Computer Aided Engineering Design

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Lecture - 15

Welcome lecture fifteen of computer aided engineering design. Here we discuss how to design bezier segments and curves, in the lay out here down to ladder. Here just done discussing how to design progressive segments and compulsive progressive curves, now we are discussing Bezier segments and curves. Why there need for other curve design models? In previous lecture we have discussed how to design progressive segments and curves, it is not very intuitive to specify the slope information or tangent information. This is especially when are designer wants to design three dimensional composite curves. The designer is more comfortable in specifying the data points as suppose two higher order information likes slope of coverage. We are looking for alternative curve design methods that allow to specifying only design points or data points, while maintaining local shape control.

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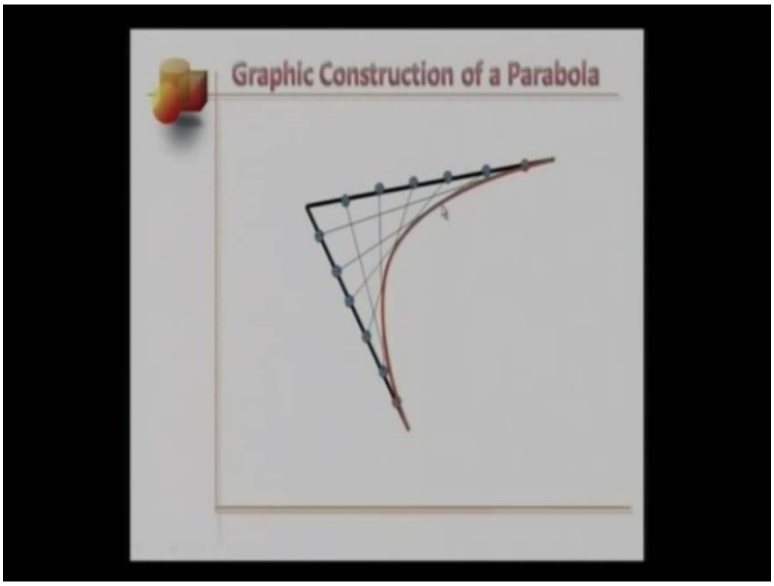

Let us look first at the graphic construction of a parabola all of you must have done this in our first year engineering drawing course given two line segments, how construct a parabola using that. The key idea is this. We divide the two line segments, such that the ratio of this length over this length is the same as ratio of this length over this length, and then joint these two points. We repeat the procedure again hear this length over this length is the same as this length over this length. Once again this length over this length is the same as this length over this length. You would keep continuing the process

And finally, you would want to draw a curve, that passes through these points, to these points an expanding to through all this line segments. The curve in the red is your parabola a constructing the previous line alludes to the three tangent theorem to construct a parabola. .We have two segments (( )) try again such that length of this segment over the length of this segment is the same as the length of this segment over the length of this segment. We joint these two points and be mark another point on the segment maintaining the length equation. We have seen before that the curve joints. This points this point and this point is parabola, to add further is curve is also that that this three line segments a tangible prove it.

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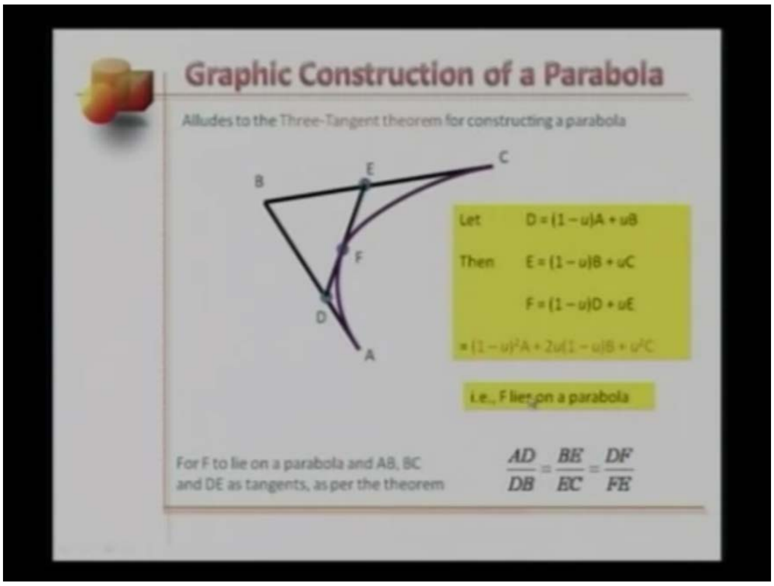

Let us normal play different point, point A point B, C, D, E in F for F to lie on A parabola and line segment A,B,B,C and D E to be tangents to the curve as per the thermo this length per shows that is D over DB should be equal to BE over EC and that should be equal to DF over FE. We have seen this relation four any points on a line segment and be expressed that D equals 1 minus u times a plus u times B u is the parameter. That assumes the value between 0 and 1 A and B are position vector and d is the position vector of a point. On the line a b for u equal 0 d would merge with a and for u equals one

D would concede with b likewise point E on the segment B C is given as 1 minus u times B plus u times C note this, how the (( )) ratios are be maintained. The value for parameter u would be identical this relation and finally, f as 1 minus u times D as u times e again the length ratios A maintained if substitute for D from hear and for E from hear we will get F equals 1 minus u squared times A as 2 u times one minus u and B u, squared times C once again A ,B and C, A position vector of this three points if u want to be very difficult for us figure using co-ordinate geometry that F would lie parabola.

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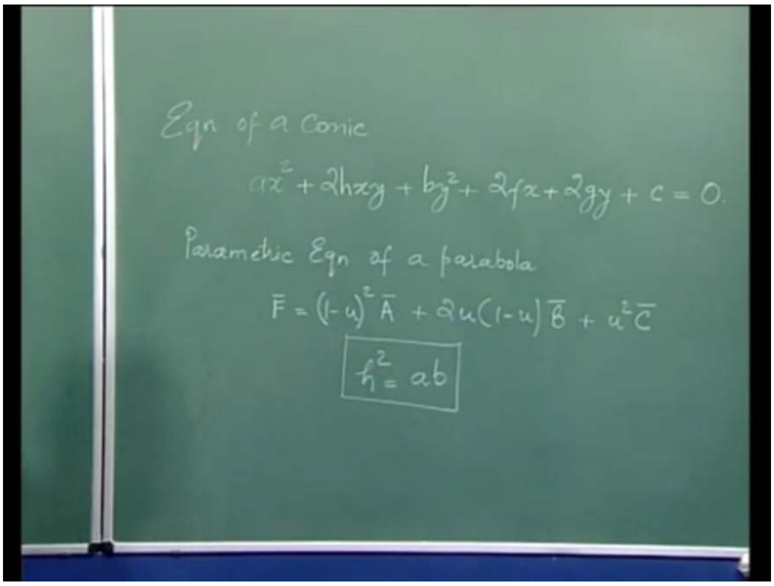

This may be one way to verify how point f would lie on a parabola the generate the equation. Of a conic is given by a x square plus 2 h x y plus b y square that is 2 f x plus 2 g y plus c equal 0. Hear a h b f g and c are none concept hear see from the construction before that the parametric equation of a parabola is given by f equals 1 minus u squared times a plus 2 u times 1 minus u time b plus u squared time c 1 can extract the x component of F and the y component of F and substitute those condition over here to determine these unknown factors and then 1 could verify for a parabola that h square equals A B.

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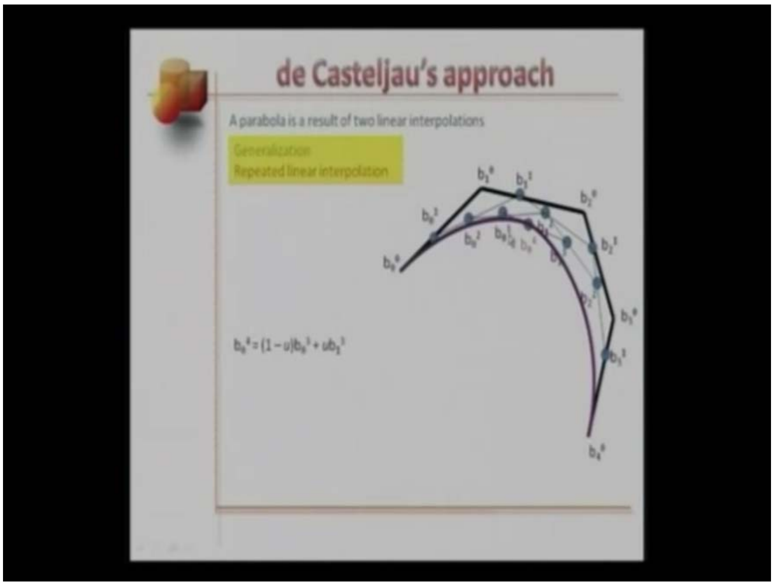

Let that be an exercise for u the de castelijau’s approach the constriction one may not constrain themselves him self’s or herself to only two segment to start with one would notice that parabola is a result of two linear interpolations a generalization that Castalia’s were done was based on repeated linear interpolations see that parameter assume that. We have this control poly line as we call it a set line segment edges with each other lets normal plate points. This point here is b subs 0 to super 0 are explain. What the subtract and superscript mean in while this point b is 1 super 0 b sub 2 0 b sub 3 0 and b sub 40 for this example the subtract. Here represents the index of this end point and the superscript represents this cage of linear interpolations .and

This time we have not perform any linear interpolations such therefore, this number superscript is 0. Lets now start performing the first stage interpolations for given value parameter u this length ratio is specified. I can mark this point as b 01. This 1 now represent this first interpolation this is the first point. I have mark 0 a 2 and maintain the length ratio and keep marking that point. This point here sub one two per one this point here is b sub 2 for 1 and then as b subscript super 1. I joint this point through this line segment and I cannot stop. I keep performing the interpolation this point here on this line segment represents the first point of stage two linear interpolation that is why it is (( )) as b sub 02 notice that. I maintaining the length ratio throughout may linear interpolation. The points on the second segment. This b 1 2 the point here is b is 22. We joint these three points and continuing the linear interpolation.

This point here as b sub 03 represents. The first point of stage three linear interpolation this point here is b sub one three and finally, with joint this two points the mark here the point here on this line segment and (( )) this point as the first point of the fourth stage in linear interpolation. We would realize that we can no farther where linear interpolation any more this curve here would very similar to parabola let discus before and this curve pass through the first point the last point and this final point as a result are repeated in interpolation also the first time segment the last time segment and this line segment over hear and all be tangent prove it. very similar to..

What we had seen in case parabola let us try to work backwards. Let start form the b 04, it is the result of linear interpolation between b 03 and b 13 and given parameter u it is 1 minus u time's b 0 3plus u time's b 13 b 03. Itself is as a result of linear interpolation between b 0 2 and b 1. We know what the relation is. We directly substitute the result for b 03 and likewise b 13 as well realizing that b 13 rights. Here is a result a linear interpolation between b 12 and b 22. We will get the final expression as 1 minus u time 1 minus u b 02 plus u times b 12 plus u times 1 minus u times b 12plus u times b 22 all .We in need to do is keep on substituting.

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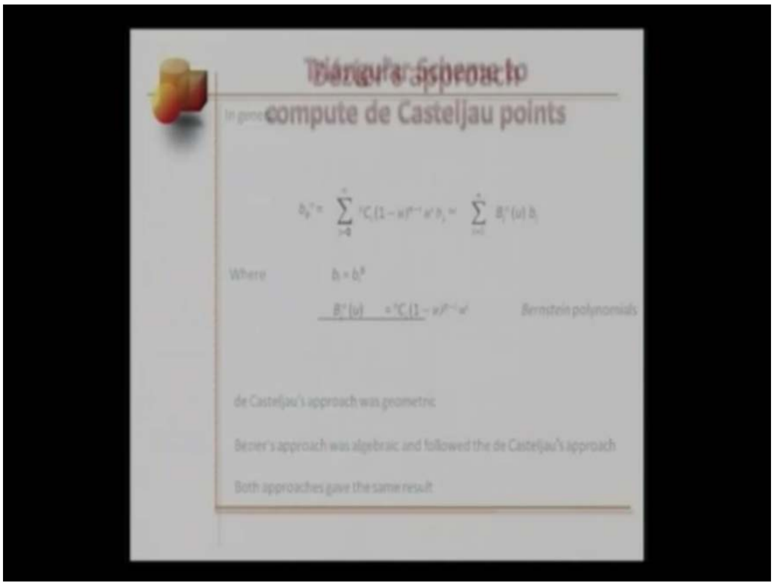

For this points here which will across the intermediate the Casteljau's approach it be keep on working the final result will look at this. The left hand slide. Which represents a point on the curve for a given value of u b 04 equals. 1 minus u raise to 4 times b sub 0 super 0 plus 4 times u times 1 minus u times b1 0 plus 6 u squared times. 1 minus u squared b 20 plus 4 u cube and 1 minus u time b 0 plus that u raise to four time 40 notice that this expression. Here is a degree 4 in u. This superscript there for represents the degree of the curve of the note that b 00 b o10 b 20 b 30 and b 40 are design point that have satisfied may be interactively by the designer.

Bezier approach is not geometry it was (( )) in general a point and an nth degree curve and written as summation. I going from 0 to n combination i times 1 minus u raise to n minus. I times u raise to I times bi, bi are the design point specified by the designer. This is equal to summation. I going from 0 to n b sub i super n as a function of u times bi, it be relate bi to the normal or curvature that. We seen before bi would be the first stage design points bi 0 and capital b subi would be function of u that be equal to m combination i times 1 minus u raise to n 1 minus i times u i this is term like here capita B sub i super n are called Bernstein polynomials as a said before the casteljau's approach with geometric, for Bezier approach algebraic and it flowed the Casteljau approach. It is interesting to note that both the approaches gave same result if u look at the expression b sub 0 super n equal. i going from 1 to n when summation capital B i super n are combination of u time bi there n points. this expression represents. Bezier curve a degree n and in fact both. Bezier approach and de casteljau's approach gave same Bezier segment.

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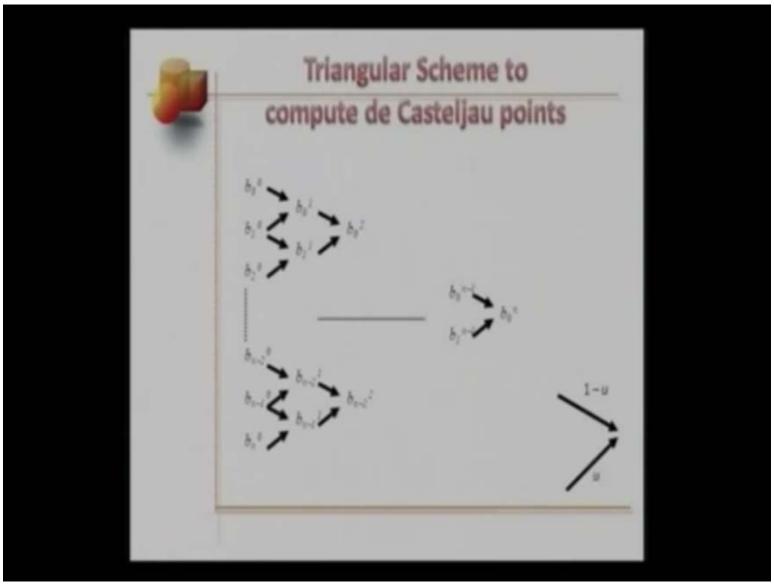

Coming back to de Casteljau's approach there is a very nice triangular scheme to compute the intermediate de Casteljau's points. Let see how first. We have need to do is a range all the design points. In the first Colum b 0, 0 b 1 0 b 2 0 the keep on point. B n minus 2 0 b sub n minus 1 super 0 and b n 0 the start placing. In the second columns.. The first stage interpolation point we using this point. Here b 0 0 and b 1 0 combined them to get b 0 1. Likewise, we use b sub 1 super 0 b substitute super 0 combined them together and get the first stage linear interpolation. In this linear interpolation being performe for unknown value of u, we keep on going to combined eight design points to get any point in this case b n minus 2 1.

And finally, we combined the last two design points. To get b sub n minus 1 super 1.The third column would have all. Intermediate Casteljau's points has a result of the linear interpolation between points. In the first column for example. We combined b 0 1 b 1 1.We get u are right b 0 2. Likewise the combined b n minus 2 super 1 and b n minus 1 super 1. We get once again u are right b n minus 2 super 2. We keep on following this game. Until we see last two points. In last but, one stage of linear interpolation as b 0 super n minus 1 and b 1 super n minus 1. And finally. This combined this two get are b sub 0 super n notice. Once again that is this point will lie,. 1 degree curves in the parameter u the arrows falling downwards represents multiplication with factor. 1 minus u and those falling upwards represents multiplication by u.

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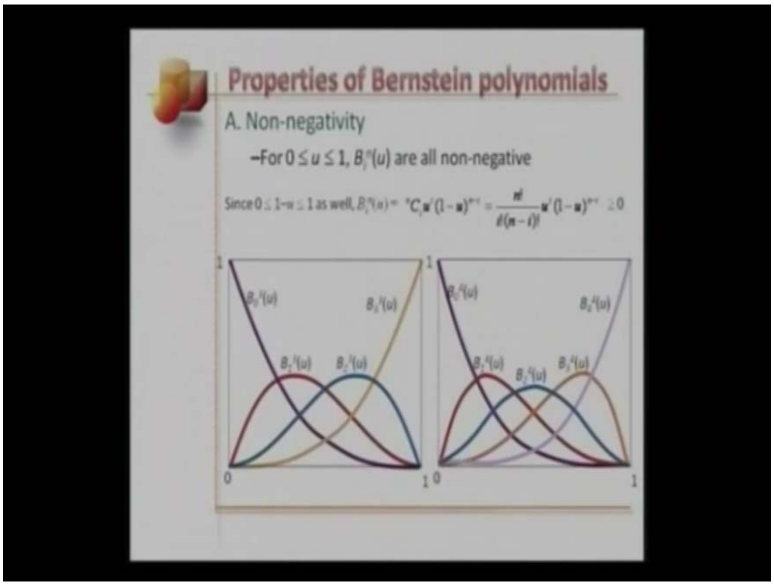

There is a point. I want mentioned here this column very first column contain a position factors of all design points by that and mean that is entire triangular seeing we working individually with x coordinates, with the y coordinates and this is e coordinates is point here second column on the third column and so on so called would be again the position factors of all the intermediate the de Casteljau’s points. One would keep that mind.

This, now discuses some properties of, Bernstein polynomials first one non negativity for parameter values between 0 and 1 all. Bernstein polynomials capital b sub i super n are all non negative well. We known that in Bernstein polynomials are given by n combination. I times u raise to i time 1 minus u raise to n minus i. We can expand n combination i in terms of n factorial over i factorial times n minus i factorial notice that for u to alive between 0 and 1. This is positive and this is positive. So this is no way in which this. Bernstein polynomials will be negative.

Let us look at few examples. These are Bernstein polynomials degree 3 Bezier curve. This is b sub 0super 3 this is b sub 1 super 3 this 1 b sub 2 super 3 and this b sub 3 super 3. We would notice that there are all non negative. Another example of degree 4 Bernstein polynomials this 1 is b of 0 super 4 this 1 here is b sub 1 super 4 this is b sub 2 super 4 .This 1 b sub 3 super 4 and finally, this 1 b sub 4 super 4.Once again all of him non negative.

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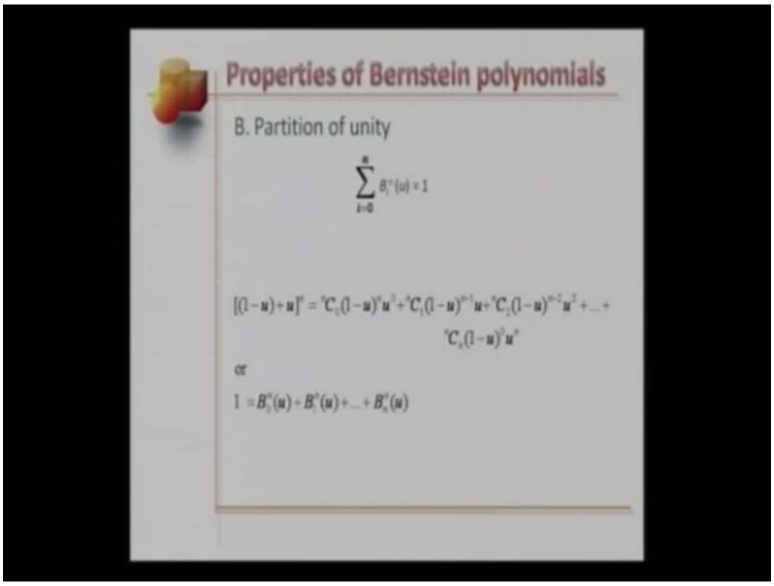

The second properties of Bernstein polynomials this one relates to partition of unity. All Bernstein polynomials this sum to one this is interesting you start with 1 and v act with minus u and plus u the result is excepted 1. What do now is raise it an exponent n as. We would know again left turn side will be 1 and right and side can be expressed as by normal expiation minus u , plus u raise to n. This is have to look n combination 0 times 1 minus u raise to n times u raise to 0 plus n combination 1times 1 minus u raise to n minus 1 times u plus n combination 2 1 minus u raise to n minus 2 times u squared and many terms. This would be the last time here n combination n times 1 minus u raise to 0 times u raise to n.

If, we notice the right run side will have term that would individually corresponded. Bernstein polynomials for example. This term here would represents. The first Bernstein polynomials of degree n this time here would be the second Bernstein polynomials again degree n and the last point here will be capital B sub n super n, for the left hand side would still b 1. We have exactly shown this relation. Once again all Bernstein polynomials of the same degree would have to 1 and this is partition of unity properties.

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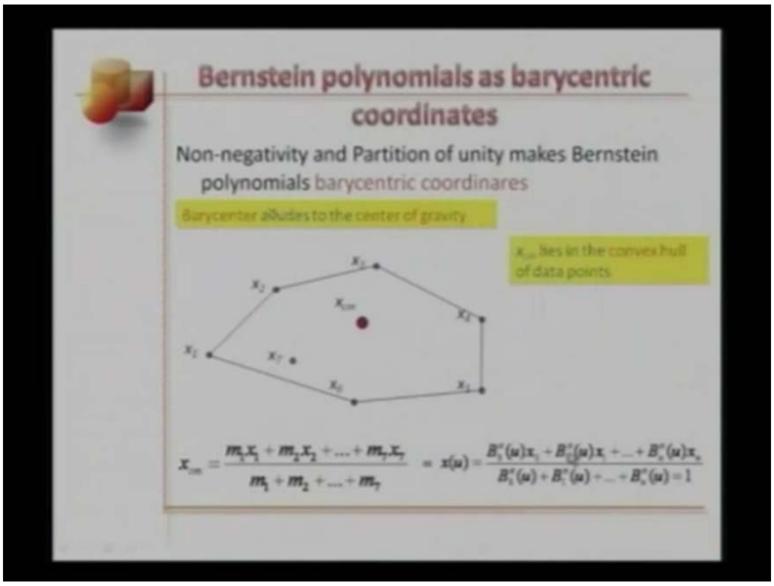

Next, Bernstein polynomials as barycentric co-ordinate. I will tell you barycentric means in a why non negativity and partition of unity properties to make Bernstein polynomials. Barycentric co-ordinates. .The term barycentre refers to the center of gravity. Let say they have a point x ,1 x 2 ,x 3 ,x 4 ,x5 ,x 6 and x 7. In terms of masses associated with these points. ..For example, m1, m2, m3, m4 ,m5, m6 and m7 a test possible for us to compute the center of mark of these points. That would be somewhere shown by the red that over here x the m the coordinate of the center of marks. We also know by a geometry that this center of marks would lie within the contextual defined by this points. So the red dot will never be outside on contextual this to re emphasizes x e n will always lie within the contextual of data points. This expression is very familiar the co-ordinates of the center of mass would be given by m 1 x 1 plus m 2, x 2, plus m 7 ,x 7 over sub all several masses. For this example here x would either v x co-ordinates or the y co-ordinates.

We replace all the masses for Bernstein polynomials. We have x is the function of the u know because all of the Bernstein polynomials are function of u respectively. We replace m 1 by say b sub 01 m 2 by say b sub 1 n and so on so four. We get a cube as b sub 0 n times x zero plus b sub 1 n times x 1 so on so four after b sub n super n times x n over this sum of all Bernstein polynomials there have seen using the partition unity property that all is Bernstein polynomials with sum to 1 and there for expression here represent of point. On the Bezier curve… What about trying to say here ,well for given value of u between 0 and 1 x of u will always be seeing to be within the contextual of data points no matter. What u is in other words a Bezier curve will always lie within the contextual of data points. And there is something that will see late. We will just for u keep in mind as of now.

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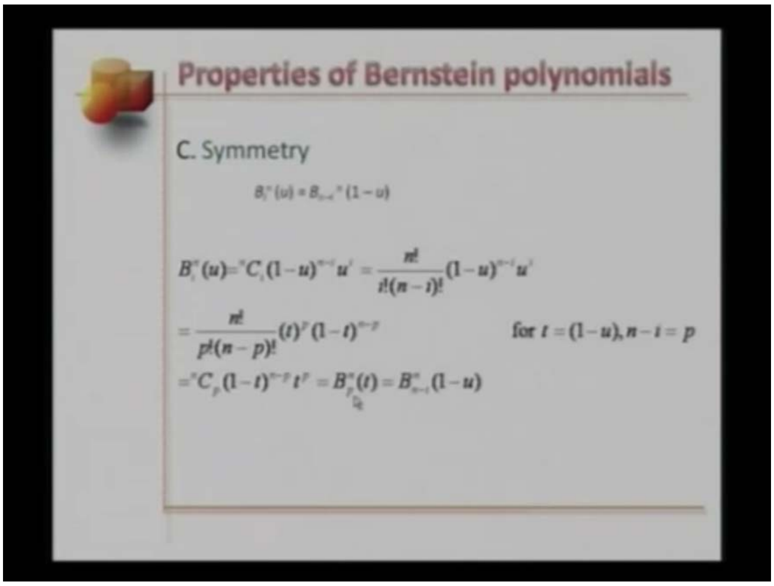

The third properties of, Bernstein polynomials symmetry capital b by super n is the same as capital b sub in minus i super n as the function of 1 minus u. This start with left urn side b sub i super n if given by n combination i 1 minus u raise to n minus i can v raise to v expand and combination i over here. This expression is equals to factorial n over factorial b ,time's factorial n minus b, time's t raise to p time's 1 minus t raise to n minus p. If we replace 1 minus u with t and 1 minus i with p. Here this expression becomes n combination p times 1 minus t raise to n minus p raise to p which is in fact the p the Bernstein polynomials a degree n and. We replace p here and t here. We get b sub n minus i super n 1 minus u which is the right urn side here.

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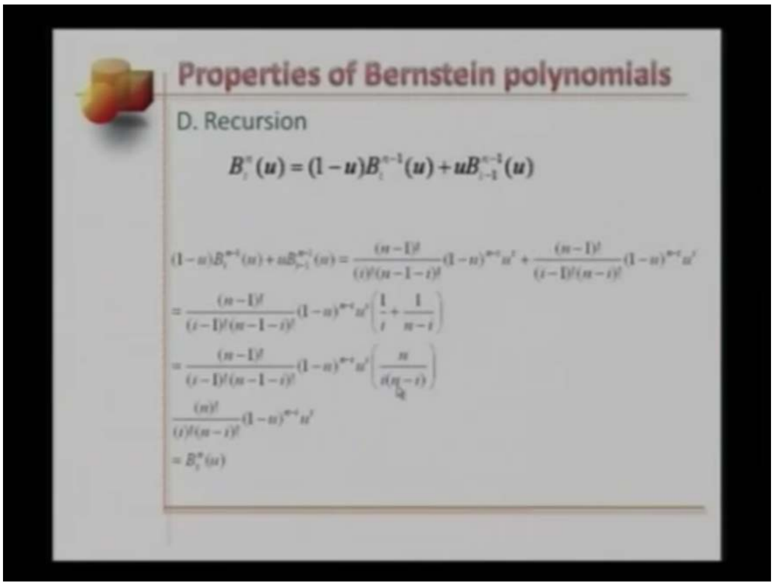

The fourth properties recursion b sub i super n function of u is equal to 1 minus u times. B sub i super n minus 1 function of u plus u times, b sub n minus 1 super n minus 1 u. What why mean here the i Bernstein polynomials a degree n and expressed as a linear combination the i th Bernstein polynomials a degree n minus 1 and i minus 1. Bernstein polynomials again of the degree n minus one the left factor one minus u and u here they prove this let us start with the right urn side 1 minus u times b i super n minus 1 plus u times b n minus 1 ,super n minus one that expand this. Bernstein polynomials to get 1 minus 1, factorial over factorial i ,factorial n minus i minus 1 times. 1 minus u, raise to n minus I, times u i plus n minus 1 ,factorial over i minus 1,factorial times n minus i factorial times 1 minus u raise to 1 minus i u. I notice that these two terms are absorbed in this expression.

We have work for the right run side what can u do u can take, this factor of common which is n minus 1 factorial over i minus 1 factorial n minus 1 minus n factorial 1 minus u raise to n minus i times u i and .what would be left with the one over i plus 1 over n minus I, a little bit of algebra is term here will be n over i over m minus i n this n minus i termed get probably here get 1 minus i factorial the psi aborted by here. We get i factorial and n times n minus 1 factorial is infectively and we have write here 1 minus u raise to n minus i times u i and this is b ith. Bernstein polynomials are degree n which is the left hand side.

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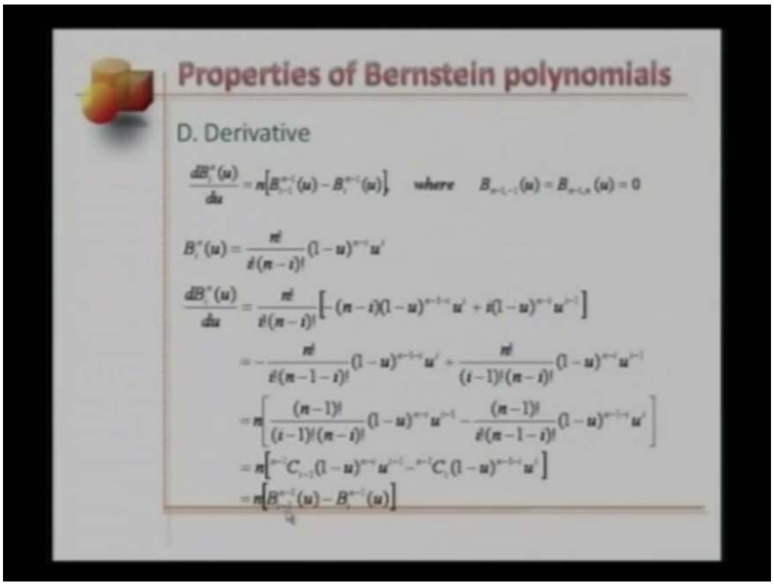

Just look at some derivative of Bernstein polynomials will need this derivative later to design compulsive. Bezier curves to impose slope and covertures continuer condition assumption point between individual Bezier segments. We already know how Bernstein polynomials are expressed. As function is give the first derivative of n nth degree. Bernstein polynomials is given by b sub i minus 1, super n minus 1, minus b sub i super n minus 1 here. We follow contextual b sub n minus 1 sub minus 1 is equal to 0 and b sub n minus 1, sub n is equal to 0, notice term using a slightly different conventional for. Bernstein polynomials over here the first term here represented the degree. Which is a same. It by represent the degree by the superscript here many books can to follow this location.

We know what a Bernstein polynomials a degree n it is n factorial over factorial i factorial n minus i times. 1 minus u, raise to n minus I, times u raise to i ,differentiate that with respect to u to give factorial n over factorial lie factorial n minus i this is n minus i times 1 minus u, raise to n minus i minus 1 and there would be a negative design of here. We differentiate this term here, to the respect to u such i times 1 minus u, raise to n minus I, u raise to n minus 1. We absurd this term in side to get factorial n over factorial i times n minus 1, minus I, factorial times 1 minus u raise to n minus i minus 1 u raise to I, plus factorial n over factorial i minus 1, times factorial n minus i multiplied by 1 minus u ,raise to i minus 1. A little bit of work will leads to common factor n that multiplied by n minus 1 combination i minus 1 times 1 minus u times u raise to i minus 1 ,minus n minus 1 combination I, 1 minus u raise to n minus i minus 1 t,imes u raise to i. and this is i minus 1 Bernstein polynomials of degree n minus 1 minus b I th Bernstein polynomials of degree n minus 1 which is same expression over here likewise the second third fourth derivatives can also be compute.

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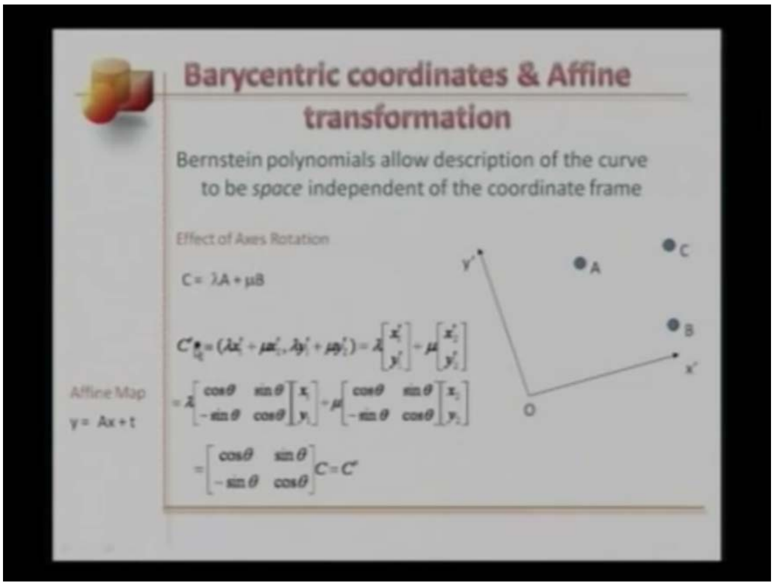

Now let discuses Barycentric co-ordinates and affine transformation. Berycentric polynomials allow description of the curve to be space independent of the co-ordinate frame, and will see how the using a simple dimension first. What is in a affine map and affine map is of the type y equal ax plus t, x would be the Casteljau vector of a point a is a transmutation metric and t is translation vector ,y is the Casteljau vector of the results.

We seen before that if u use margin co-ordinates for x and y this translation vector get observed in the 4 by 4 transmutation metric and enough lets first investigate how acts with rotation will effect relation. We have 2 dimension participant spaces. We have point A another point B and point C. Which is represented as a waited liner combination of points A and B is the waits are scalars lambda and mu if u rotate a axis to expand. y frame maintain the position of the origin and the rotation angel here what happen well a frame this point. Here will now be expressed in term of new co-ordinates its one point and one point as two dimension rotation metric with terms coos sin theta sin of theta minus sin of theta coos sin theta times original position of a given by co-ordinates x 1 and y 1 likewise. The new co-ordinates of B will be given by x 2 frame y 2 frame which is equal to same rotation metric. Here 3 multiplying a original co-ordinates of B given by x 2 and y 2.

If, we maintain this linear combination the frame start will have be x co-ordinates as lambda times. X 1 frame plus mu times x 2 frame and the y co-ordinates will be lamed times y 1 frame plus mu times y 2 frame in make it form. This will be lambda times. The column vectors x 1 frame y 1 frame plus mu time. The column vector x 2 frame y 2 frame try to relate this expression. This expression here if i substitute for x 1 frame x 2 frame y 1 frame y 2 frame the right urn side would become lambda times.

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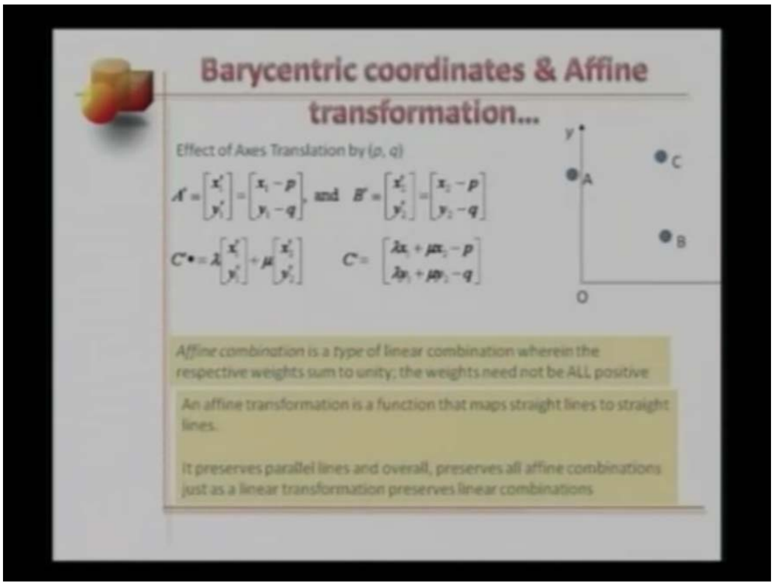

This rotation matrices times. The original co-ordinate away plus new times rotation metric times original co-ordinates of B, i can take the rotation metric common and will have lambda times. X 1 y 1 plus mu times x 2 y 2 which is nothing but, C so we have observed that the coordinates of C one times star is the same as the coordinates in other words the rotation of the axis does not affect the linear combination.

Now let study the effect of axes translation by vector p q. We have the original axis point a point b and point c let translation. This axes by vector of p q the new co-ordinates of a with respective. These translated axes will be given by frame. Which is x 1 frame y 1 frame? Which is equal to x 1 minus p y 1 minus q? The x co-ordinates of b frame will be given by x 2 minus 3 and y co-ordinates of B frame will be given by y 2 minus q. If we maintain the same the linear combination will have the frame start equals lambda times A frame plus mu times B frame. While C frame will be given by lambda times. X 1 plus mu times x 2 minus p is the x co-ordinates and the y co-ordinates will be lambda times y 1 plus mu y 2 minus q. How they did a get this expression.

Well not that c was expressed initially as the linear combination between a and b and this translation A axes would directly effect that relation let for the see C frame star is the same C frame. We start with C frame start. We substitute the co-ordinates. X 1 frame y 1 frame and x 2 frame y 2 frame from here. We get c frames star equals lambda time x 1 minus p y 1 minus q plus mu times. X 2 minus p y 2 minus q the x co-ordinates C frames star is lambda times. X 1 plus mu times x 2 minus of lambda plus mu time p and the y co-ordinates is lambda time y 1 plus mu time y 2 minus lambda plus mu times q .What have done here as. I have added in subtracted p form the x co-ordinates and likewise added and subtracted q from the y co-ordinates infect this two expression are respectively the same as the two expression now if see wants to expressed in the new that is translated co-ordinate system. We should have expected. The co-ordinates of C frame to be identical those of C times star.

This would mean that this expression should be the same as this expression and y co-ordinates c frame should be the same are the y co-ordinates of C frame star and this can happen only. When the 2 scalars sum 2 1notice. That be never mentioned that lambda and mu cannot negative or 0 for the map all. We see here is the 2 scalars. They should sum 1 affine combination is a type of linear combination. Where their respective weights sum to unity as a mentioned for these way need not be all positive an affine transmutation is a function that map straight line to straight lines it preserves parallel lines and overall preserves all affine combinations just as a linear transformation preserves all linear combinations.