Computer Aided Engineering Design
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Lecture - 10

Hi and welcome to lecture ten on Computer Aided Engineering Design, this lecture is on transformations of solids.

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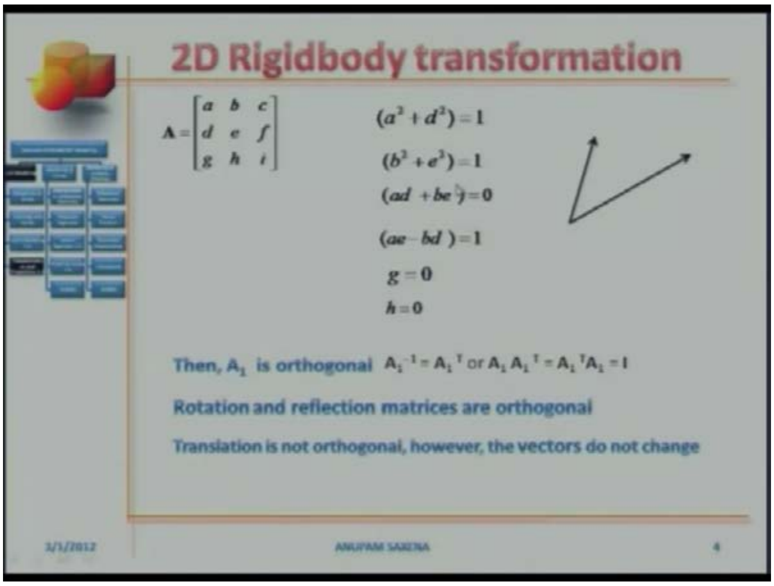

We will start from where we left half last time, we have discussing two-dimensional rigid body transformations. We have this matrix a if you remember which had nine elements; a b c d e f g h and i, this was a three by three matrix that represented a generic two-dimensional transformation. We applied this transformation on say up solid at which let us say we have these two edges represented by vectors v 1 and v 2. Recall that when performing rigid body transformation, the lens of these vectors do not change and also the angle between these two vectors remains constant. We had in co-operated both this conditions, in terms of the duct and cross product between the two vectors. We said that before and after rigid body transformation, the duct products did not change and also the cross products remained the same.

Then we applied these constrains on the duct and cross products, we came up with the following conditions on the elements of this transformation matrix. The first one was that a squared plus d squared equals 1; the second was that b squared plus e squared equals 1;

the third was a times d plus b times e equals 0; and the fourth was a times e minus b times d equals 1. We had additional conditions g is element here equal 0 and h equals 0, we did not say anything about the element i. If you look at these four conditions, they involved this sub matrix away, in a sense the four terms a b d and e, if or write this sub matrix of a as a 1 then last time we found that a 1 is orthogonal. In other words, the inverse of this two by two matrix a 1 was a same as the transpose of the matrix or a 1 times a 1 transpose was the same as a 1 transpose times a 1 and this was equal to a two by two identity matrix. If we look at rotation and reflection matrices, the cases that we have discussed in lecture 9 we find that rotation and reflection matrices are orthogonal.

However, translation is not an orthogonal operation; in that case the vectors do not change. Let us go back to these conditions, once again these four conditions relate these four elements of a generic transformation matrix; a into dimensions and we have information about these two elements as well g equal 0 and h equal 0. We do not have any information on i and also we do not have any information on c and f but that is not true. If we look at translation matrices, c would be an element that would represent translation along the x coordinate and f would be an element that would represent translation along the y coordinates in the x-y plane.

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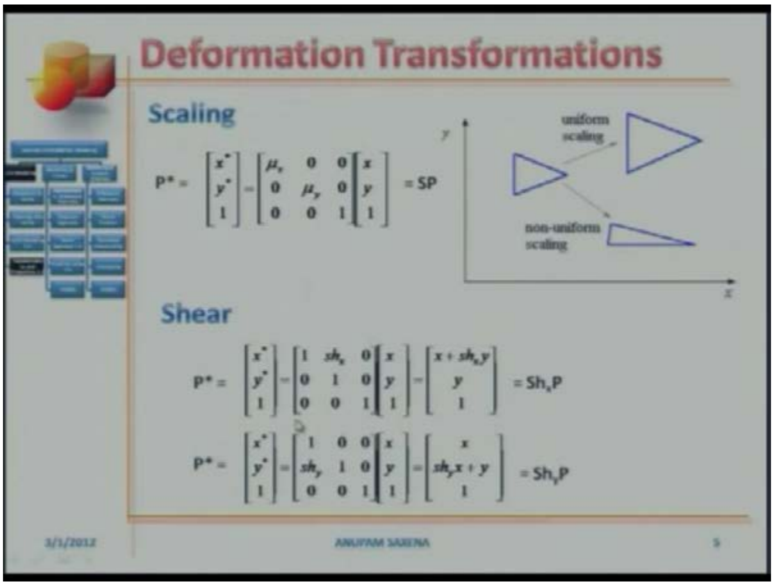

Now let us discus deformation transformations, these are the transformations if applied two solids will deformed solids, going back, the vectors will no longer have the same magnitude and also the internal angels between the vectors would change if we apply deformation transformations. The first of these is a scaling transformation, let us say we have a position vector p with coordinates x y and 1 and of course, these the homogeneous coordinates. We can scale the x component individually and the y component separately, mu x and mu sub y will be the two scaling factors.

So this three by three matrix represents the scaling operation, in chart that is represented by s. It is not very difficult to note, that if we free multiply this column vector by the scaling matrix, we will get x star as mu sub x times x and y star as mu sub y times y, x star and y star will represent the scaled position vector p star. We can go for either uniform scaling or non-uniform scaling and here is an example. If the set mu x equals mu y, we will then have a case of uniform scaling. Scaling will be the same in both orthogonal directions x and y, if however we set mu x and mu y to be of different values then will have a case of non-uniform scaling.

The second of the deformation transformations is Shear, let us say that the on to shear an object along the x direction. An object will have vertices, each of x will be represented by this column vector, in short by p and this three by three matrix over here, would represent a shear transformation. Once again shearing is happing along the x direction, this three by three matrix will have components 1, s h sub x, 0, 0, 1, 0, 0, 0, 1. If we free multiply this column x with this shear matrix, we have x star as x plus s h sub x times y and y star as y the original coordinate of point p and of course, x star and y star are the new coordinates p star. Realize what is happened to the new x coordinate, it has of course, (( )) to new value x plus s h sub x times y, the y coordinate however remained the same.

Once again, this is a case shear along the x direction, if you want shear a two-dimensional object along the y direction; we need to make minor modifications in the shear transformation matrix. Here we would be introducing a shearing factor s h sub y in the first column and the second row, so for shear along the y direction the transformation matrix is 1, 0, 0 s h sub y, 1, 0, 0, 0, 1.

Let us see what happens when we free multiply the column vector x y 1 with this transformation matrix multiplication. So x star would be x, the x coordinate of the transformed point does not change, y star will be s h sub y times x plus y, it is the y coordinate that would change. In short this transformation matrix that shear a point along the y direction is represented by s h sub y and of course, this transformations they act on points p.

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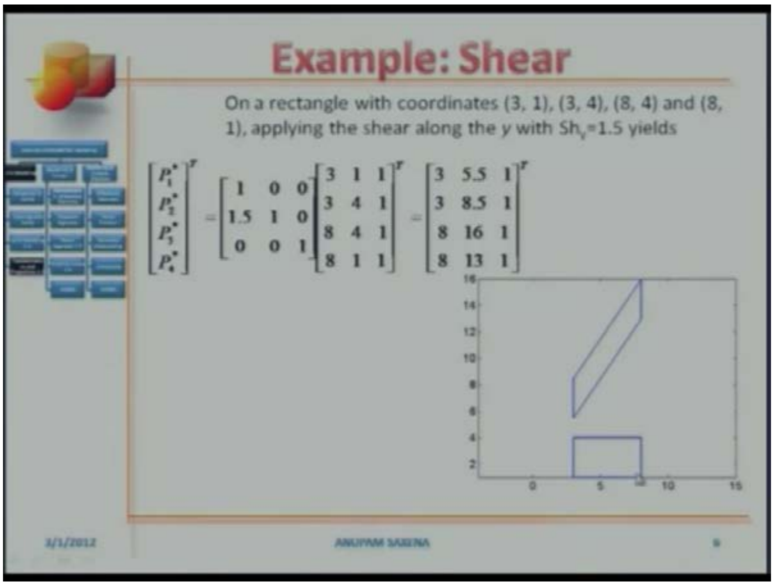

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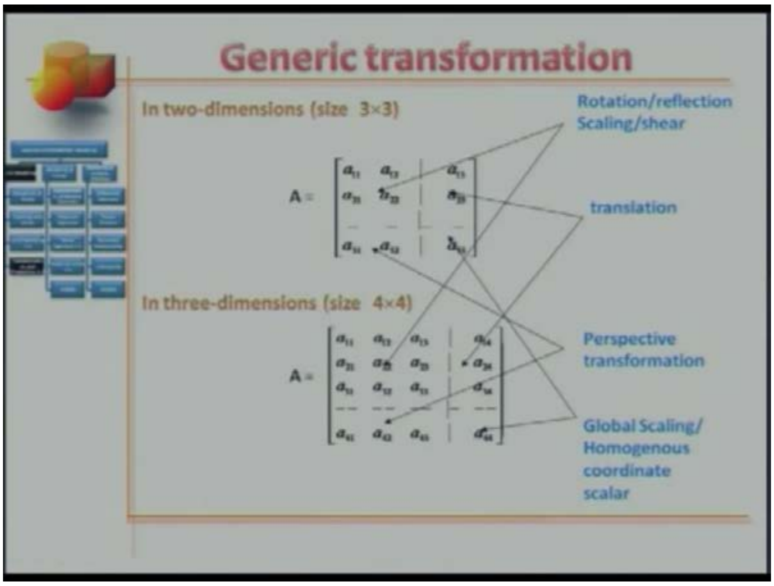

A few examples of shear on a rectangle with coordinates (3, 1), (3, 4), (8, 4) and (8, 1) if we apply shear along the y direction with the shear factor s h sub y as 1.5. We get the new coordinates p 1 star p 2 star p 3 star and p 4 star as the common shear matrix, note that this is the shear factor, that appears over here and this common shearing matrix will be free multiplying the original coordinates. The results are 3, 5.5, 3, 8.5 8 and 16, 8 and 13, the new coordinates after shear transformation along the y direction is performed. That is how this original two-dimensional object is changing both in shape as well as sides, if you notice carefully there is also a component of rigid body transformation.

Now let us try to generalize two-dimensional transformations and let me also use this opportunity to generalize transformations in three-dimensions as well. In general two-dimensional transformation matrices are off size three by three and those in three-dimensions, they are off size four by four. It is possible for us, to sub divide both these transformation matrices into 4 different sub matrices.

The top left sub matrix is off size two by two in case two-dimensional transformations and for three-dimensions this top left sub matrix is off size three by three. These sub matrices contain entries pertaining to rotation, reflection, scaling or shear transformations. We have noted in lecture 9, that this two by two sub matrix is orthogonal in nature that is this matrix times as transpose is equal to an identity matrix off size two by two.

Similarly, in case of three-dimensions these three by three matrix is also orthogonal, well when I say that they are orthogonal I am referring to rotation and reflection cases. In case of scaling or shear these matrices well of course, not be orthogonal. The top right sub matrix in case of two-dimensional transformations is off size two by one, in case of three-dimensions this top right sub matrix is off size three by one. You have seen before in case of two-dimensions that, these two elements represent translation along the x coordinate and along the y coordinate respectively. In case of three-dimensions, these three elements would represent translation along the x y and z coordinates.

In summary the top right matrices will contain entries that pertain only to translation. The bottom left sub matrix which is off size one by two, so we have two elements here a 3 1 and a 3 2 and correspondingly in three-dimensions the bottom left matrix that has three elements. These sub matrices represent perspective transformation; we will investigate perspective transformations in detail in the following lecture.

In general if you are not referring to perspective transformations, these entries will be 0, but otherwise if you are dealing the perspective image of an object these entries will be non 0. The bottom right sub matrix in both cases, in case of two-dimensional and three-dimensional transformations as off size one by one. In fact, these matrices are simple scalars and in fact these elements represent global scaling, they are also called homogenous coordinate scalar.

In summary, any two-dimensional transformation will have the size three by three, this matrix can be sub divided in to four parts. The top left will represent rotation, reflection, scaling and shear; the top right will represent translation; the bottom left will be invocated in case we need to view an object in its perspective transformation and the bottom right number is used essentially when we are performing global scaling. In case three-dimensions the four sub matrices have identical meanings just that their sizes are different.

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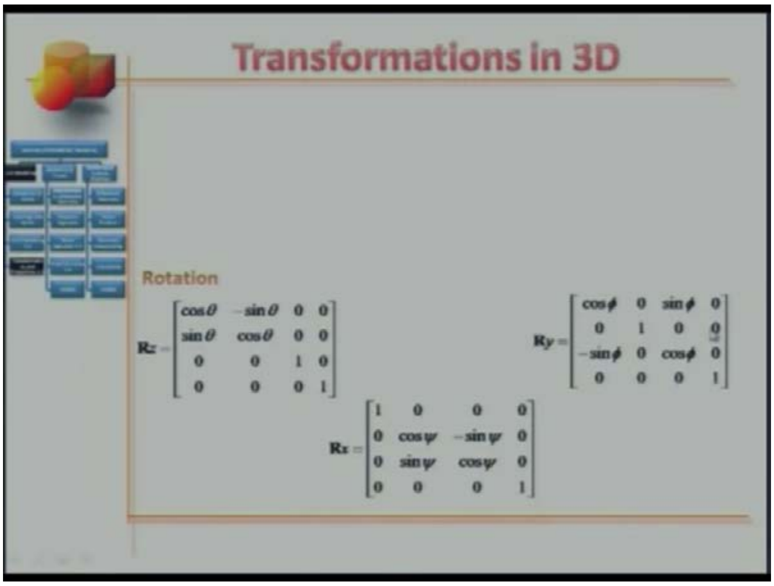

Let us study transformations formulae in three-dimensional, we first study translation. A translation matrix in three-dimensional is represented by a four by four matrix, where this sub matrix as the identity matrix we have the top right sub matrix or the column vector with non 0 entries. These entries a 0 and this fourth by fourth entries 1. p represents translation along the x direction, q the translation along the y direction and r translation along z direction. This is an example of the steroid being translated from this position to this position along this vector.

Rotation in three-dimensions let us first try to figure the rotation matrix, if you are trying to rotate an object about the z axis. This is a four by four matrix with entries, cosine of theta minus sin of theta 0, 0 sin of theta cosine of theta 0, 0, 0, 0, 1, 0 and 0,0,0,1 no translation and no perspective and the global scaling factor is 1. You can verify that the top left part of this matrix is orthogonal, this is the way I remember when I just drawn rotation matrices about different axis.

For example if I am to rotate an object about the z axis, I will have the entry 1 in the third row and the third column. And then I will start putting in cosine of theta minus sin of theta, sin of theta and cosine of theta in the corresponding x and y entries which are of course, the first and second rows and the first and second columns. When I do that, I did not forget this static order x y z, y z x and z x y. If I maintain the static order of coordinates, I will hopefully not make a mistake in writing the cosine and sin terms.

Let us why this for case, where I am trying to rotate an object about the x axis. As we did here, we will have an entry 1 corresponding to the x row, that is a first row and the x column that is a first column. And then I will start writing the cosine terms and sin terms maintaining the cyclic order of the x y and z coordinates. So my cosine term will be in the second row and second column and minus sin psi, this term will be in the second row and third column. And then for the rest I will have sin psi and cosine psi, on the other entries will be 0 except for this fourth by fourth entry which is 1.

Let us try the same for case when I am find to rotate an object about the y axis. (( )) The result in front of you, the entry corresponding to the second row and second column will be 1 and then I have to maintain the cyclic order. So after y comes z, so I will have cosine of phi here and in the same row, I will have minus of sin phi in the first column. And then I will have sin phi term here and the same row first column, I will have cosine phi; all the other entries would be 0 expect for this entry. If you practice a little bit and if you understand the concept writing rotation matrices in three-dimensions is not difficult. This is an example of an object being rotated about an arbitrary line l that passes through the origin. We will study rotations about arbitrary line formulae in the following slides.

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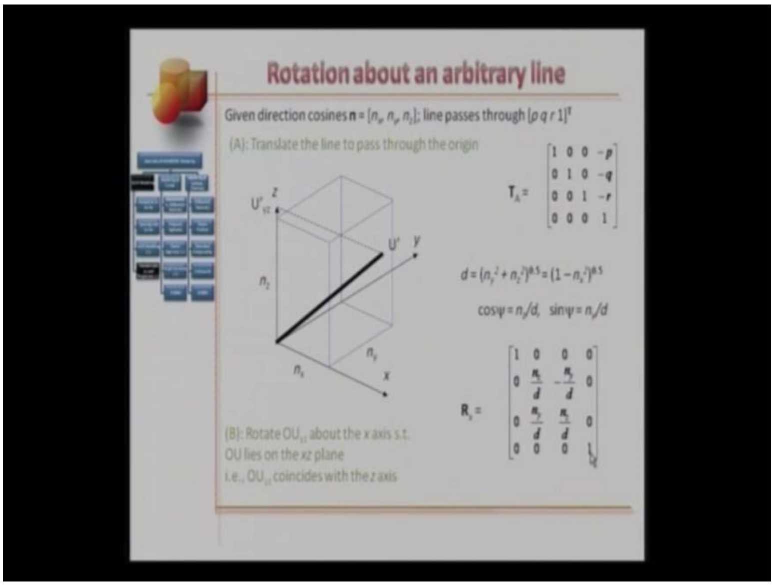

So this is the problem statement given direction cosines n which has components n x, n y and n z of a line that passes through a point p, q and r. Now this point here is expressed in terms of homogenous coordinates. If we are to rotate in (( )) above this line will have to follow some steps. The first of them is to translate the line such that, it passes through the origin and to do that we have to apply this transformation. Notice the negative signs here, so that brings us to the case where we have translated this line that is represented by o u. So that one end of this line is sitting on the origin of a three-dimensional coordinate system.

Let us draw a block around this line, so n x is this magnitude here, n y is this length here and n z is this length. o u is one of the diagonals of this block, let us project you to a point u y z on the y z plane. Let the distance between o and the point u y z be d. In terms of the direction cosines n x, n y and n z, it is not why difficult to find what d is. d is n y squared plus n sub z squared raise to half and sense the direction cosines are such that, n x squared plus n y squared plus n z squared equals to 1, d as 1 minus n x squared raise to half. Let us mark this angle and name it a psi, then cosine of psi is n sub z over d, n sub z over d and sin of psi is n y over d, n y over d.

Just in case, if I am boring you, you can always ask me and go for tea break or coffee break, I am resume the lecture after 5, 10 minutes. What? I am going to be saying would be important if, I rotate line o u y z about the x axis what will happen? Well you get surprised, o u y z will coincide that is not the same as the original line, o u coinciding where any of the principle axis.

We will come to that later but if you performed this rotation, that is rotate o u y z about the x axis such that o u lies on the x-z plane. I will decided once I rotate this line o u will lie on the x-z plane and o u y z will coincide with the z axis, u prime is the new position of u and u y z prime is a new position of u y z. The corresponding rotation matrix will be r x equals 1, 0, 0, 0, 0 n z over d minus n y over d, 0. Note that these two terms are the cosine psi and sine psi from these respectively. The third row will be 0; n y over d n z over d 0 and of course, the final row will be all 0 except for the fourth by fourth entries which is 1.

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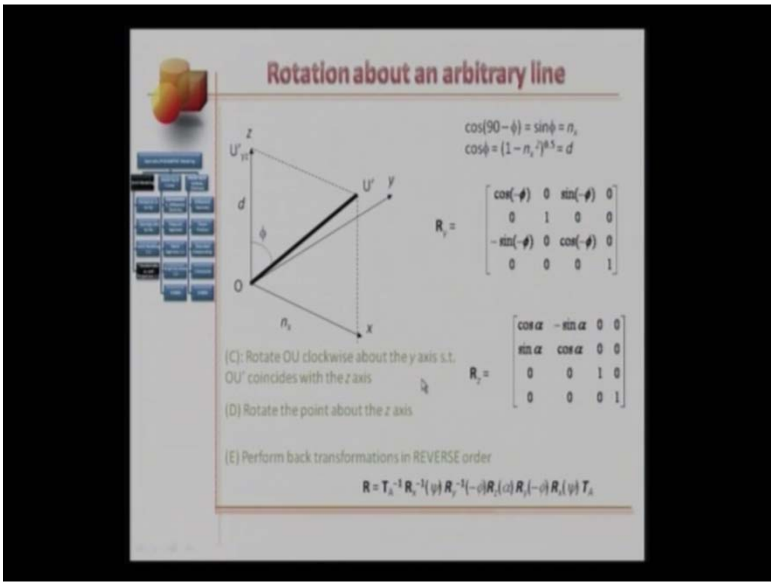

Now to make the line o u prime coincidence with left say z axis, it becomes a little easier for us. If o u prime y z it is coinciding with the z axis, remember they are performed rotation (( )). So this length will remain d, the direction cosine n x will not change, because of previous that was rotation about the x axis. All we need now is to rotate o u prime by an angle phi about the y axis. So that the line coincides with the z axis, what is phi? From trigonometry this angle is 90 minus phi and so the cosine of this angle is n x over o u prime length of edges 1. In other words cosine of 90 minus phi equals sine of phi, which is n x and cosine of phi will be 1 minus n x square the entire thing under root which is d. While once we have the cosine phi and the sin phi terms or we need to do is to construct a rotation matrix about the y axis.

I would want you to stand on this arrow held and figure out how whether clockwise or anti-clockwise would o u prime be rotated about the y axis. You get surprised it will be clockwise, I would you say this rotate o u clockwise about the y axis such that, o u prime coincides with the z axis. So if we consider the rotation matrix r y will have 1 in the second row and the second column and then will have cosine minus sin, sin and cosine terms. And remember this is minus phi why?

Because we are rotating this line clockwise about the y axis; once we have line coinciding the z axis or we need to do is perform the (( )) rotation about the z axis. I let say a desired angle as we know the rotation matrix for that, r z is cosine of alpha, minus sin of alpha, 0, 0 sin of alpha, cosine of alpha, 0, 0, 0, 0, 1, and 0, 0, 0, 1. Now replace this line back to its original position, how do you do that? We perform back transformations similar to the example we saw in a two-dimensional term.

If I consider the overall transformation process, this is how it will look like. Once again recall, that I have to go from right to left. First we perform translation, so make a point on the line coincide with the origin, then we perform rotation about the x axis by an angle psi and then we perform rotation about the y axis by an angle minus psi which is clockwise. By this time, line is coincident with the z axis, we perform the rotation about the z axis which is equal in to performing rotation about the z line and then inverse transformation or inverse rotation about the y axis phi minus phi that will bring the line on the z axis back to o u prime and then inverse transformation rotation about the x axis. That would orient the line in its previous orientation, and then translating one of the points on the line back to its original position, right to left a little bit of factors and this will (( )).

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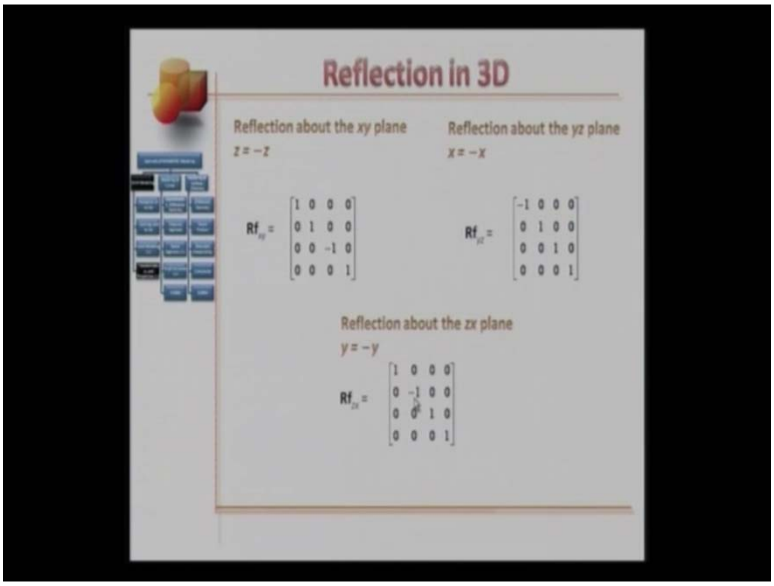

Let us quickly constructs a few more transformations of element reflection. If we consider reflection about the x-y plane, only it is z coordinate (( )), the x and y coordinates remain the same. Accordingly r f sub x y reflection about the x-y plane will be a four by four transformation matrix, which will look like this; entries 1, 0, 0, 0, 0, 1, 0, 0, 0, 0 minus 1, note the sign change in the z coordinate and 0, 0, 0, 0, 1.

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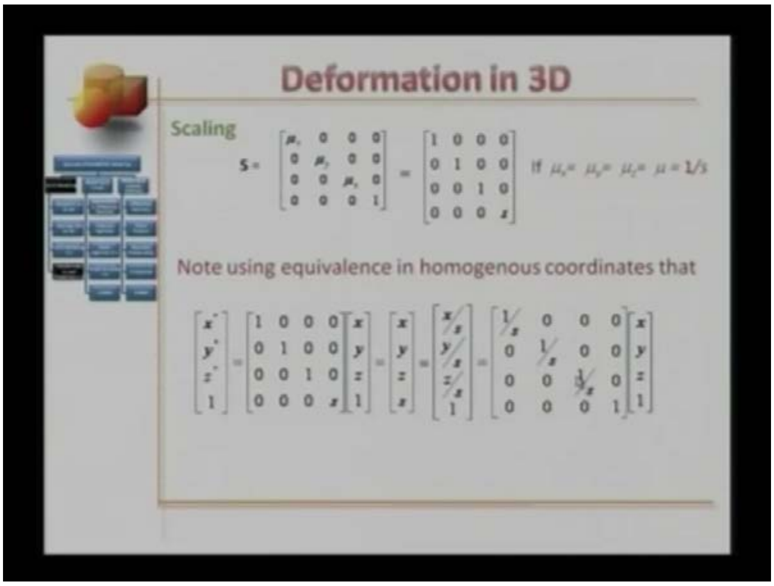

Likewise reflection about the y-z plane will make the x coordinate change in sign, reflection about the y-z plane will be a four by four matrix. Well you can consider it to be an identity matrix with the difference, that the first term here will be negative 1. Finely, reflection about the z-x plane for which the y coordinates, which in sign r f z x, reflection about the z-x plane will be a four by four dimension matrix. Again very similar to an identity matrix but with the second by second term here as negative 1.

Deformation in three-dimensions, first scaling; the scaling matrix s looks like this. The scale factors mu x mu y and mu z are in the positions 1 1, 2 2, and 3 3 on the principle rather this are non o values and this scaling matrix is equivalent to this one. With elements in the first row as 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0 and 0, 0, 0 s this equivalence happens under special circumstances. When the scale factor mu x mu y and mu z they are all equal say a mu and mu equals 1 over s.

Let me explain the equivalence on board, let say the homogeneous coordinates of point p are x, y, z and 1. If I multiply this column vector by s, this would give me s x, s y, s z and s. In homogeneous coordinates the column vector s x, s y, s z and s would represent the same point as x, y, z and 1. We say that these two column vectors are equivalent, because these represent the same point in the Cartesian on a three-dimensional space. Noting the equivalence in homogeneous coordinates we see the following.

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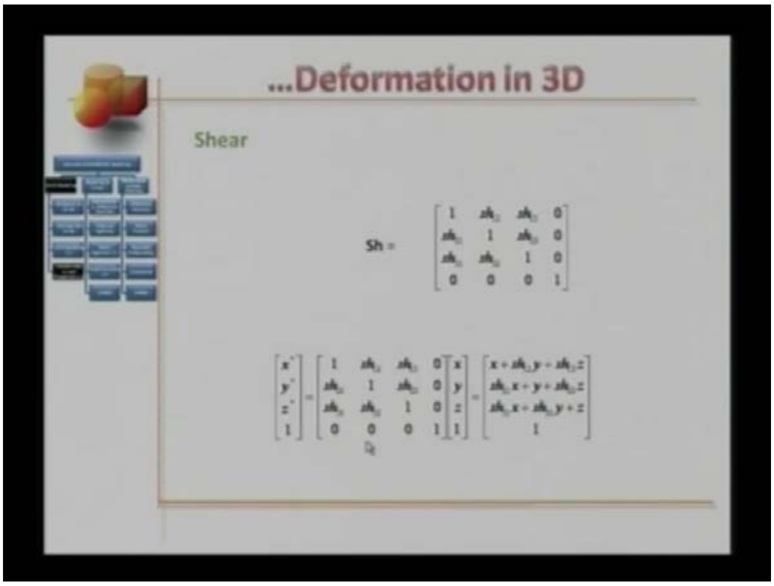

The left most column in the case a new position 3 star, this make rights is obtained by making all the scale factors equal to 1 over s. If I free multiply the old position of the point by this scaling matrix or uniform scaling matrix, I get x, y, z and s and you seen before that this column vector is equivalent to this one. If I divide the entries in the column vector by s, I will get x over s, y over s, z over s and 1 and this will be similar to multiplying the original position of the point by this four by four matrix, with entries 1over s at the free positions of the principle (( )).

The next is shear transformation in three-dimension, the generic transformation matrix for shear is given by this four by four matrix with entries 1, s h sub 1 2, s h sub 1 3, 0, s h sub 2 1, 1, s h sub 2 3, 0, s h sub 3 1, s h sub 3 2, 1, 0 and the fourth row we have all these entries as 0 and the last entry is 1. These entries are non 0 and they result in shears along (( )), these 2 entries will result in the shear along the x direction, these 2 along the y direction and these 3 along the z direction.

We cancels this out by multiplying the column vector of the original point x, y, z, 1 by this four by four matrix to get the new position. This multiplication (( )) much apply you would see that the first entry x star corresponds to x plus s h sub 1 2 times y plus s h sub 1 3 times z; making these 2 non 0 entries affecting the shear along the x direction. The second entry y star is s h 2 1 times x plus y plus s h 2 3 times z, these 2 entries resulting in shear along the y direction. The third entry z star is s h 3 1 times x plus s h 3 2 times y plus z, these 2 entries here resulting in shear along the z direction and forth entry of course, is 1.