Computer Aided Engineering Design
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Lecture - 17
Design of Bezier Curves

Hello and welcome; this is lecture number 17 of CAD, we continue with the discussion on design seconds on curves. In previous lecture, we have discussed Bezier segment designs. We had also noted that, if we change the position of any control point, we observed a global change in shape in the Bezier segment. Let us concentrate on shape change Beziers today, and let us look at certain methods to which we can somewhere introduced local shape changes in Bezier segments or minimize the global shape changes.

There are two such methods that we will discuss today; one is based on the subdivision procedure that is, we try to subdivide the Bezier segments into two segments, try to change the shape of one of them. In the second method, without changing the shape of a Bezier segment, we will try to raise the degree of Bezier curve. In a sense, we will try to increase the number of control points, so that the shape change is at least minimizing even if is globe.

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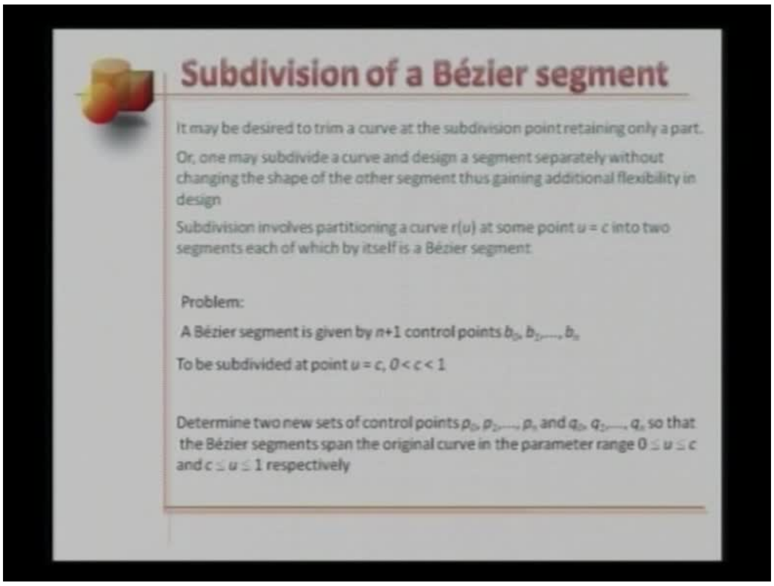

Let us take look at the first method, the sub division procedure of Bezier segment. It may be desired to trim a curve at the subdivision point retaining only a part of Bezier segment or one may subdivide a curve and design a segment separately without changing the shape of the other segment and therefore, gaining additional design flexible. Subdivision involves partitioning a Bezier segment r u at some point u equals c. Note here, at the value of c will be in the c 0 and 1 and it is going to be a particular value. Well subdivision involves partitioning curve into two segments, each of which by itself is a Bezier segment of the same type.

So, the problem now is a Bezier segments is given by let us say n plus one control points or get up points or design points, b sub 0 b sub 1 until b sub n. This Bezier segment is to be subdivided at point u equals c. As I mention the code the value c is should be in between 0 and 1. We need to determine two new sets of control points namely p 0 p 1 until p n. So, this is the first set and q 0 q 1 until q n, this is a second set.

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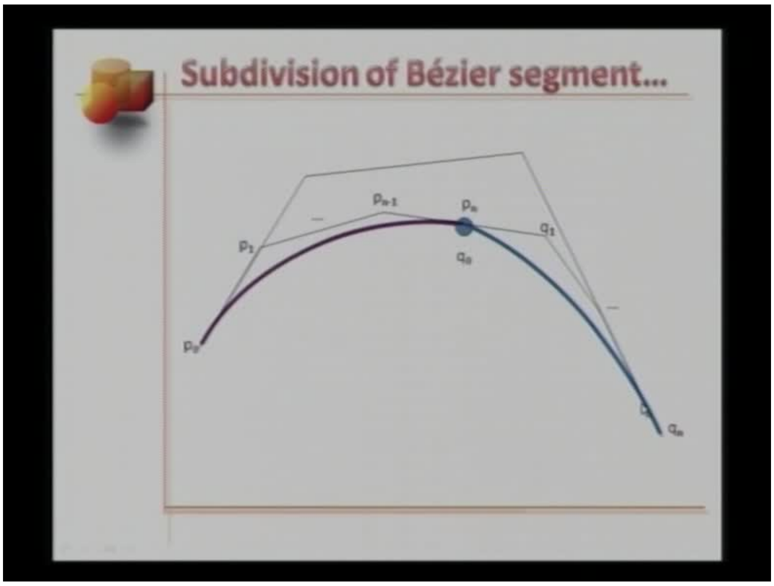

So, that the Bezier segments span the original curve in the parameter range 0 u c and c u 1 respectively. For a sense now, is to be able to identify two sets of control points, this one and this one, in a manner that a Bezier segments can be represented by two separately segment and original shape of Bezier segment remains the same. Let us see, how we can determine the control point? Let us say we have disc control polyline, well this is right.

This is for a cubic Bezier segment and let us say this is a Bezier segment. I plot a point for u equals c right here and I would like to subdivide this segment at this point. So, that the two parts are this one and this one respectively these parts independent each other. As I mentioned before, you as a designer might want to change the shape of this segment locally without changing the shape of the segment or you might one to tram a Bezier curve at this point, retain may be this segment and eliminate this segment or vice versa.

Anyhow, here many (( )). This is the first segment, this is the second segment. Next, try to sketch the control polylines for the first segment and this one for the second segment. Notice that the control polylines will change for the Bezier segment retain its shape. So, the first control polylines is given by points p sub 0, p sub 1 so and so forth, p sub n minus 1 and p sub n. The second one is given by q sub 0, q sub 1 so and so forth as q n. Notice few features here.

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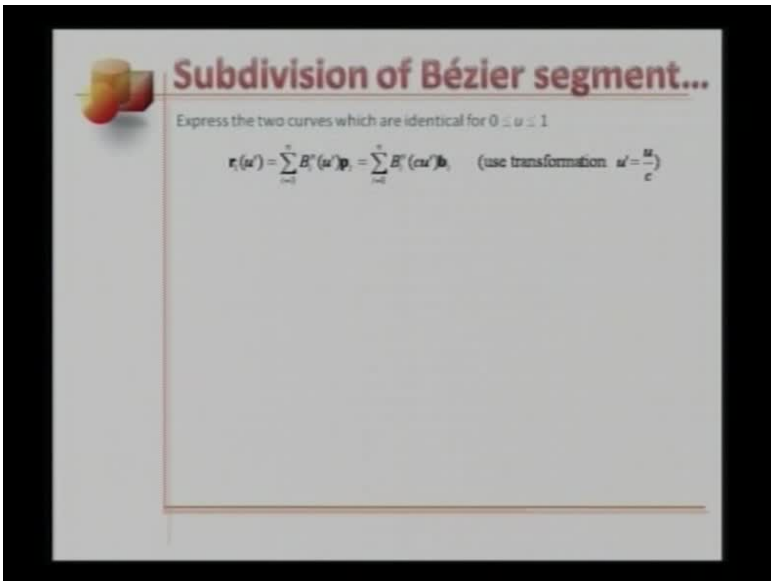

One that p sub n, q sub 0 are the same and the last tangent or the last segment of the curves for a line, a lines with a first segment of the second polygon, in the sense that the slope here are the same. The same control points here and the same direction slope here well maintain position and slope continent at this point. The question now is, how to determined these control points p’s and q’s? Well first express the two curves, which are identical for values of u between 0 and 1.

So, for the first segment r sub 1 will be expressed in terms of parameter u frank as summation i goes from 0 to n, b i constant along with a degree n, a function of u prime now times the control points to sub i. Notice this Bezier segment will be the same as summation i goes from 0 to n, the i constant along with or a new parameter segment which is new frank times the original design points occurs specify by u. Here we have used the transformation u prime equals u over c, what does that mean? When u is equals to 0 u prime is equals to 0 and when u is equals to c which will be the case for the first piece of segment, rather first is divided for subdivided piece of segment u prime will be 1.

Let us go back and try to understand this better. At this point, the parameters u and u prime will have the same value 0. At this point the original parameter u will have the value c, but new parameter value, u prime we have the value 1, due to that transmission. Now, this expression is going to be through for all values of u prime in between 0 and 1. How to be extract condition to obtain the new control points p i. Notice that p i will be n plus one in number, so we need n plus 1 in such conditions.

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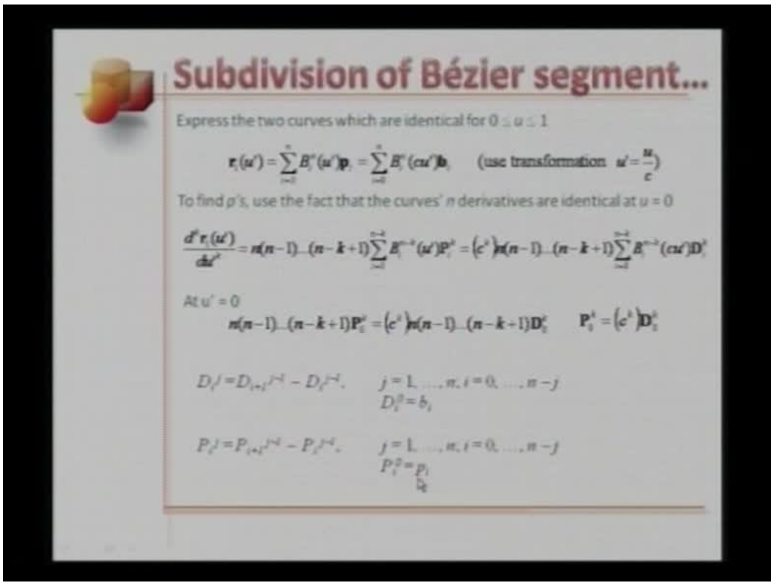

Well to find a new control point’s p’s, we use the fact that the curves n derivatives are identical at u or u prime is equal to 0. That would mean that the n derivatives of this segment and this segment, which are the same shape they will be equal at u or u prime equals to 0. We have seen this relation before the k eth derivative of a Bezier segment in this case r sub 1 as a function of u prime is given by n times n minus 1 times up till n minus k plus 1, summation i going from 0 to n minus k p i stands for along with for degree and minus k as a function of u prime time capital P sub i super k.

This is equal to c raise to k times n times n minus 1 after n minus k plus 1 summation i going from 0 to n minus a b sub i n minus k function of c times u prime times capital D sub i k. What is capital P sub i k? And what is capital D sub i k? It is something that we would know from previous lectures. Now, at u prime is equal to 0, this equation becomes n times n minus 1 times a few more terms, the last term in the multiplication being n minus k plus 1 times p sub 0 k is equals to c raise to k n times n minus 1, a few more terms times n minus k plus 1 times D sub 0 super k.

And note here, this is a k derivative of this Bezier segment. This is a k th derivative of this Beizer segment and both these derivatives are with respective u prime. Capital P sub i k, I am giving you a clue here will be in terms of the new control points P sub i while capital D sub i super k, will be in terms of the original derivatives P sub i. Well coming back to this expression, we will have these two terms getting cancelled out which should mean that should mean that P sub 0 k is equals to c raise to k times D sub 0 super k. In terms of the original data points P i D sub i super k are define such that D sub i super k is equal D sub i plus 1 super j minus 1 minus D sub i super k minus 1, where the values of j are going from 1 to n and the values of i are going from 0 n minus j.

While D sub i super 0 is equals to D sub i the original example. This is as you have guess it right, are reverse relationship. Likewise capital P sub i super k are define in terms of the new data points that we need to find. P sub i super k is equals to P i plus 1 j minus 1 minus p i j minus 1. Again j is going from 1 to n and i goes from 0 to n minus j. P sub i super zero is equals to small p sub i.

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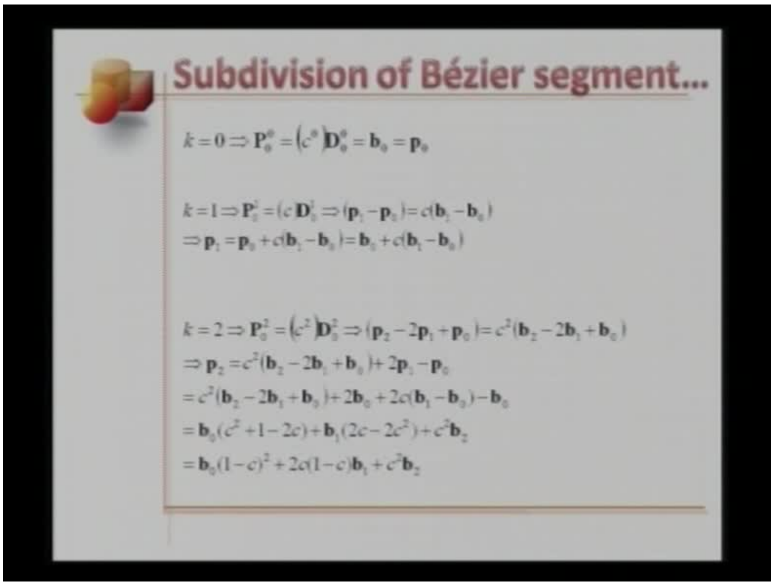

Now, for k is equals to 0 equation gives P sub 0 0 is equals to c rise to 0 times D sub 0 0, which means that the first control points is the same as the first whole control point. For k is equal to 1 P sub 0 super 1 equal to c raise to 1, which is c. Hence, D sub 0 super 1, which implies that small p 1 minus small p 0. It defense in between the first two new control points is equals to c times the difference between first two whole into minus 1. We already know what P sub 0 is? We can find what P sub 1 is. That is equals to b sub 0 times c b sub 1 minus b sub 0.

For k is equals 2, we have p sub 0 2 equals c square d sub 0 2. This would mean that D 2 minus 2 P 1 plus P 0 is equals to c square b 2 minus 2 b 1 plus b 0. You can compute what P 2 is? That is c square b 2 minus 2 b 1 plus b 0 plus two prime P 1 minus p 0. We can substitute for P 1 and p 0 from these expressions. Eventually after some algebra, we can find P 2 as b sub 0 times 1 minus c square plus 2 c times 1 minus c b 1 plus c square b 2.

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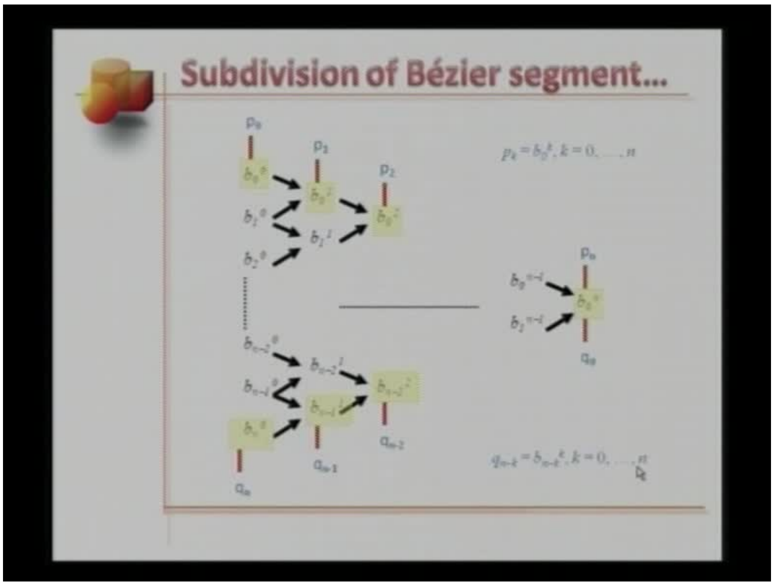

Now, this is interesting. Physically if we notice, then P 2 looks like it is a point on the Bezier segment rather a point on the quadratic Bezier segment define by control points b sub 0 b sub 1 b sub 2. Your c appears to be a prime, likewise over here P 1 would be like a point Bezier segment of degree 1. And you can this physical observation and extended to be get P 3 P 4 and so on and so forth. What we saw previously it was a little involved algebra to compute the new control points for the first subdivided Bezier segment. We have an alternative, which is surely based on geometry.

Now, recall this (( )) of intermediary a Casteljau points from the De Casteljau. If we recall the column corresponds to the original data points (( )) would had specified. The first column here the second column here rather correspond to stage one intermediate to De Casteljau points. These are stage two intermediate De Casteljau points. If u keep one power eventually get to stage n De Casteljau points, which would be a point on the n th Beizer curve for a given parameter value u.

Now, keep the expression for keep sub i in mind and remember what I told you is regard to the observation. Now, try to identify those p's in this converging prime. You would notice, that your first data point is actually is b of 0. Next you will notice, that stage one first intermediate De Casteljau points as point P 1. Stage two first intermediate De Casteljau point is a point P 2. Likewise the final point for a value of c now, so remember that d is intermediate at De Casteljau points or for u equals c. Stage n, which is the final point on the b is second for u equals c is horizon point P 7.

What have we observed without going through hints taking algebra, we can geometrically identify what a new control points P sub 0 p sub 1 p sub 2 p sub n are to the first subdivided is a segment and they correspond to the top edge of this. Now, using the geometric information that we have here and we get without using the algebra, what the new control points for the second subdivided Bezier segment would be? I am just asking you to guess, so you can be wrong. My guess would be at this point b sub 0 super n will be q 0. This point here will be q 1 and if I keep going from right to left, this intermediate De Casteljau point b sub n minus 2 super 2, which is the last point in stage two (( )) this would be q sub n minus 2. Likewise, the last point of the stage one repeated (( )) this intermediate De Casteljau point will be q sub n minus 1.

This one of course, the last data point will be q sub n. In summary therefore, the topped edge would represent the new control points for the first subdivided p is second, for u equals c. This bottom edge here of this triangle, will represent control polylines for the second subdivided p is second. Again for u it will c, notice that these are n plus 1 data points, so the corresponding segment here (( )). Likewise is again r n plus 1 data point for which this segment as well will be as well will be of degree n. Let us try to investigate the story further geometrically but before that we try to summarize this observation mathematically. p sub k is equal to b sub 0 super k, for values of k going from 0 n and q sub n minus k is equal to b sub n minus k super k, again a going from 0 to n

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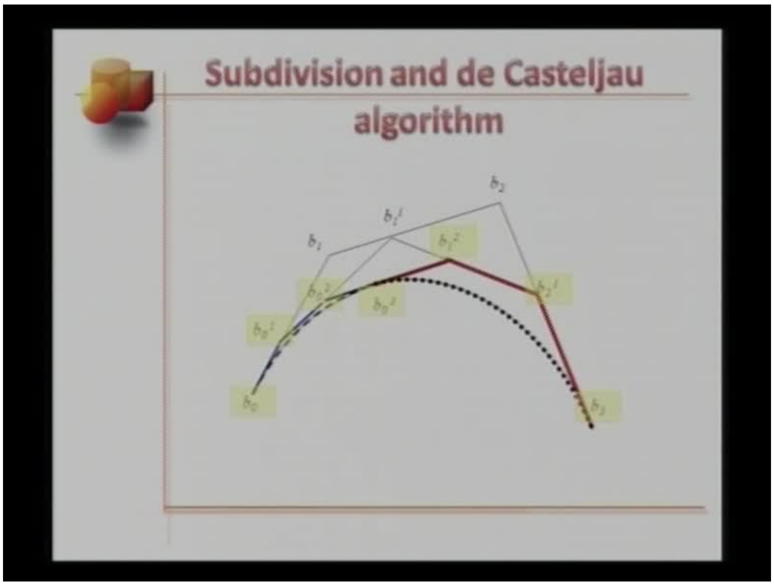

Now, let us try to see this geometrically. How subdivision and De Casteljau algorithm are related? We start again with a cubic piece of curve, we have this control polylines and we start a performing repeated in interpolations. This one the stage one repeated interpolation and this one second stage repeated interpolation. The curve that I just sketched is the original Bezier curve. These are original control points b sub 0 b sub 1 b sub 2 b sub 3 stage one, intermediate De Casteljau points b 0 1 b 1 y b 2 1 stage two intermediate De Casteljau point b 0 of 2 b 1 2. The final point the value of u again equal c b 0 3. You would had observed at this Bezier curve is tangent to the last segment joining piece of b sub 0 2 and b sub 1 2 and also b sub 3 but joining the Bezier.

An emphasizing these throne in b sub 0 b 0 1 b 0 1 and b 0 2 and b 0 2 and b 0 3. I am also emphasizing using a different colour these line segments, b 0 3 to b 1 2, b 1 2 to b 2 1 and b 2 1 to 3 segment is the first half individual curve. At this point here, is the second part of the design. The control of polylines in blue are define by the points p, p 0, p 1 until p n. Each other and the control polylines in red are define at point b sub 0, q sub 1. And from this figure, you would already know what the p's and q's are, given without going to be (( )). I just point as the value of u and must emphasize again equals c.

I am highlighting the control points p's and q's before we go on the next method let me pose a few questions. Try to figure what will happen if I change the location of any of these points? I must tell you here that subdivision procedure essentially let us go of this original control polylines and replaces that two sets of the data points again p 0 and p n and q 0 and q n. As a designer what would this mean? You would now, if free to work with to the data point on the control line in blue and a the data point in control line is red. In the sense we can move any of this control points to wherever you want or we can move any of these control points again to wherever you want.

Now, if I move p 1 which is identified by b sub 0 super 1, let say at different location here, which segment of the b 0 curve, the original Bezier curve will change? Would it be this segment or would it be this segment or would it be both segments. Think about the that for a little while or maybe while you are thinking I pose another question. What if I decide as a designer to move this point may be to a different location here? Again which one of these segments, which change in shape? This one or this one or both?

In a sense, what I am trying to ask is what you think now, that after subdivision or after subdividing the original Bezier curving into two parts, what I see is the local change in shape. My answer in the first question, if I move these points to different location, only this segment changes in shape. For that we (( )) answer to question two, if I move this point to different location, the segment on the right will change in shape, while the segment on the left will retain its shape, only if I move this control point, identified by a p sub n or to sub 0 because both this points are the same. I would witness the shape change in both segments, this one and this one. So, in way would you are agree in with me that subdivision introduces local shape change features in the piece of segment. If I were you, I would say yes.

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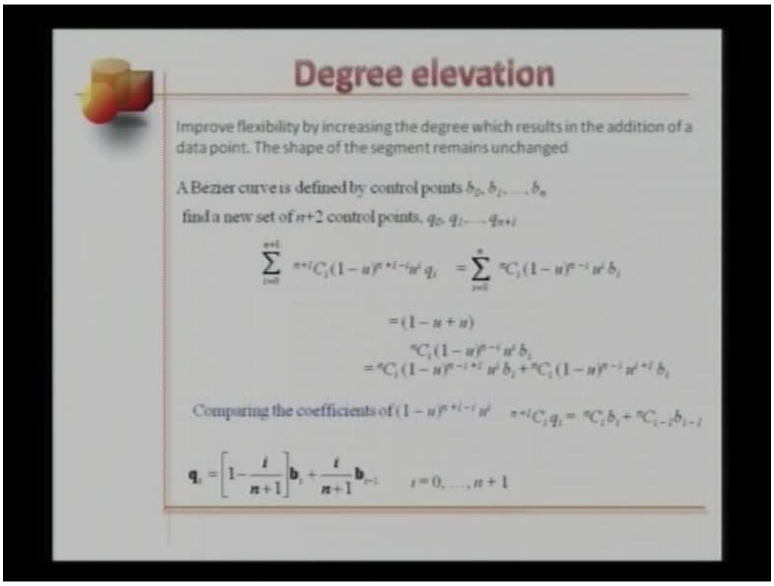

Just move one the second method, degree elevation. The idea here is to improve flexibility we design by increasing the degree of the original Bezier segment, which result in the addition of a data point. The shape of a segment well however remain unaltered or unchanged. We all know this, a Bezier curves is defined by controlled points b sub 0 b sub 1 b sub n, let us say these are n plus one control points here. Say, and the problem is to find a new set of n plus 2 control points given by q sub 0 q sub 1 until q n plus 1. So, that the original shape remains unchanged.

The new Bezier curve is going to give by summation i going from 0 to n plus 1 n plus 1 combination i times 1 minus u raise to n minus 1 minus i times u super i times q 1. This is how, the Bezier segment will be represented using new control points q i. You guessed this is the right, this is the i eth constant polynomial of degree n plus l. This entire Bezier segment will be the same as the original segment, as i have mentioned here before that the shape on the segment remains unchanged.

The original segment as we know is given by i going from 0 to n, n combination i times 1 minus u raise to n mines i times u super i times the original data point b i. Now, this expression is true for any and all values of u between 0 and 1. Using this expressions can we determined what q sub i will be, for values of i between 0 and n plus 1? Here we need to do a little mathematical trick. Notice, that this polynomial here is the degree n plus 1, while this polynomial here is degree n. How do I make this polynomial or degree n plus 1?

If you listen what if I say that, you multiply this expression by the term 1 minus u plus u. In a sense you were multiplying this by 1, but it is the terms 1 minus u and u which will elevate the degree of this expression or best Beizer curve by 1. This is precisely, what have I just said. If I see one term in this expression any term, then that is equal to 1 minus u plus u times n combination i 1 minus u raise to n minus i times u raised to I hence, b i. Absorb this term into this, to get n C i times one minus u raise to n minus l plus 1 times u raise to i times b i plus n combination i times 1 minus u raise to n minus i. Hence, u raise to i plus 1 hence b i.

All we need to do now is try to compare into visual (( )) of the term 1 minus u raise to n minus i plus 1 times u raise to i. Well even though I have not express, the summation sign here expressively, it is there. So, what is the coefficient of 1 minus u raise to n plus 1 minus i times u i in this expression, that is q sub i times n plus 1 combination i. From here, that is n combination i times b I, which is this term and from here it may not be happening as of now, but if you try to change this index i 2 i minus 1, this would become i minus 1 plus 1, which is i and this would become n minus i plus 1. So, there we have this expression also contributing the term and that term will be n combination i minus 1 times b minus 1.

We can work the algebra out. You can expand n plus 1 combination I, in terms of respective factorials. We can rearrange term to eventually that q sub I equals 1 minus i over n plus 1 entire n times b i plus i over n plus 1 times b sub i minus 1. But you notice here u i is express as a linear interpretation between points b i and b i minus 1. Also notice, that terms here i by n plus 1, which is the same is this term changes, as we decrease the index from 0 to eventually interest 1.

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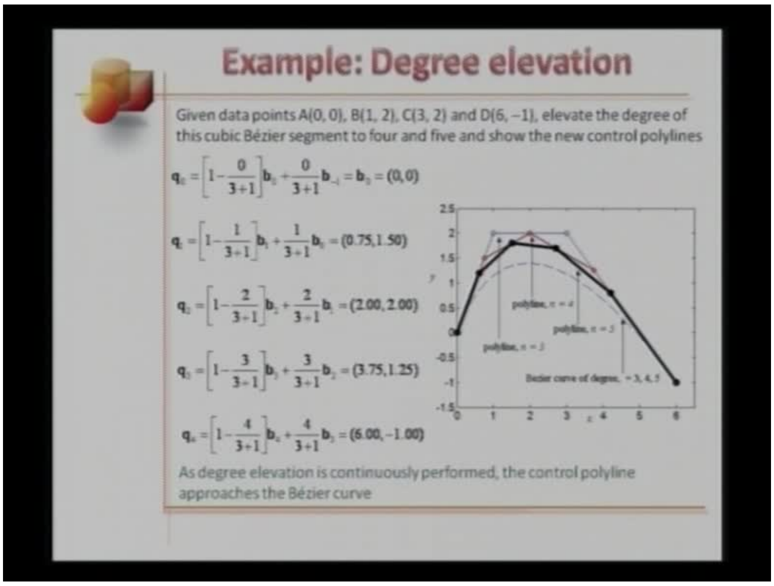

So, like in case of these segments, it repeated linear interpretations. We had held the parameter value constant, if you recall the De Casteljau. In this case, we perform the similar procedure and it is to be consider this to be parameter value, this would change and infact increases as I increases. Again this index i will go from 0 n plus 1. Let us take an example, the given data point A, B, C and D, the two dimension of coordination's are for A 0 0, for B 1 2, for C 3 2 and for D 6 and minus 1. We raise the degree or we elevate the degree of this cubic Bezier segment 2, 4, 5 and so on so forth.

We try to determined and show the new control points and therefore the new control polylines. We have just seen this expression all we need to do is plug in the values and gets the q ones. For example, q sub 0 is 1 minus 0 over 3 plus 1 times b 0 plus 0 over 3 plus 1 times b sub minus one1. Now, this is interesting, although we do require the original index points with index minus 1, well this because this i is 0 that we do not actually need. In the sense q sub 0 will be equal to the original data point, the first data point b 0 0 in this case. q sub 1 keep on increasing the index, this is 1 minus 1 over p plus 1 times p 1 plus 1 over 3 plus 1 times b 0 compared as 0.75 and 1.5 as the x and y co ordinates.

q 2 is equal to 1 minus 2 over 4 times b 2 plus 2 over 4 times b 1 x and y co ordinates are 2 and 2. For q 3 its 1 minus 3 over 3 plus 1 times b 3 plus 3 over 3 plus 1 times b 2 co ordinance three 3.75 and 1.25. Likewise we can conclude the final point q 4. The co ordinates are 6 and minus 1, which is the last point here. Let us take local spread the control polylines in blue, the original 4 data point polyline for witch this is a cubic curve. The control polyline in red here which has this data point, this one, this one, this one and this one represented respectively by q 0 q 1 q 2 q 3 and q 4, represent the polyline for a degree for Bezier segment.

Notice at the (( )) for Bezier curve would be the same in the shape as degree free Beizer curve. There is no shape change. Likewise using these q's I can raise the degree of this Bezier curve by 1 again. I will have 6 data points and those data points will be shown by the control polyline in black line 1, 2, 3, 4, 5 and 6. I can keep on elevating the degree of Bezier curve without changing the shape. So, from the desire prospective, what is happening? I am introducing additional data points or desire points and retaining the shape of the curve.

Try to compare two scenario. Had I work with the polyline with 4 data points at 2 polyline here and had I move this point to different location? Will this change in the curve shape the significant compare to the following case that I am taking about. The case where I move this point to may be a different location. I would say that had i moved this point to the local change in shape over here would have been left compare to had I moved move this point to different location. While I have just said local change in shape, I know and you also know that the actual change in shape will be globe. What i mean by a local here was the change in shape in this region in the neighborhood of the data point, which I had moved. Another point if I keep on raising the degree of Bezier curve, I can do it infinitely many times.

As I keep point raising the degree of this Bezier segment, notice how the new control polylines becomes closer and closer to this Bezier curve? Blue one the is little far and red one with 1 degree is higher is at little close and black one with 2 degrees higher as suppose to the blue control polyline, which is even closer. If I keep on raising the degree I willl closer the closer to this curve, of course again without the changing shape of the origin deception. In fact this curve by itself would represent the control polyline constituted by many, many points, if you think about it from the prospective degree elevation. In summary, as degree elevation is continuously performed the control polyline approaches the Bezier curve.