Bio – Microelectromechanical Systems

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Module No. # 01

Lecture No. # 28

Hello and welcome back to this 28th lecture on Bio-Microelectromechanical systems. Let us begin with a quick preview of what we have done last time.

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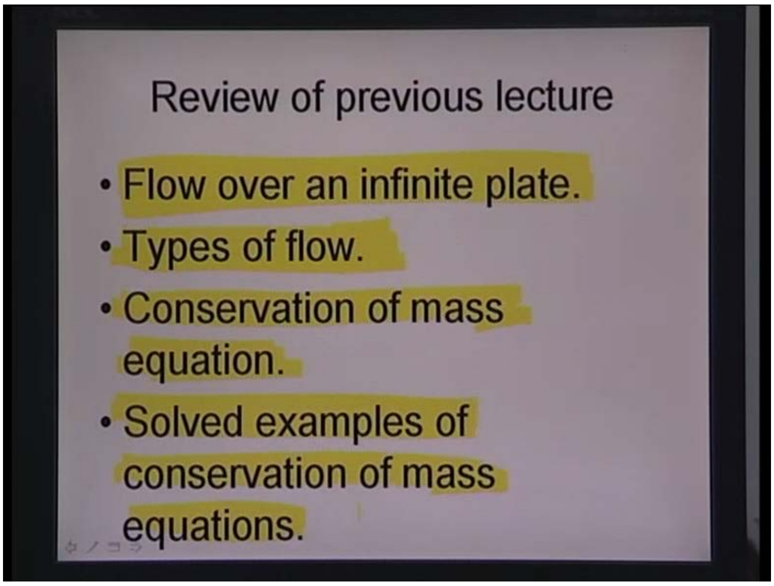

We talked about the flow over an infinite plate; the way that the boundary layer would be created as a difference between the fully developed region of the flow and the layer adjacent to the plate, which has shear deformation. We talked about different types of flows regarding 1, 2, 3 dimensional flows and tried to calculate the velocity fields. We also discussed about a control volume dx, dy, dz and tried to derive the conservation of mass equation, which is also the first Navier-Stokes equation. We solved examples for ascertaining the velocity component or the density with respect to time or space for compressible and incompressible flows. Again to recollect, compressible flows are those, where density changes with time. Incompressible are those, where the density does not change as a function of time.

Today, we will try to derive the conservation of momentum equation, which is also known as the second Navier-Stokes equation. Essentially, all this theory is very important because when we translate this theory to the micro scale, the interesting part is that the mass transport becomes time independent. It essentially reveals, if there are two side by side flowing streams of fluid in a micro channel, the seldom x because owing primarily because of this reason

If we look at how to derive the conservation of momentum, there is something to do with acceleration of a fluid particle. Newton's second law of force is nothing but mass into acceleration. So, let us first calculate the acceleration of a fluid particle in a velocity field.

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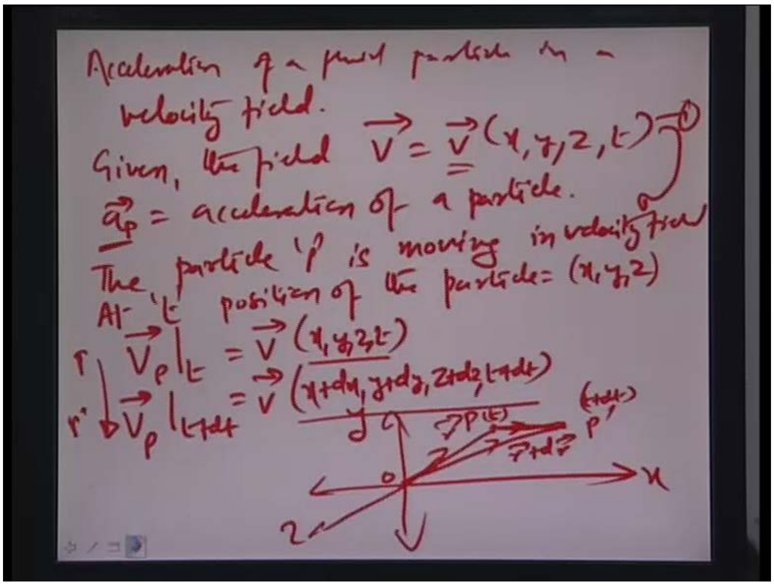

Given the field v as a function of x, y, z and space-time, let us assume a p is the acceleration vector of a fluid particle. So, we want to find out what a p is in terms of velocity. Let us say, the particle p is moving in the velocity field as represented here above in equation 1. At time t, the particle is at position x, y, z. So, let us say at t position of the particle is x, y and z. This particle has a corresponding velocity at that point in space at time t, which is represented by v p at t equals v into x, y, z, t.

Let us say, it has a velocity at t plus dt, which is represented as x plus dx, y plus dy, z plus dz, t plus dt. Essentially, what it really means is that you have this rectangular coordinate system with x, y and z components. You have a radius vector r, somewhere

here at point p with respect to the origin o. This changes to another point p dash and the new radius vector becomes r plus dr. Essentially, this is the position at time t, this is the position as t plus dt. The vector connecting these two is the position vector; the particle has traversed from point p to p dash. It is definitely a function of x plus dx. This point here is x plus dx and that means the traversing of the particle in the x direction is by an elemental distance dx. The traversing of the particle in the y direction, but this path is being executed. So, it there are three components of this path. The particle traverses dy and similarly the traversing of the particle in the z direction, the particle traverses dz from point p to p dash. There are two different time instances, t and t plus dt. If we consider or if we try to find out what is the change in velocity as the particle moves from p to p dash.

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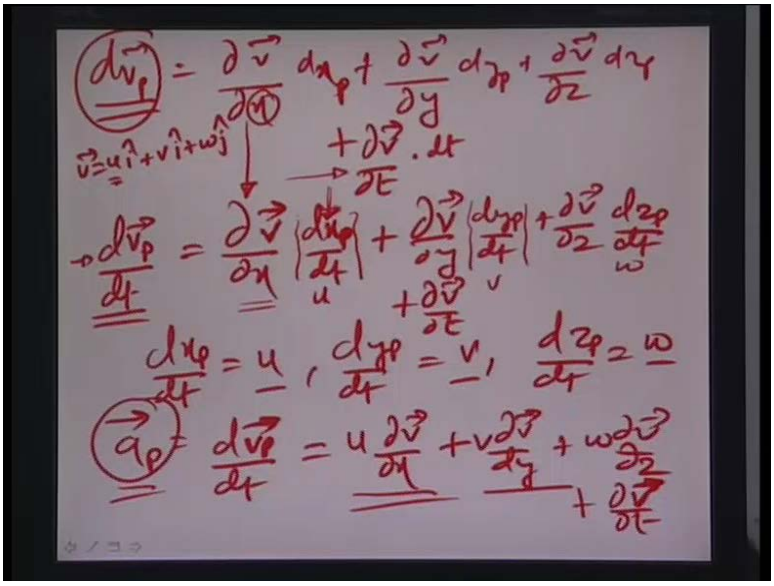

If we consider or if we try to find out what is the change in velocity as the particle moves from p to p dash, the velocity dv p can be effectively written as the rate of change of the velocity vector p with respect to x times of dx plus dx p because x p is the position vector or the x component of the position vector at of the particle p. So, dv vector by dx is the rate of change of velocity in the x direction of the particle times of dx p plus rate of change of velocity of the particle with respect to y times of dy p plus rate of change of the velocity of the particle with respect to z times of dz p.

You have a time component here; the time is varying with respect to t. So, the rate of change of velocity vector of the particle with respect to t times of dt. So, this is how the differential element of velocity dv p can be written from vp. If we differentiate this with respect to time, we will have dv p vector with respect to dt. It is essentially the rate of variation of v with respect to x times of… These are again special components, so it should have only… Suppose your v vector comprises of three components u, small v and small w. Therefore, u really is differentiable only with respect to x and not with time. This is how you basically represent the velocity vector. At the position p, the vector is invariant. So, delta v by delta x times of dx p by dt plus delta v by delta y times of dy p by dt plus delta v by delta z times of dz p by dt is plus delta v by delta t. This is essentially your rate of change of velocity vector of particle p with respect to t.

Now, we know dx p by dt is nothing but the velocity of the particle in the x direction, dy p by dt is the velocity of the particle in the y direction and dz p by dt is the velocity of the particle in the z direction. We can represent them by u, v and w. Essentially, the acceleration vector is nothing but the dv p by dt, the rate of change of velocity vector of particle p with respect to t is nothing but u delta v by delta x plus v delta v by delta y plus w delta v by delta z. Why we did not use the chain rule here for differentiation by taking the derivative of dv by dx? Because v is essentially,… at a certain time point t.

Again, let me just repeat this point once more (Refer Slide Time: 10:01); v vector is the velocity at time point t and it changes to the velocity at time point t plus dt later on. This is essentially x, y, z and t and this is related to x plus dx, y plus dy and z plus dz plus times of t plus dt. So, once you have differentiated this vector with respect to x. The time component is automatically gone out. As time is concerned, the differentiation of v with respect to x is independent of time and therefore, the first component is not differentiated with respect to time. The second component is a representative of the position of a particle x p. What we are doing here is dx, p by dt, which means it is a instantaneous velocity of the particle in the x direction, when it is at position p.

Similarly, dy p by dt is essentially a representative of the instantaneous velocity of the particle in the y direction, when it is resting at p. So, these are essentially nothing but u, v and w. V is already independent of t, so it is a constant as far as differentiation with respect to t is concerned. Here is the new term, which comes out. X p can of course be differentiated with respect to t and this gives the instantaneous velocity of the particle in the y direction, in the x direction at the point p. This is how this equation emerges out of acceleration from the previous equation.

So, acceleration of the particle can be represented as u dv by dx plus v dv by dy plus w dv by dz. I may just quickly rub this and write this again. This is what the acceleration vector ap is and of course, there is a delta v by delta t and the velocity of v with respect to time t.

Thus, you have defined the acceleration. The fluid may be accelerated, as it is convicted into a region of higher velocity. Let us look at that; suppose you have a rectangular block of fluid, what kind of forces that block will feel because of stresses, which are taking place in the medium with respect to that control volume? If you look at the type of deformations that such a small control element would have in a shearing stream, assuming the volume to be present somewhere in the stream, the kinematics of such a mechanism can be illustrated as shown in this particular figure.

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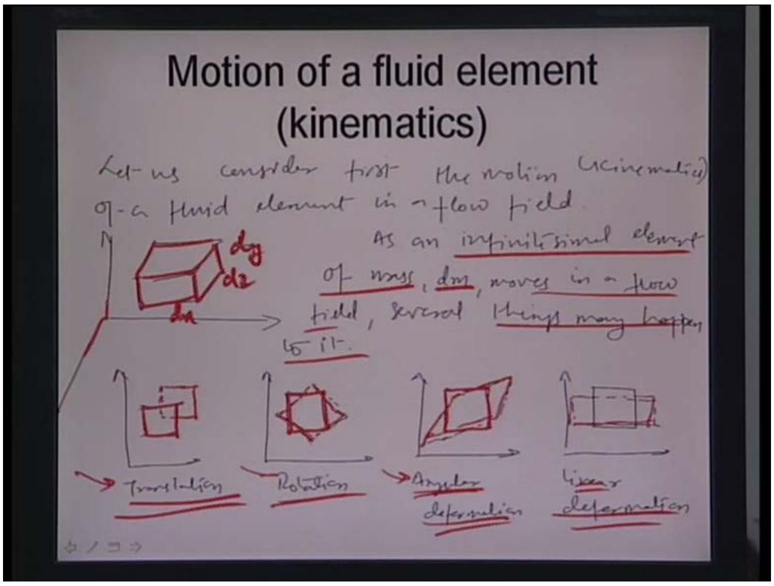

This is the box present in a rectangular coordinate system. The volume element has a volume of dx, dy, dz. Essentially, this is an infinite decimal element of mass dm, where rho is the density. We assume the density rho is to be constant. It is a incompressible flow, so rho dx, dy, dz.

Essentially, the total mass of this small volume element is called dm and it moves in a flow field. So, the number of deformations or the number of kinematics states of a block is simply translated. It means that this was really the initial position of the block. It changes to this new position here. So, this is simply translated motion. It may rotate, which means that the block is integral in shape or not changing or losing its shape. It rotates at the same point; it may deform, which means that the block actually starts changing its shape from a square to more like a parallelogram. Something like this or it may actually deform linearly. This is linear deformation and that means a cube will become a cuboid because of volume continuity. Increase the length of the block by pulling it together. The area of cross-section will definitely decrease.

There are four different kinds of kinematics states that this block can have while moving in a fluid volume. One is translatory and another is rotary. As you can see here, third is angular deformation, where you are actually trying to compress the block and make a square into a parallelogram. The fourth is linear deformation, where you are actually trying to pull the block and trying to deuce cross section and increase length.

Consider the first aspect that is fluid rotation, how this small fluid element would be rotated? So, let us assume that omega is the angular velocity of rotation. If you again recall, the rotation is actually a block at one angle and one position. It actually rotates about its own place retaining the center, goes to a different location and rotates to a different location.

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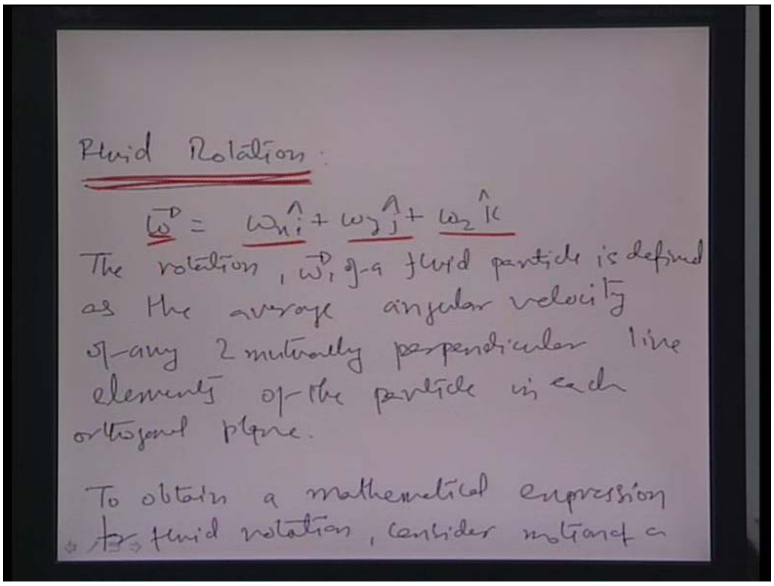

Let us assume angular velocity as omega. So, omega will again have three components: omega x i, omega y j and omega 2 k, which means that these omegas have different axis and they have independent rotations. How do you define the rotation in this kind of a case? Suppose, you have two mutually perpendicular planes like this. You are rotating the planes all together retaining the perpendicularity between them. It means the planes are not really deforming along its own with respect to each other.

They are just keeping in the same angle and then trying to rotate in the average velocity of rotation. In this case, it would be nothing but the velocity of plane 1 plus velocity of plane 2, which is orthogonally placed by two planes. This is how you do the average velocity of rotation of average angular velocity of rotation of two mutually perpendicular line elements of the particles in each orthogonal plane. Let us obtain a mathematical expression for this. It will be a generic expression because it is later on translated to the case, where the angle between the plane changes as the rotation goes on and it is also the case of shear deformation; it is the third case.

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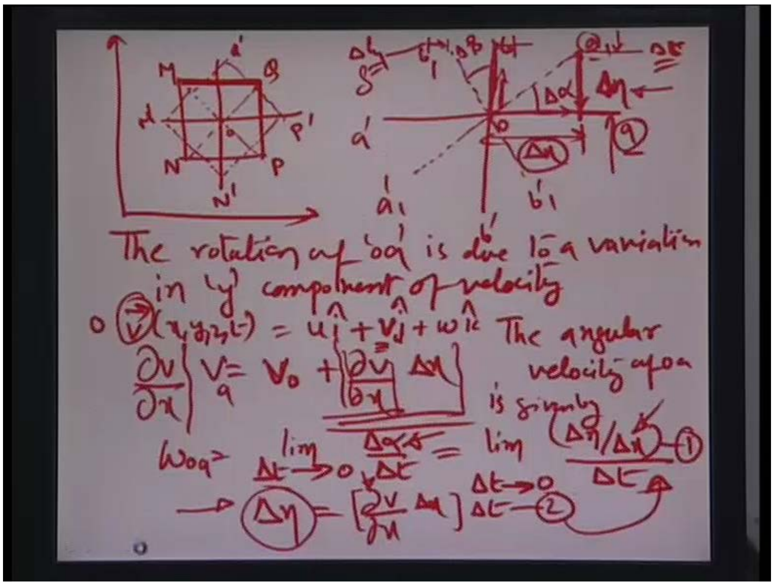

Let us actually look at their aspect. In this particular case, you have a coordinate system x y. We are just considering rotation only in two-dimensional plane. You have a block somewhere in position O. These are the two diagonals of the block. The block rotates in a position, where basically these diagonals come like this. There is a new location of the box created because of this rotation in this manner.

Basically, you have the block placed at coordinates M, N, P and Q. Now, the new position coordinates become M dash, N dash, P dash and Q dash. It rotates about the center O. This rotation can be figured out in terms of two angles. The angle, which is in plane 1, it is able to define while it rotates and plane 2 - M Q, which is able to define as it rotates.

In other words, we can also consider two diagonals, which are perpendicular to these planes. They have a fixed angular relationship with respect to the plane. We consider the rotation of the two diagonals. In order to consider the average velocity of rotation in this particular instance, let us assume and proceed. So, you have a case, where you have diagonals intersecting at point O. In the first instance and after rotation, these diagonals change positions to the new value, where the perpendicularity between the diagonals is maintained; it is a case of pure rotation.

Let us call these with different names. Let this be o a, o a dash and this be o b and b dash. Essentially, these move to new positions a 1, a dash 1 and b 1, b dash 1. The rotation vector is definitely related to the angles that these two planes would move in. Let us assume this angle to be delta alpha, this as delta beta. Let us also assume that the amount of distance that this particular plane o a moves, as it goes to a 1 is delta eta. The distance is about delta x and essentially, here again, the distance that this position b 1 is spaced from this position b is delta zeta.

Let me just rewrite this a little. So, essentially, as the diagonal has moved by delta beta, the position here was b. Before this, it is b 1 and the distance between b 1 and b is given delta zeta. So, this is delta eta the distance between a and a 1 and the distance between b and b 1 is given by delta zeta. So, these are some of the presumptions that we make to find out the relationship between the angles and the velocities. One important point to be considered here is that the rotation of vector o a at this particular line is that o a is due to a variation in y component of velocity.

Suppose, this particular point o is having or is defined by a velocity vector v(x, y, z, t), where v in the x direction is given by u i, v in the y direction is given by v j and that in the z direction is given by w. Let us say the small v here represents the velocity in the y direction. There is definitely going be a change of this velocity from point o in this particular region 2. At this point a, the velocity is different. If you assume that change to be delta v, this is small v and mind this is not the v vector. Here, it is delta v by delta x.

We can assume that the velocity at o in the y direction is v zero, the velocity at a would be equal to v zero plus delta v times delta x times of delta v by delta x times of delta x. So, this is the additional velocity, which is coming by the variation of the y component of the velocity when move from o to a in this particular rotation case. So, the angular velocity of o a is really given by the rate of change of angle. Limit of delta t tends to 0 into delta alpha by delta t. We assume that two points a and a dash are placed by a time point or they are placed in time by a difference of delta t.

So, delta alpha by delta t is the angular velocity - omega o a at this particular instance of time. Therefore, this can also be represented by limit delta t tends to 0. What is delta alpha? It is equal to nothing but delta eta by delta x. We assumed this to be the length of the arc. This distance is delta eta to the length of the arc. We assumed the radius vector to be delta x. So, the angle delta alpha, which it moves is delta eta divided by the radius vector delta x.

This is essentially delta alpha divided by delta t. Let us now try to figure out what this quantity becomes or what is the relationship between the rate of change of velocity and the y component of the velocity with respect to x that we derived that is delta eta. Again, it would be nothing but the differential velocity change of v. The y component of the velocity goes from y o to a, which is nothing but delta v by delta x times of delta x divided by or into times of delta t, the time that it takes for this point to go from a to a 1.

The distance that the point a covers in order to go from a to a 1 is nothing but the differential change in the y velocity with respect to x times of delta x. This is the change or due to which the velocity at point a is different than the point o. This change times of delta t that is delta eta would be the amount of distance that a traverses to a 1. Let us say this is number 2 and this is equation 1. If I put this value of delta eta from equation 2 to equation 1, let us see what the value of omega o a finally becomes.

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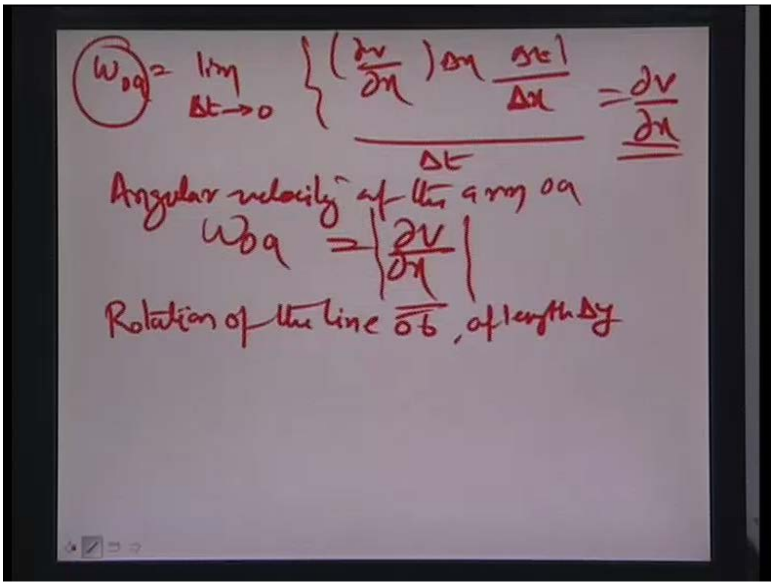

Omega o a is limit delta t tends to 0 times of delta v by delta x times of delta x times of delta t by delta x divided by delta t. It is nothing but dv by dx. So, this essentially is dv by dx into delta x into delta t. It is basically the delta eta value divided by delta x divided by delta t. So, this comes out to be dv by dx. Omega o a or the angular velocity of the arm o a comes out to be equal to delta v by delta x. The rotation really translates into a rate of change of y velocity with respect to the x direction. If I do the same thing for the other component of the diagonal, which was in the y direction, we may need to ascertain what is the change in the x direction or what is the change in the x velocity, y direction as it goes along y. What is the change in the x velocity? It would essentially be omega o b.

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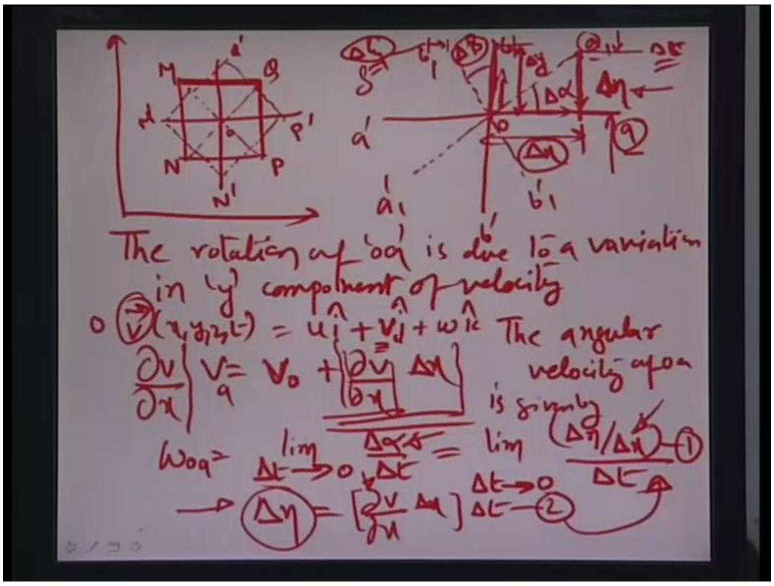

So, let us derive that expression as well. The rotation of the line o b of length delta y and it has become too crowded here. You can say that this (Refer Slide Time: 28:09) from here to here. This distance is delta y and you can also say that this distance is delta zeta. The angle moved is delta beta. So, there is a similar kind of relationship to describe omega o b.

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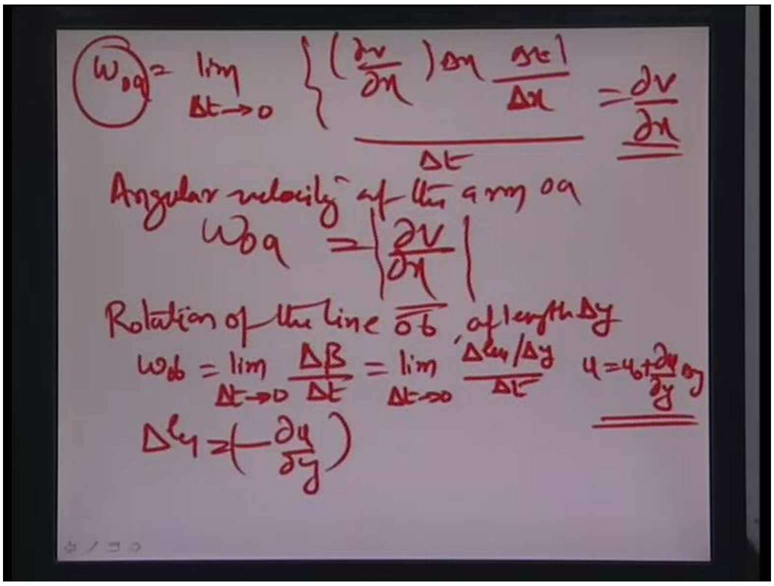

There is a similar kind of relationship to describe omega o b and you can write that as limit of delta t tends to 0 delta beta by delta t. This can further be expressed as limit of delta t tends to 0 delta zeta by delta y by delta t. Further, we have the velocity x at the origin. Here, the velocity is in the opposite direction and it is u 0 plus du by dy times of delta y.

So you have let us say u is equal to u 0 plus du by dy times of delta y, pretty much in the similar manner as we did for the x variation. We are doing the y variation in this particular case. So, the delta zeta in this case is nothing but minus delta u by delta y. This is actually minus because it will be in the reverse direction. So, essentially this would be minus.

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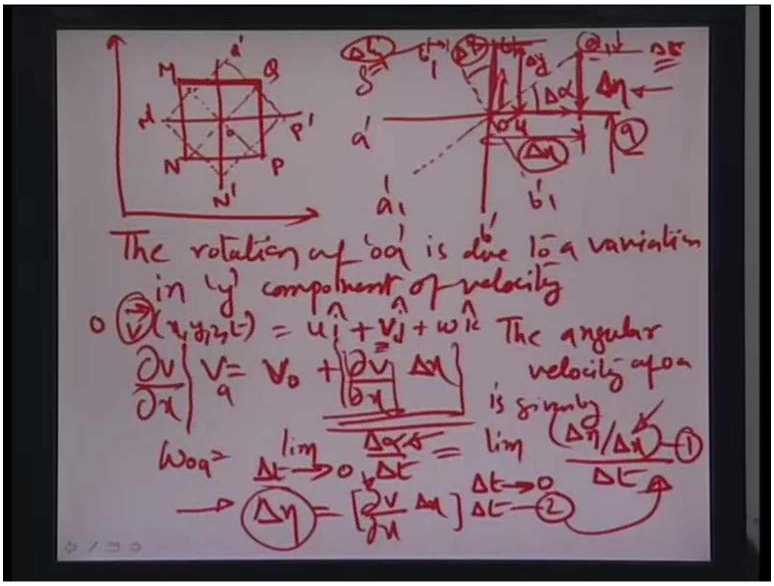

As you see here in this figure, u is in this direction, but delta u by delta x is the differential because of this rotation in the opposite direction. The value here would be u minus delta u by delta x times of delta y delta u by delta y times of the delta y. So, this is opposite to the direction of the positive u that you have taken.

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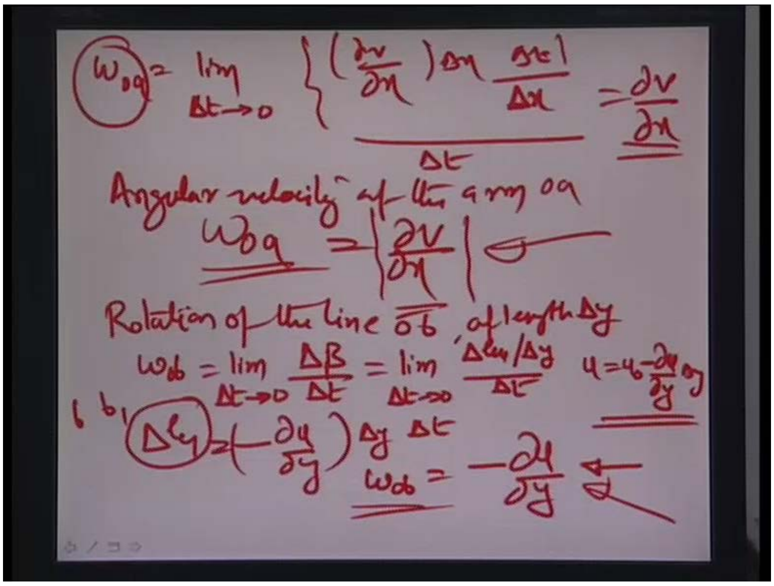

Now, we are left with delta zeta equal to minus delta u by delta y times of delta y times of delta t and the amount of distance that b takes to move to b prime or b 1. Therefore, omega o b in this particular case would be nothing but minus delta u by delta y in a pretty much similar manner, as we did before for the x component. Therefore, we found out that omega of the two arms or the diagonals of the two arms are dv by dx and du by dy with the minus sign.

So, this is a very interesting derivation. We will keep this in mind for proceeding ahead with the momentum equation. So, this is what the rotational component would do. So, anything related to shear, which causes such a rotational component would essentially give you values in terms of dv by dx and minus du by dy.

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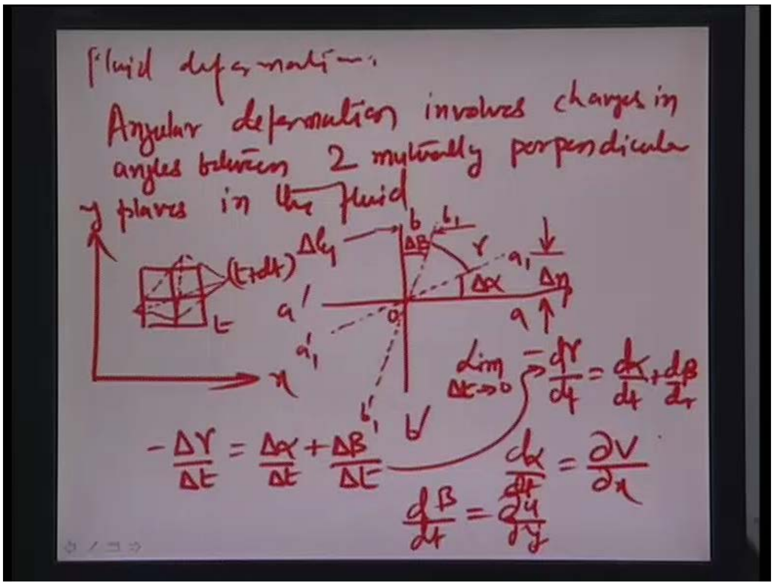

Now, let us look at what is fluid deformation. Case 2 is regarding fluid deformation. Let us just refer back to that particular diagram here (Refer Slide Time: 31:40). Fluid deformation is this second case angular deformation. As you can see, the element actually changes its shape from regular square into a more like a quadrilateral or a parallelogram. Basically, there is a relative angular change between the two perpendicular planes. They do not remain at 90 degrees because the fluid moves.

Let us actually calculate using mathematics. What is this angular deformation? Angular deformation involves change in angle between two mutual perpendicular planes in the fluid. You have x and y coordinates here, you have a square with two diagonals. You are angularly deforming this, so that the new shade assumes that the diagonals moved away from perpendicular to this new angular shape.

Essentially, you are doing deformation in the particular shape. What essentially you are doing is that there is a diagonal o a, o b,o a dash, o b dash direction and these diagonals are actually now changing in the o a 1, o b 1,o b dash 1,o a dash 1 direction. Let us assume that we have two angles, which is similar to the last case, it was delta alpha and delta beta. Let us also assume that the distance between a and a 1 becomes delta eta. The distance between b and b 1 becomes delta zeta. We also assume another angle gamma, which is in between. It is your delta gamma, what really is delta gamma? It is gamma and not the delta gamma. This angle is gamma and any change in this angle is delta gamma. It is negative; the angle is decreasing as you go along with time. so, minus delta gamma is nothing but delta alpha plus delta beta. If you just divide the whole by delta t assuming that this shape was at time instance t and this shape came at t plus dt. So, there is a time interval of delta t in between or dt in between. You get minus gamma; delta gamma by delta t is equal to delta alpha by delta t plus delta beta by delta t. so, taking limits on both sides as delta t approaches 0, this whole expression becomes minus d gamma by dt equal to d alpha by dt plus d beta by dt. So, this is what angular deformation would really mean.

As we already know from earlier derivations that delta alpha by delta t or d alpha by dt is nothing but the rate of change of the y velocity in the x direction. Similarly, delta beta by delta t or db by dt is nothing but delta of u by delta y or the change in x velocity in the y direction. We have just done these two proofs. If you can recollect that in the last slide, we were trying to see the rotation of a component without deformation. In this case, there is a rotation and deformation simultaneously happening. Actually, the rotation is happening because of the deformation of the two sides.

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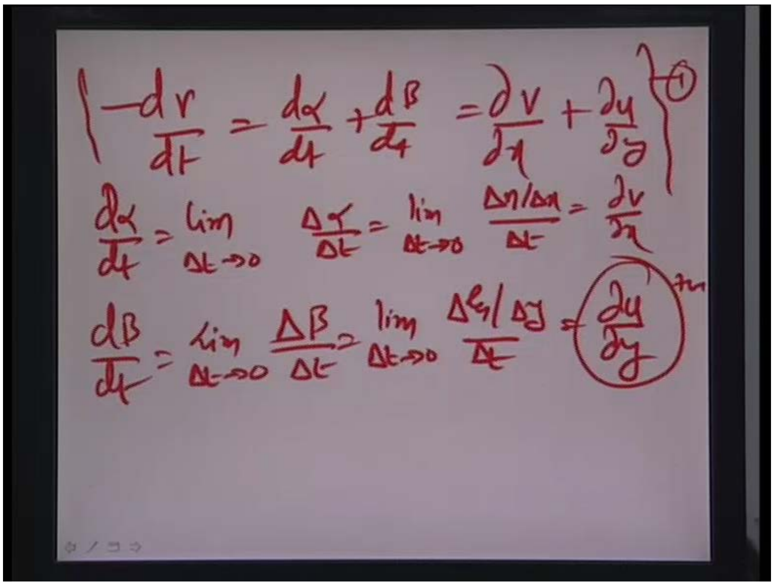

Essentially, this minus d gamma by dt, which is also d alpha by dt plus d beta by dt. It would become same as delta v by delta x plus delta u by delta y. You may retreat that d alpha by dt is nothing but limit of delta t tends to 0 delta alpha by delta t, limit of delta t tends to 0 delta eta by delta x divided by delta t, which is delta v by delta x. Delta beta by delta t is nothing but the limit of delta t tends to 0 delta beta by delta t, which is equal to the limit of delta t tends to 0 delta zeta by delta y by delta t. It is nothing but delta u by delta y. So, this d u by the d y is positive because if you look at the deformation, it is in the same direction as the u. So, u was u 0 here and it is u 0 plus delta u by delta y times of delta y. It is in the same direction as u. Therefore, it is a positive addition and thus the relationship one holds valid. It is minus d gamma by dt and it is delta v by delta x plus delta u by delta y.

We have now found out relationships of what would happen in case of fluid element rotating by itself and a fluid element deforming by itself. With all this knowledge in mind and also the way we did the acceleration of a point vector in a fluid space, we combine all these things together to find out what dynamically alters the cube as it moves along in flow field. There would be stresses: there would be principle stresses, there would be shear stresses, which take place. Some of them we had illustrated in our last class, when we were talking about the conservation of mass equation. You will have sigma xx, tau yx, tau zx as the three stress components because of the force in the x direction. Similarly, three stress components force on the y and z directions.

We had a stress matrix or a stress tensor that we have defined in this manner. Let us see that if all these things are individually implemented. We can somehow find out the force by multiplying it with the area element of the different stresses. We can equate that to the mass of this cube into acceleration and find out what is the equation, which is emerging from that. So, this is essentially the conservation of momentum that is the Newton's second law - rate of change of momentum is nothing but the force that takes place in the direction of the force. So, applying Newton's second law here, try to see what the final states of these relationships between the different stresses like shear, principal stress and so on; the acceleration, the mass and the area elements.

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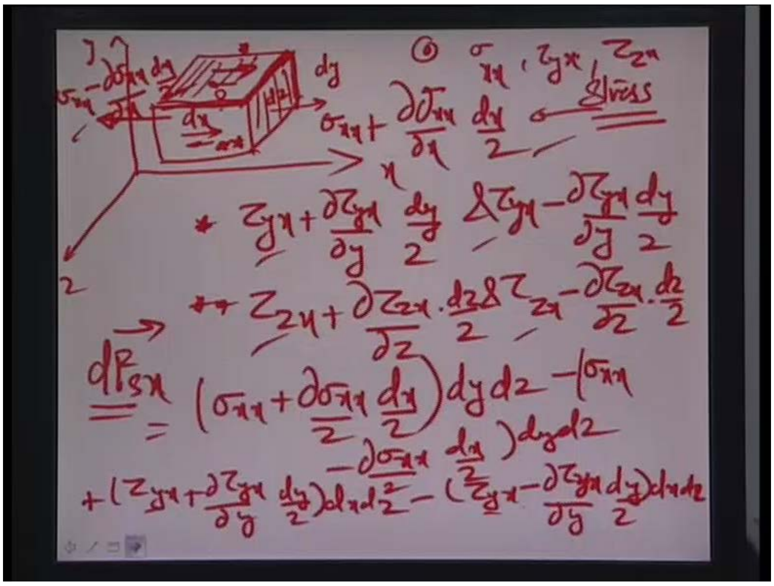

Let us look at our cube and the control volume again. You have a small cube, which is of volume dx. So, this is x, y, z direction and dx is the distance of this particular dimension of the cube, dz is the distance of this particular dimension of the cube and dy is distance of this particular dimension of the cube. You have different forces, which are acting in the x direction. Let us say you have a force let us just for the time being delete these arrows in the interest of space. So, we delete this we by and large know what these d x d y d z are so here we try to again do stress components we have this component of stress We assume that this cube is centered about a point o, where you have sigma xx, tau yx, which means the area vector is in the y direction and tau zx.

At this particular edge of the cube or this particular phase, your stress components would defer. This would be sigma xx and let us say we have a variation in the x direction. So, delta sigma xx by delta x times of dx by 2 would be the stress component in this phase. The stress component similarly in the other phase would be sigma xx minus delta sigma xx by delta x times of dx by 2. Similarly, the shear stress components would be in the opposite direction. In the lower phase, put this is a star and it would be tau yx plus delta tau yx by delta y times of dy by 2 and tau yx minus delta tau yx by delta y times of delta y by 2 or dy by 2.

Similarly, we will have the z components, which are essentially in this phase. In the backside, these components would be there. So, these double stars tell that tau zx plus tau delta tau zx by delta z times of delta z by 2 or dz by 2 and tau zx minus delta tau zx by delta z times of dz by 2. Therefore, dF sx, which is essentially the total amount of force in the x direction. These are the result of all different components like component 1, 2, 3, 4 and 5 6 all put together and these are all stressed terms.

So, you need a certain area of cross-section, across which these stresses are applied in order to find out the net force on the particular cube of interest. So, the net force in this case would be equal to sigma xx plus delta sigma xx by 2 times of dx by 2. The area vector here is dy dz that is dy times of dz. As I have pointed out, you have a stress in the opposite direction, which is same as sigma xx. Of course, minus dell sigma xx by 2 times of dx by 2 times of dy dz. Similarly, you have components related to the shear and I will just try to illustrate this here as plus tau yx plus delta tau yx by delta y times of delta dy by 2 times of dx dz minus tau yx minus delta tau yx by delta y times of dy by 2 times of dx dz. So, these are the various components in the x direction and this tau yx again has a cross- sectional area dx times of dz.

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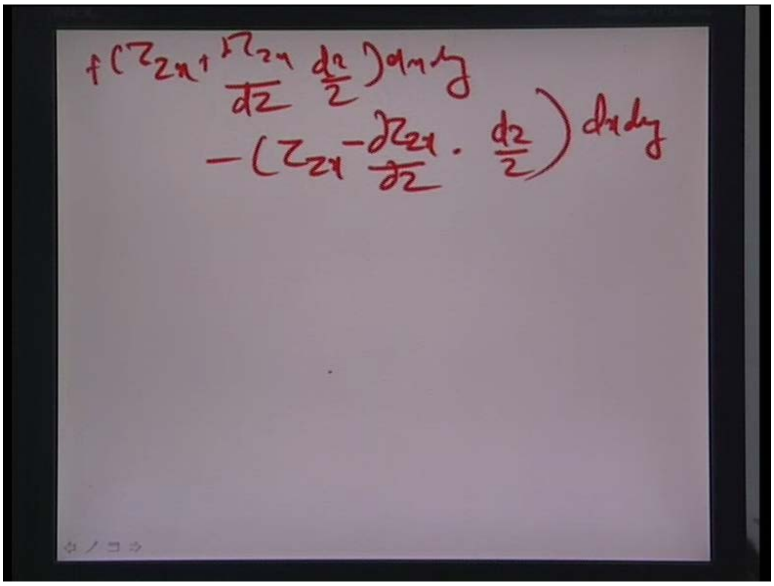

You also have a third component as a part of the whole deal. This tau zx plus delta zx by delta z times of dz by 2 times of dx dy minus tau zx minus delta tau zx by delta z plus d z by 2 into dz by 2 times of dx dy, which is the cross-sectional area.

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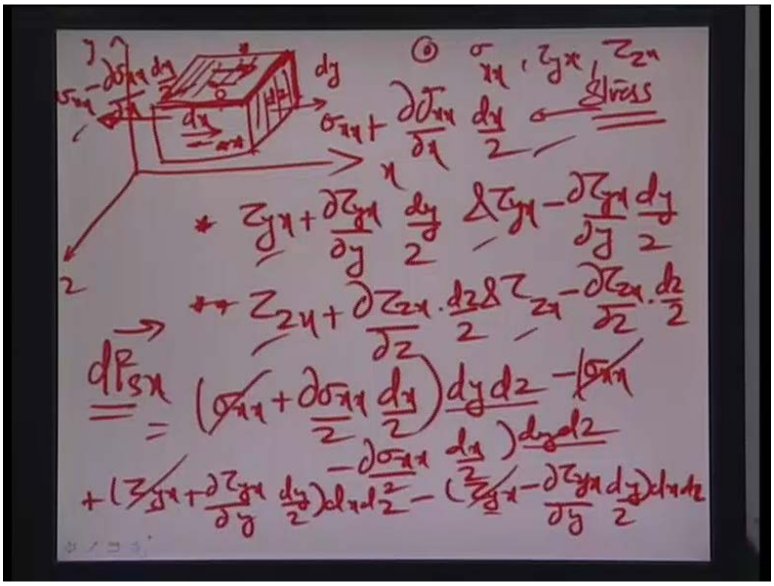

If I take a simplified form of all these from this expression, you can easily find out that the sigma x is actually cancelled because dy dz are similar. Similarly, the same goes true for the tau xy's. Only the differential components are retained.

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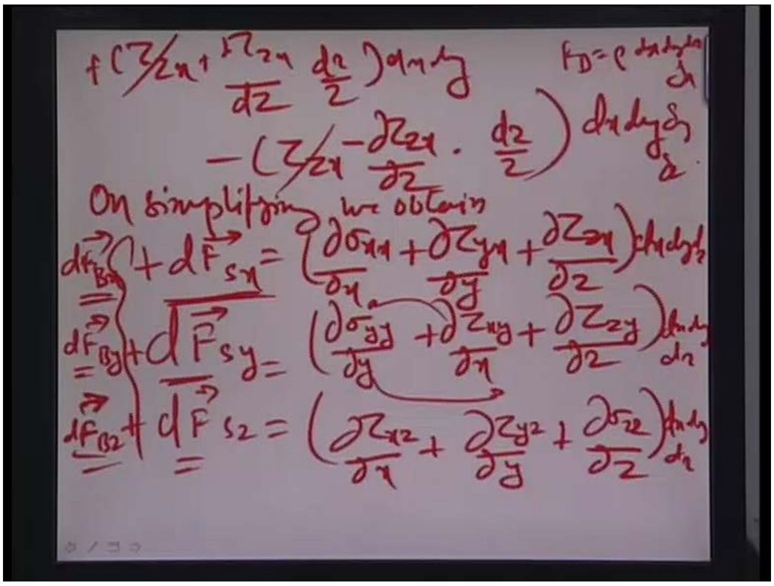

Similarly, here tau zx and minus tau zx are cancelled. If you assemble together all these different terms, then on simplifying, we obtain dF sx. It is nothing but delta sigma xx by delta x plus delta tau yx by delta y plus delta tau zx by delta z times of dx dy dz as the total amount of the force in the x direction. Therefore, on a more simpler note, similar kind of expressions can be generated for the different force components in the y and the z direction. Let us just write this down here and so in the y direction, it should be sigma yy times of delta y plus delta times of tau xy by delta x plus delta times of tau xz by delta z times of again dy dx dz. Similarly, you have the other component like a stress matrix, so, this comes here and this actually goes here.

In the other component, you have the delta sigma zz with respect to delta z. You have the component delta tau… Now, you have the y components and this is actually zy. You have delta tau yz by delta y plus delta tau. It is xz by delta x times of dx dy dz as a three different components of the force. If we involve the body force at this stage, to figure out how overall it would look like. We will have this equation slightly modified because we will then have to add the body forces along the x direction, the y direction. Let us call dF Bx, dF By and dF Bz as the different forces.

Therefore, if we have all these different forces together, we can say that this is actually the body force f B. It can be represented as the rho times of the volume, which is dx dy dz times of g. It has three components: gx, gy and gz. So, let us write all these equations together in the next page.

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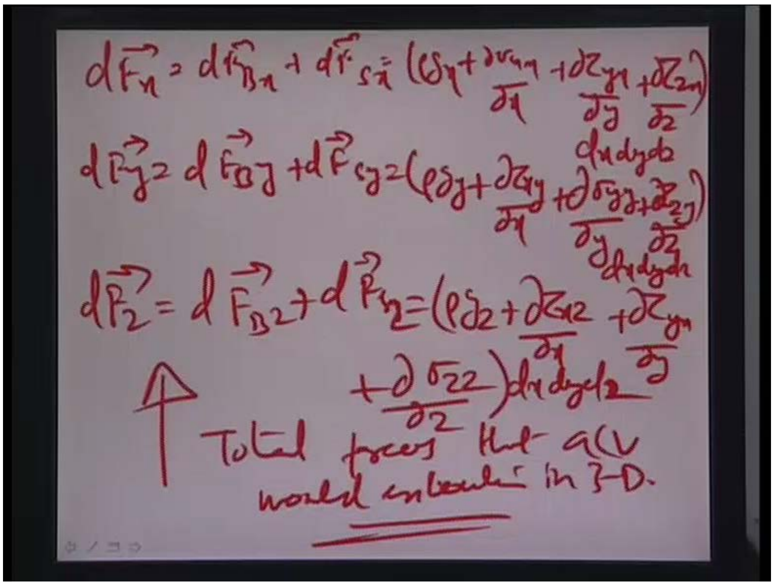

The total force is in the x direction because of the body force f Bx and the stress force. The force due to the stress vectors can be represented as rho gx plus delta sigma xx by delta x plus dell tau yx by delta y plus tau delta tau zx by delta z times of dx dy dz. Similarly, d Fy is again having a body force in the y direction plus a stress force in the y direction, which can be rho gy plus delta tau xy with respect to x plus delta sigma yy with respect to y plus delta tau zz y with respect to z times of dx dy dz. Then dF z is equal to delta body force in the z direction, the stress force in the z direction. It is rho gz plus delta tau xz by delta x plus tau d tau xy by dz dy plus delta tau or delta sigma zz because it is not a shear for this principle stress sigma zz by delta z times of dx dy dz.

So, these are the equations for the total forces that control volumes cv would encounter due to stresses and as well as its own weight in all the three directions. So, this brings us to the end of this particular lecture. In the next lecture, what I would like to illustrate is that we will move ahead with this force equation and try to compare it with mass times of acceleration and see what the final form of equation would look like. Thank you.