be corresponding to first differentiation of this term for values of t i and t i plus 1. This is column vector comprising coefficients and these are the respective y and y plus 1 value. Also the values of the two slopes s i and s i plus 1. We know this column, we know this coefficients of x of this 4 by 4 matrix. We can compute what a 0, a 1, a 2, a 3 are? It would be this 4 by 4 matrix inverse of that times y i y i plus 1 s i and s i plus 1, but wait a minute. When did I tell you that s i and s i plus 1 are knowns? We just said let the slopes be unknowns at these two points. Let us see where with this takes us?

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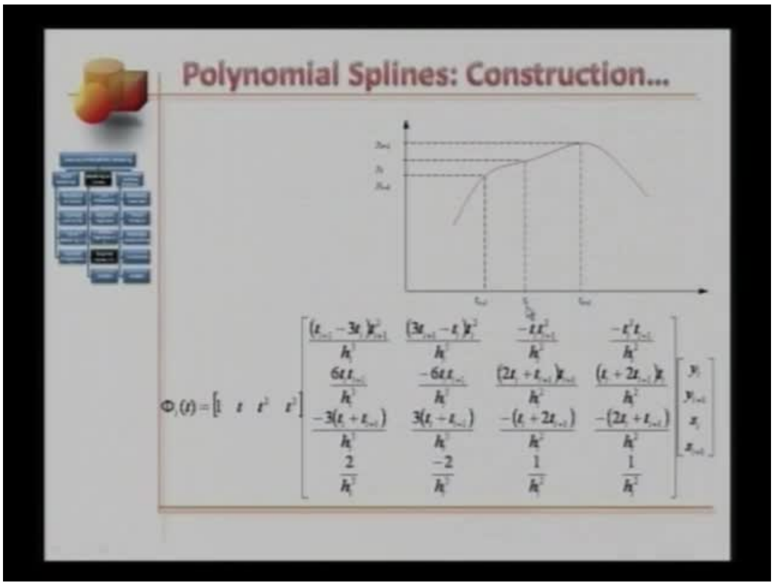

This is little more involved expression here. Phi i t can be represented in matrix form or in compact form. So, the first matrix will be 1 by 4 in terms of parameter values t ,1, t t square and t cube. This is a very complex looking of 4 by 4 matrix and this is the geometric matrix pertaining to the geometric definition here of the two points of the two slopes. What is this 4 by 4 matrix the inverse of this matrix here? Let me read out the terms first term t i plus 1 minus 3 t i times t i plus 1 square over h i 2 3 times t i plus 1 minus t i minus t i square over h i cube minus t i t i plus 1 square over h i square minus t i square t i plus 1 over h i square second row 6 t i t i plus 1 over h i cube 2 times t i plus t i plus 1 times t i plus 1 the fourth term t i plus 2 times 2 times t i plus 1 over h i square.

The third row we have minus 3 t i plus t i plus 1 over h i cube 3 t i plus t i plus 1 over h i cube minus t i plus 2 t i plus 1 over h i square minus 2 t i plus t i plus 1 over h i square. In the fourth row we have 2 over h i cube minus 2 h i cube 1 over h i square and 1 over h i square. What is h i? h i is difference t i plus 1 and its t i. Let us continue with this maths.

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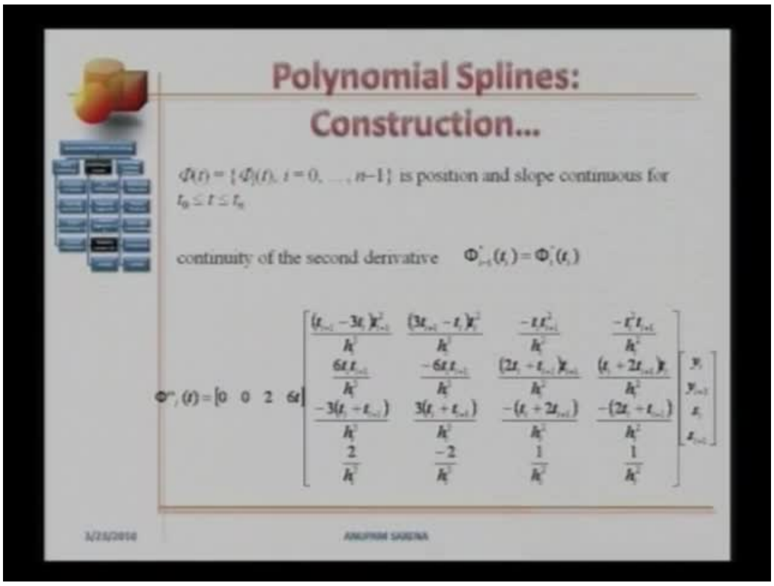

Now, the entire to be explained is essentially a set of each cubic segment phi i t i going from 0 2 n minus 1. This spline is position and slope continues within the entire not span till t 0 until t n. Notice that, since Phi i is a cubic spline it needs to be having unique second derivatives at each value of t, in the entire span. This would mean that phi i minus

1, which is the i minus 1 segment, the second derivative of that, at the naught value t i should be equal to phi the I th polynomial segment, the second derivative that at t i.

From that previous complex relation, it should be possible for us to compute for the second derivatives. This coefficient matrix will not change, this geometric on vector will not change, what will change would be the first 1 by 4 row matrix. If I differentiate by previous relation phi the first term is 0, the second term is 0, the third term is 2 and the fourth term is 6 times t. These two matrixes are constants in the differentiation. All i need to do is plug in. These expression here in this equation for value t equals t i for both i minus 1 cubic segment and the i cubic segment and figure what that relation would look like? You have to bear with me here, because the math is really quite involved.

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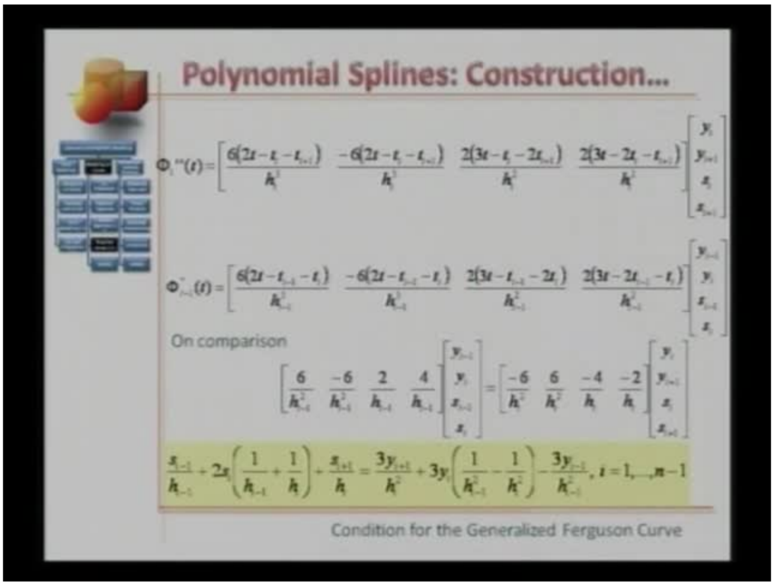

Let us continue if you work out and simplify the math will have the second derivative of the i th cubic segment written as 6 times 2 t minus t i minus t i plus 1 over h i cube minus 6 times 2 t minus t i minus t i plus 1 over h i cube 2 times 2 t minus t i minus 2 t i plus 1 over h i square and 2 times 2 t minus 2 t i minus t i plus 1 over h i square. This is over 1 by 4 matrix multiplying its column vector here. y i y i plus 1 s i and s i plus 1, remember the second derivatives of the i minus 1 segment and i th segment are the common knot t i are equal.

All I need to do is set i equal i minus 1 to get this expression. The composition of each term will be similar, is that t i is going to be replace by t i minus 1 and t i plus 1 will be replaced by t I, h i gets replaced by h i minus 1, y i gets replaced by y i minus 1, h i gets replaced by h i minus 1 and so on. On comparison that is when you impose t 2 continuity condition will have this expression 6 over h i minus 1 square minus 6 over h i minus 1 square 2 over h i minus 1 4 over h i minus 1 times y i minus 1 y i x i minus 1 x i is equals to minus 6 over h i square 6 over h i square minus 4 over h i minus 2 over h i times y i y i plus 1 x i x i plus 1.

Look how that complex math turn simple, as we go on. We can further simply this and have this relation x i minus 1 over h i minus 1 plus 2 within parenthesis 1 over h i minus 1 plus 1 over h i plus x i plus 1 over h i equals 3 times y i plus 1 over h i square plus 3 times y i times within parenthesis 1 over h sub i minus 1 square minus 1 over h i square minus 3 y i minus 1 over h i minus 1 square. Now, this is for i going from 1 to n minus 1, there is nothing important about this relation. Now, while we were doing the complex math that which is did, let us not try to forget what was the physic behind entire discussion?

Remember that I had mentioned that the slopes in fact at all junction points were are knot implying, that these slopes x i minus 1 x i x i plus 1 there are all knots. What is knot to us? It is the values at each junction point y i minus 1 y i y i plus 1 so and so forth. Also the knots at each junction point, that is t i minus 1 t i t i plus 1 so and so forth. So, here we have a relation that relates three consecutive slopes. X i minus 1, x i and x i plus 1. Have you seen such an expression four? Yes, when we are discussing composite for this curves rather C 2 composite curves, this condition right here is a condition generalized Ferguson curve.

Going back, how many equations will we have here? i going from 1 2 and minus 1 will have therefore, n minus 1 such equations. How many unknowns will we have, will possible have n plus 1?

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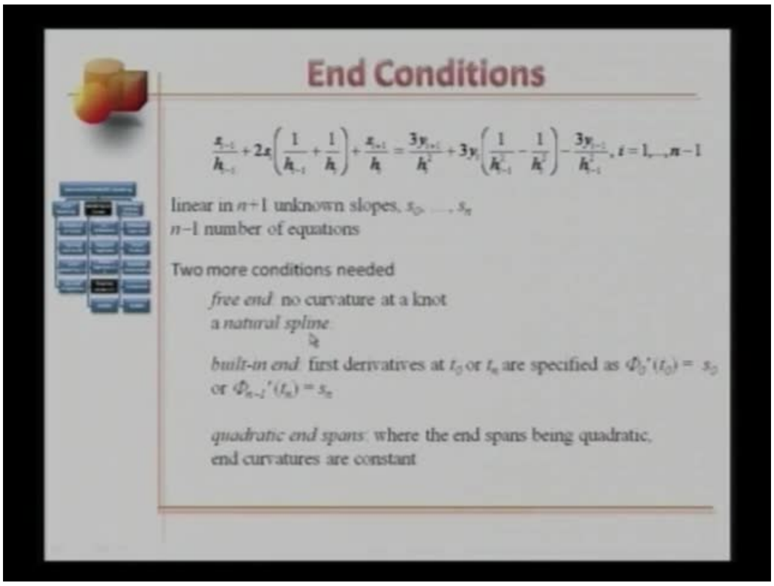

Let me copy previous equations right here and let me also observe few more things about these equations. These equations are linear in the slopes x i minus 1 x i and x i plus 1, the slopes which are unknowns are x 0, x 1 up to x n. These are n plus 1 slopes are unknowns. I just mention a while ago. This equation right here are n minus 1 in number will have therefore, 2 3 choices are two more conditions that we need to generate. Those conditions any one of the following. One, three end that is no curvature at a knot, we call it a natural spline.

Condition two built in end. What do I mean by that? The first derivatives at the first knot or the last knot are specified as phi 0 prime t 0 which is x 0 or phi n minus 1 1 t n minuss n here. In other words slope s 0 and s n are specified. Condition three, quadratic and spans. Here if the ends span of the quadraticv the end curvatures are constant. So, the conditions that we need to solve the system n minus 1 equations and the n 2 of this three conditions. Remember for C 2 composite position curves, we had specified the two end slopes as free choices. This is here a more general set of end conditions.

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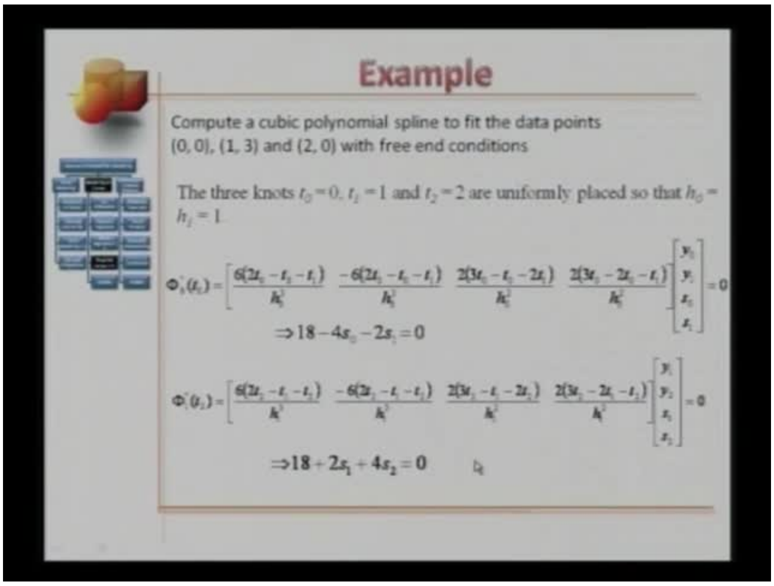

Let us work on an example, compute a cubic polynomials spline to fit the data points. 0 0, 1 3 and 2 0 with free end conditions. Let us assume three knots values t zeros equals 0 t 1 equals 1 and t 2 equals 2. We have uniform non splines h 0, which is t 1 minus t 0 is 1 and h 1, which is t 2 minus t 1 is again 1. I would point you to remember the mathematics that we had gone through, with that will have the second derivative of the first spline segment evaluated at t 0 as 6 times 2 times t 0 minus t 0 minus t 1 over h 0 cube minus 6 times 2 times t 0 minus t 0 minus t 1 over h 0 cube 2 times 3 times t 0 minus t 0 minus 2 t 1 over h 0 square 2 times 3 times t 0 minus 2 times t 0 minus t 1 over h 0 square the calling vector will have increased y 0 y 1 s 0 and s 1.

We know what y 0 is what y 1 is and s 0 and s 1 are unknowns at this time. We will have an equations let us say 18 minus 4 times s 0 minus 2 times s 1 equals 0. For the second spline segment phi 1 will have a very similar expression phi 1 double time evaluated at t 2 will get 6 times 2 t 2 minus 3 1 minus t 2 minus h 1 h 1 cube minus 6 2 t 2 minus t 1 minus t 2 over h 1 cube 2 3 t 2 minus t 1 minus 2 t 2 over h 1 square 2 3 t 2 minus 2 t 1 minus t 2 over h 1 square the column here. We have y 1 y 2 x 1 and x 2 working one this equation, it give us 18 plus 2 s 1 plus 4 s 2 equal 0.

We have these equation s 0 and s 1. We have this equation in s 1 and s 2. For a moment I was confused as to what I was doing? Let us go back, we were saying that we are going to be using with free end condition. That would mean second derivatives at t 0 equal 0 and t 2 equals 2 will be 0. So, the second derivative of phi 0 at t 0 is 0 the second derivative of phi 1 at t 2 is 0. So, these are the two condition corresponding to that.

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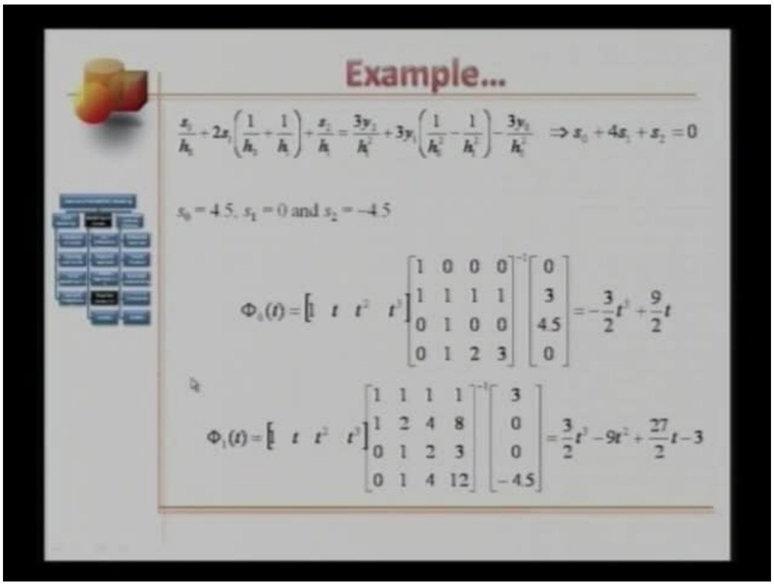

Now, this condition is familiar one, you have seen this before, relates the slop s 0, s 1 and s 2. In fact the value h 0 h 1 y 2 y 1 and y 0 will have s 0 plus 4 s 1 plus s 2 equal 0. So the three condition will gave as s 0 equal 4.5 s 1 equal 0 and s 2 equals minus 4.5 we compute what phi 0 t is 1 t t square t cube. We would be able to compute for the coefficient matrices, you need to compute inverse of that. We will have y 0 y 1 s 0 and s 1 here. We solve of this simplify this expression we get phi 0 t as minus 3 over 2 t cube plus 9 over 2 t. Likewise phi 1 t will be 3 over 2 adds t 2 minus 9 t square plus 27 by 2 t minus 3. We now know, what are the analytical expression of individual cubic segments, phi 0 and phi 1 are?

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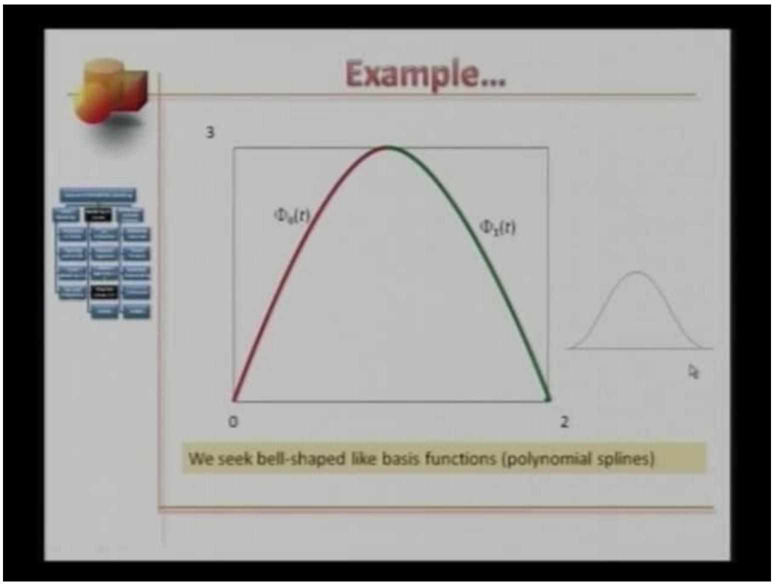

Let us (( )) red curve here and the green curve here represents the first cubic segment and the second cubic segment. We get a shape very similar to a bell shape, but Iam not saying that these would correspond to our basis function. We still have to do lot of work to be able to get nice well shaped. In the lectures that will follow this one, we would like to design bell shaped like basis functions, which will be composite polynomial splines, which will be locally barycentrics. Suppose to be global barycentrics, which are (( )) in the sense, we will be concentrating essential design these basis functions.