Computer Aided Engineering Design
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Lecture - 13

Hello and welcome to CAD series of video lectures. This is lecture 13 on Differential Geometry curves. The layout they are in the second column here. Why? Do we need to study differential geometry or differential properties of curves. Well as I mentioned in the previous lecture. We are going to be fading segments of smaller degree through sub groups of these design points.

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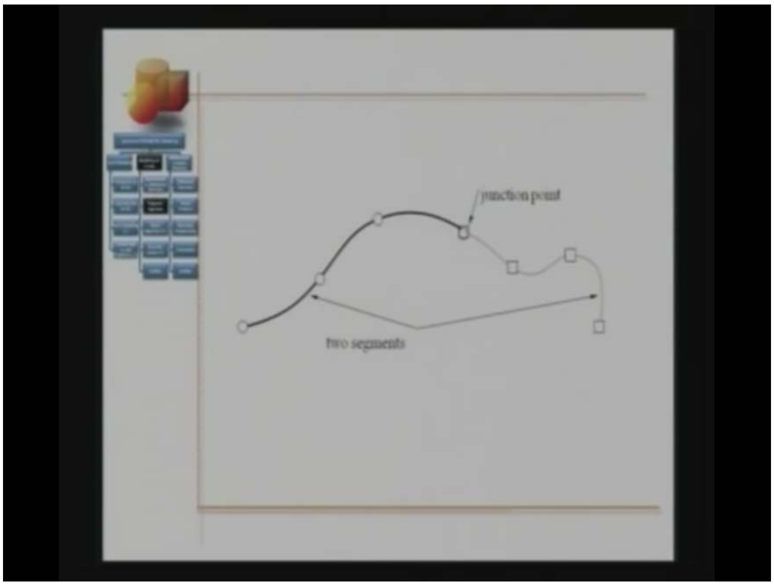

For example, here we have interpolated this point using a cubic polynomial. We have a second sub group again of four zero points, and we had used different cubic segment for interpolation; this is the junction point, which would be critical in curve design. Here you would need to ensure position continuity; continuity of slope and curvature. You would be matching zero order, first order and second order implement; it is these points in particular, that would motivators to study differential geometry curves.

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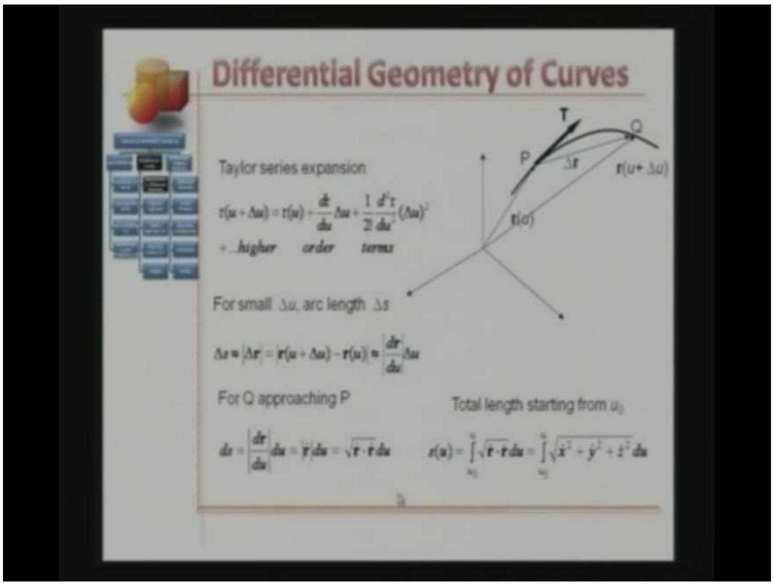

Now let us draw the Cartesian space, let us draw curve space. It is to be point P on the curve another point Q on same curve, the position vector P is r u; remember you choose to work with parametric representation, and the position vector Q is r u plus delta u, this vector here is the different vector delta r. This bold of here represents the tangent at point P is curve, the tangent is expressed the capital teen bold.

We can use the Taylor series expansion and express position vector Q, which is r u plus delta u as r u plus d r over d u the total derivative of r with respective u times delta u plus one over two factorial, here d 2 r over d u square second derivative times delta u square plus they would be higher order terms. For small delta u delta r represent the arc length delta s, so delta s is approximately equal to the absolute value delta r, which is r of u plus delta u minus r u, which is approximately equal to the obsolete value of d r over du times delta u. Here we have ignore the higher order terms; it Q approaches point P and delta s is the differential form d s, which is equal to modules of d r over du times d u; this derives of r with respect u is noted by r dot d s is the absolute value of r dot times du and mode of the r dot can be written as r dot dotted with r dot the root that time d u.

Now if we wish compute total arc length from let us say point P at which the parameter value is u sub 0, 2 of point Q at which the parameter value say u, then the total arc length is given by set u which is equal to integration from u sub 0 of u, d s which is r dot dotted with r dot within the square root sin time d u and if r is expressed in terms of skill of functions, x of u, y of u, two of u and r dot will be x dot u times psi plus y dot u times j plus t dot u times k r dot dotted with r dot is given by x dot square plus y dot square plus t dot square within radical sign d u, so this integration and computed to get overall arc length.

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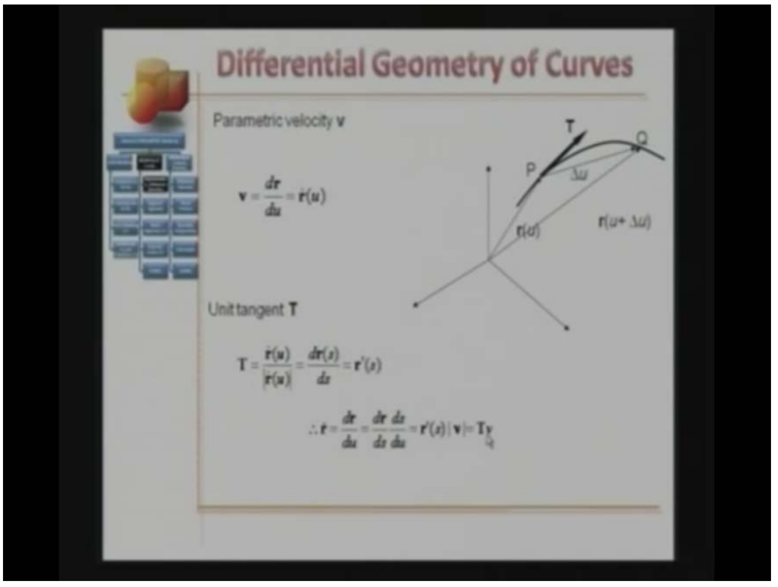

Continuing further parametric velocity bold V, is given by the first derivative r the respective u or r dot u. The unit tangent T is along the direction of the parametric velocity to v, the capital T bold is given by r dot u over the absolute value of x, which is equal to d r over d x now the first derivative of r with respect to x is represented y r point s, s as you seen before is be arc length parameter or the natural parameter.

First how? To be get from here to here; let me explain this to you on board. You have seen that unit tangent, t equals vector r dot u over the obsolete value of same vector, which is equal to d r over d u over the absolute value d r over d u. You seen from before, that d s is equal to d r over d u the absolute value times d u; we plug in for this value here, we will get d r vector over d u here will have d s here will have d u, u will cancel, this would be d r over d s, which is r prime s. Recall that we are using dot rotation to represent derivative with respective u and using the prime rotation represent derivative with respective x; now r dot is equal to d r over d u by change row this is equal to d r over d s prime d s over d u, d r over d s is over prime as d s over d u is the absolute value of the parametric velocity or from s is the unit tangent T times the magnitude of the parametric velocity.

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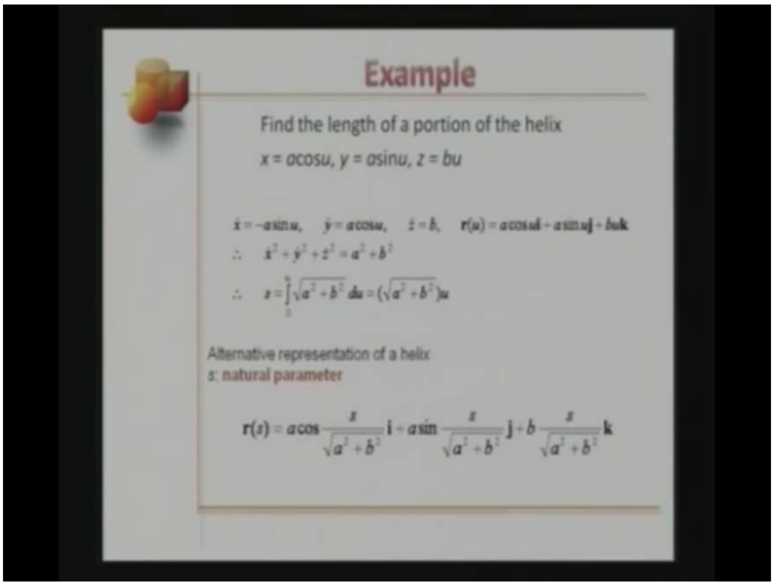

Take an example, find the length of a portion of the helix given by x equals a co sin of u y equals a sin of u and z equals b times u, a is the radios of the helix, b is related in the pitch and u is the parameter. All unit to do that is compute x dot derivative of x with respective u, derivative of y with respective u and derivative of z with respective u, x dot equals minus a sin of u, y dot equals to a sin of u, z dot equal b. Note in the parametric form of r u is a co sin of u and i plus a sin of u and j plus b u and k; x dot squared plus y dot squared plus z dot square can be determined to be a square plus b square. You seen for how to compute the arc length; s equal here we taken u sub 0 as 0, integration from 0 to u under root x dot square plus y dot square plus z dot square, which is under root a square plus b square times d u and clearly the integration consign to be a square plus b square with in the numeric sign times u.

What we observed here is an alternative, which to represent a helix using a different parameter; s note that this equation relates s with u for this helix. All within do is substitute for u and write this expression in terms of s, as a co sin s over under root a square plus b square times plus a times sin s over under root a square plus b square and j plus b hence s over under root a square plus b square times k; this expression and this expression there all u. The defense on a convenience whether we would want to choose do work with parameter u or parameter s compute different deferential properties of the curve.

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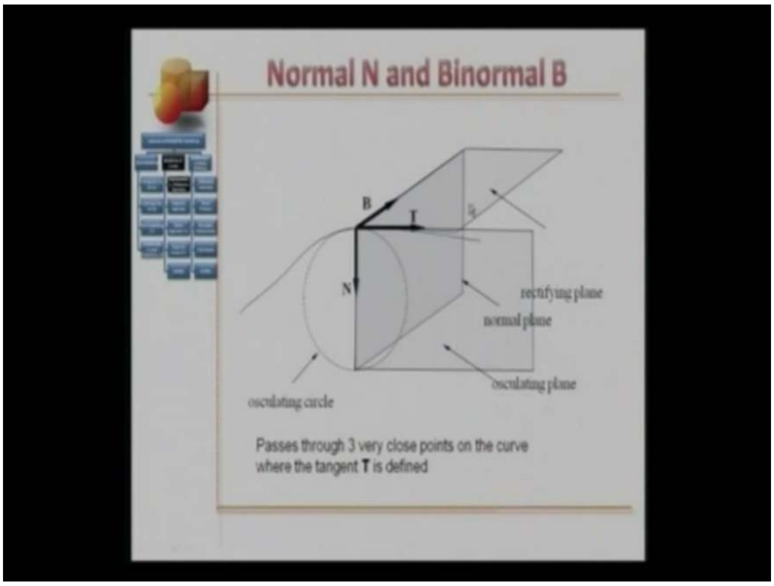

The Normal and Binormal and point in the curve. Normal is given by N and Binormal is B; Well we have this curve, we have this unit tangent T and we have a plane which is perpendicular to this unit tangent, are sum point p; this plane will disband by two vectors both are which will be orthogonal each other and also orthogonal to the unit tangent t. This is the first one represented by N and this is second one represented by B. N is called the normal and B is called binormal.

The plane containing two normal N and B this called normal plane; the plane containing the unit tangent and the binormal is called the rectifying plane. And the plane stand by N and T called as the osculating plane. If you considered two points very very close to this point; as we know, a circle can pass through three points we can construct a circle that passes through this points; the circle is called osculating circle. Notice well the normal points towards the centre circle, as said earlier this osculating circle passes through three very close points in the curve where the tangent T is defined.

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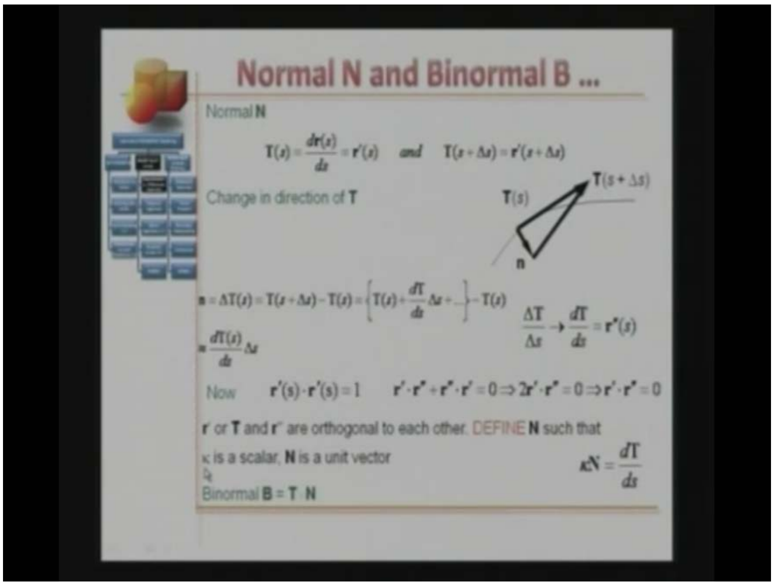

Some more on the normal and binormal, normal N how to compute that? There we have T which is evaluated as d r over d s, in short represented by r prime of s at a point very close to this point will have T of s plus delta s which can be computed as r prime at s plus delta s. Let us see how the direction of the unit tangent changes, we have this curve we have T s. Here the unit tangent at some point and then arc a point very close to this point, we have another unit tangent T of s plus delta s. What we do? We move T s to that the steps to these tangents rejoin this vector here represents the difference vector, n equals T s plus delta s minus T s. Notice that this is T s for n is delta T of s which is T of s plus delta s minus T s, we can use the Taylor series expansion this expand this expression.

Here s plus d T over d s time’s delta s plus some higher order terms minus T s, these two will cancel all and we ignore the higher order terms delta T s will be approximated by d T over d s time’s delta s. For two points very, very close to each other, delta T over delta s will take the differential form d T over d s which will be given by r double prime s. Remember that we are using the prime rotation to represent derivates to respect to the natural of the arc length parameter. Now we know that t is the unit tangent and so its magnitude is 1 in a sense r prime dotted with r prime is equal to 1.

If we differentiate this equation to respect to s, we get r prime dotted with r double prime plus r double prime dotted with r prime equals 0 which implies eventually. That r prime dotted with r double prime is equal to 0, what would this mean basically? This would mean that r double prime is a vector which is orthogonal to r prime, r prime is a unit tangent T. In other words r double prime will be orthogonal to the unit tangent T. We can use this fact and define normal N such that, some scalar kappa time's N equal r double prime which is d T over d x. Kappa is a scalar use so that n happen to be a unit vector, here kappa is a scalar and N is a unit vector. Finally, the binormal B is given by the cross product between the unit tangent and the unit normal. We know that both T and N are orthogonal to each other; it is the scalar kappa that we need to investigate. (())

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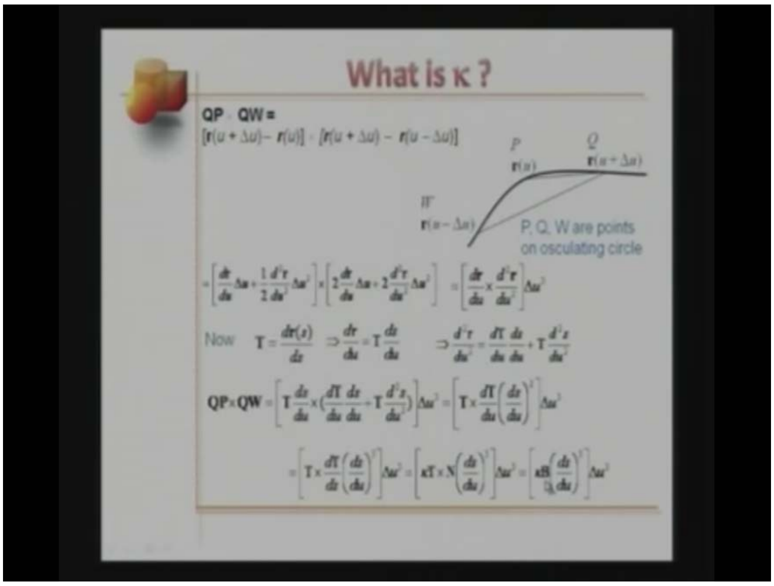

Question; what is kappa? Does it have any physical elements or significance, we have a curve here this point is P with position vector r u, this point here Q with position vector r of u plus delta u. This is point W with position vector r u minus delta u, now P, Q and W are points very close to each other, we can say they all lie on the parameter of the osculating circle. Now let us compute Q P cross with Q W, Q P cross with Q W; Q P is given by r of u plus delta u minus r u and Q W is given by r of u plus delta u minus r of u and minus delta u. We use the Taylor series expansion and considered terms up to second derivatives, r u plus delta u will be r u plus d r over du times delta u plus half of d 2 are over du square times delta u square; the term r u will cancel with this. So these two terms are what which you left and r u plus delta u minus r u minus delta u then both expanded using Taylor series will gave us 2 times d r over du times delta u plus 2 times d 2 r over du square times delta u square.

Let us try to simplify this cross product further; well what happens with this term gets cross with this term? Note that the direction d r over d u is the same in both terms, so the corresponding cost product will be 0. Next this term crossing with this term, will have 2 times d r over d u cross with d 2 r over d u square times delta u cube. Now this term crossing with this term, this half cancels to this 2; will have d 2 r over d u square cross with d r over d u. If a reverse the two terms along the cross product, I will introduce a negative sign. If we further work out the algebra, this cross product will reduce to d r over du cross with d 2 r over d u square times delta u cube.

Now we know that the unit tangent is given by d r over d s which is r prime s, this for imply using change rule that d r over d u is equal to T the unit tangent times d s over d u. All I need to do is multiply and divide the expression by d u and rearrange this equation to get this result. Now if I compute the second derivative of r respect to u that is if I differentiate this expression again with respect u, I get the d 2 r over d u square which is equal to d T over d u time d s over d u plus T times d 2 s over d u square.

All I can do now is, replace this term here by this term on the right hand side of this equation. Therefore, Q P plus Q W is given by T times d s over d u; notice that I am replacing d r over du as well by this expression. So T times d s over du cross with d T over d u times d s over d u plus T times d 2 s over du square, this entire things time's delta u. If I work on this cross product further and notices that the direction of this term and this term the two directions are the same, so the corresponding cross product is 0. So all is left is, T cross with d T over du which is this term and this term times d s over d u whole square these two terms getting multiply times delta u cube.

Now let us see how this term can we written in terms of the other normal N and B; the normal and binormal, well as you see you retain this scalar d x over du u here. We know from before that d T over d s is kappa times N to we see this two terms here, kappa N and we retain this term T here. So this expression becomes kappa times T cross with N times this scalar d s over du whole cube times delta u cube and T cross N is binormal B. In other words Q P cross with Q W, now this kappa times the binormal vector B times the scalar d s over d u the whole cube times delta u cube. Let us retain this result that can be using this in the next slide.

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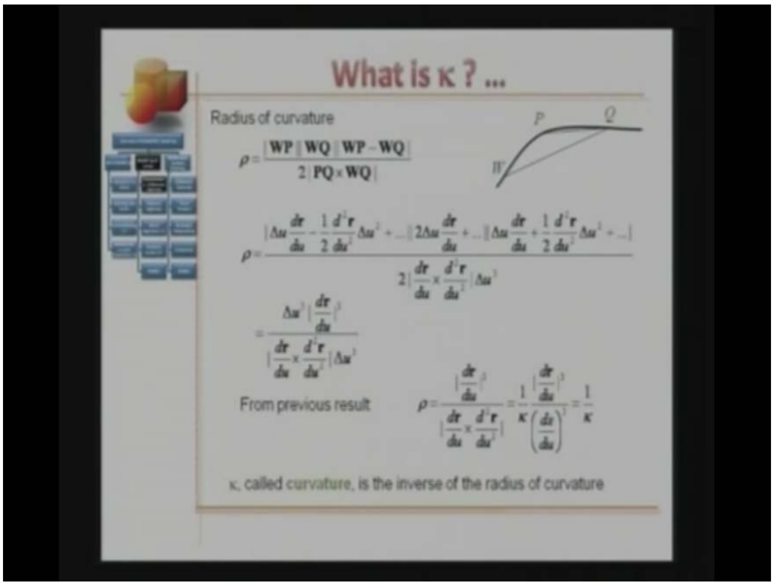

Back to this figure here, we have curve and points P Q and W; where position vector r u, r u plus delta u and r u minus delta u. Using vector algebra radius curvature is given by rho which is equal to modulus vector W P times modulus of vector W Q times modulus vector W P minus W Q, W P minus W Q over two times modulus on the cost product between the vector P Q and W Q. Well we can compute this vector in terms of the corresponding position vectors.

This looks a little tedious, this factor work it out; rho equals the mark W P, which is delta u times d r over d u minus half d 2 r over d u square times delta u square plus some high order terms. Mod of W P which is 2 times delta u times d r over du plus again some higher returns. Mod of vectors W P minus W Q which is delta u hence d r over d u plus half of d 2 r over the d u square times delta u square plus some higher returns. These expressions can be obtain using Taylor series expansion with some additional algebra, we denominated given by 2 times d r over du cross with d 2 r over d u square the mortals that times delta u cube. You may want to work as an exercise, as to have the radius curvature is computed using this expression.

Now starting with this complicated looking expression, rho can be simplified to be delta u cube times d r over du, the absolute value the cube that over d r over du cross with d 2 r over du square the absolute value this times delta u cube. We see from this expression that delta u cube can get cancel now and rho becomes modulus d r over du the whole cube over modulus of the cross product it mean d r over d u and d 2 r over d u square.

From the result in the previous slide, we can replace this expression and rewrite this equation as rho equals 1 over kappa mod of d r over d u the whole cube over d s over d u the whole cube. And noting that mod of the d r over the du is the same as d s over du, the 2 terms gets cancel the rho is equal to 1 over kappa. The kappa after roll has some physical significance it is called curvature and it is inverse radius curvature rho.

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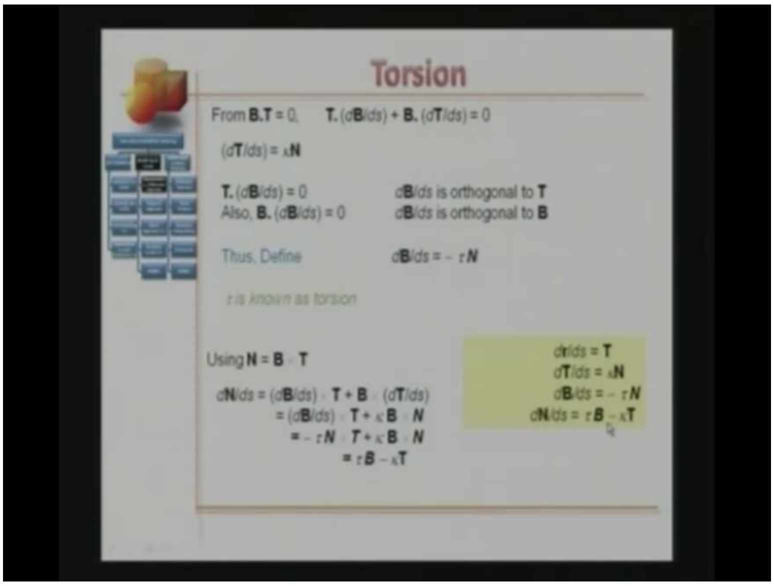

Next torsion, another differential property of curves; we know that the binormal b and tangent t orthogonal. So, b dotted with t is equal to 0. What we can do is, we can differentiate this result with respect to as to get T dotted with d B over d s plus B dotted with d T over d s, right hand side is going to be 0. And we can use the definition, d T over d s equal kappa times N; and substitute is expression here. For defined, when we substitute this thing here is that B dotted with N will be 0, which would make T dotted with d B over d x equal 0; at here T dotted with d B over d s equals 0, which physically implies that the first derivative of binormal with respect to the arc length is orthogonal to T.

Also if we differentiate these are B respective s, we will see that B dotted with d B over d s equal 0; implying that d b over d s is orthogonal to B. Interesting, d B over d s is orthogonal to T and also d B over d s is perpendicular to B d B by d s. Therefore, is bound to be aligned with the unit normal N, we can use this pack define d B over d s as minus of sum scalar tau time the unit normal B, this scalar tau is known torsion. Using the pack that the unit normal N, is expressed as the cost product between the binormal and in the unit tangent. We can write d N over d s as d B over d s cross with T plus B cross with d T over d s. Following the algebra further, the right hand side here and written as d B over d s cross with T plus, will we can use this definition here d T by d s is kappa times N. So this time here the kappa times B cross the N. Now this term here can be replaced by this term here, so we get minus tau times N cross with T plus kappa times B cross with N, N cross T minus B and B cross with N minus T. So, d N over d s equals tau times binormal minus kappa time the unit time.

In summary, we get four relations; number one d r over d s is defined as unit tangent T. d T over d s is defined as scalar kappa which we have seen to be inverse of various curvature, times the unit normal N. d B by d s is defined as minus of tau times N the scalar tau is called the torsion. And d N by d s is tau times the binormal minus kappa times unit times. If you see the left hand side of these relations r, T, B and N; the left hand sides express the first derivatives with respect to s, and the right side respectively express the corresponding results. These four relations are known as Frenet-Serrate Formulae. In subsequent lectures, we are going to be using these relations to compute the differential properties of curves at different points.