**Acoustics**
**Professor Nachiketa Tiwari**
**Department of Mechanical Engineering**
**Indian Institute of Technology, Kanpur**
**Lecture 4**
**Module 2**
**Waveguides, Transmission Line Equations And Standing Waves**

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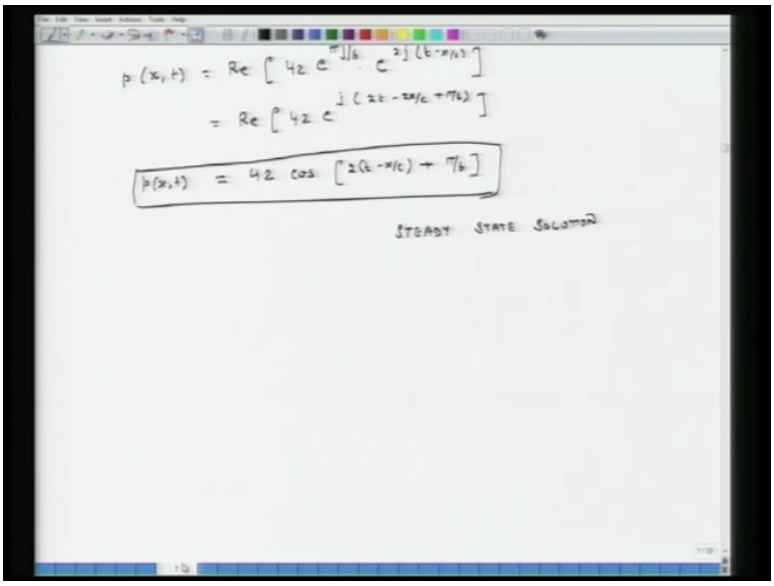


So a subsequent question which could be asked is that okay, now we know how to find pressure in waveguide a sound is travelling through waveguide, but what about velocity? What is the velocity of the wave or velocity of particles as sound is getting propagated in the medium, especially in context of a waveguide. So for that now we are going to develop a velocity equation and we are going to find its relationship with pressure and through that mathematics, we will figure out how are pressure and velocity related.

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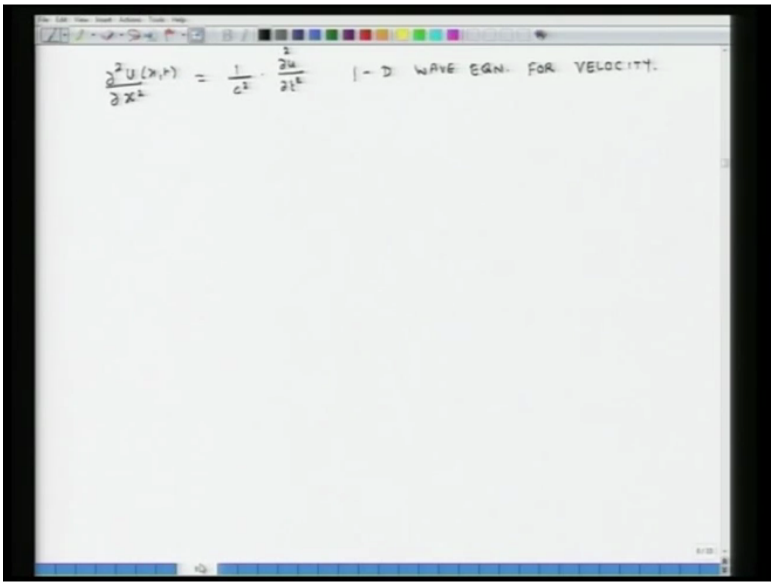


So the 1-D wave velocity equation is like this, so here velocity is a function of x and times. So del 2 u over del x square equals 1 over c square times second derivative of u, with respect to time. So this is my 1-D wave equation for velocity and the form of this wave equation is pretty much same as the one-dimensional wave equation for pressure.

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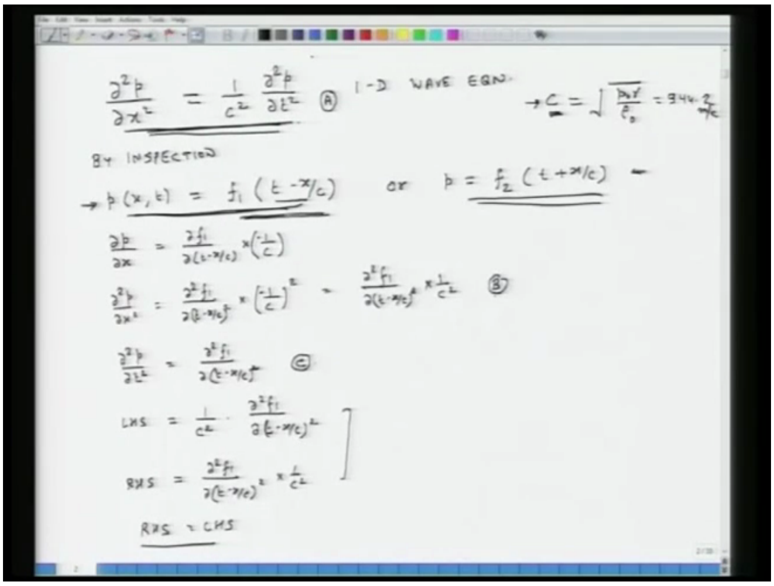


So here also I have del 2 P over del x square equals 1 over c square times del 2 P over del t square and for the velocity equation the form is exactly the same.

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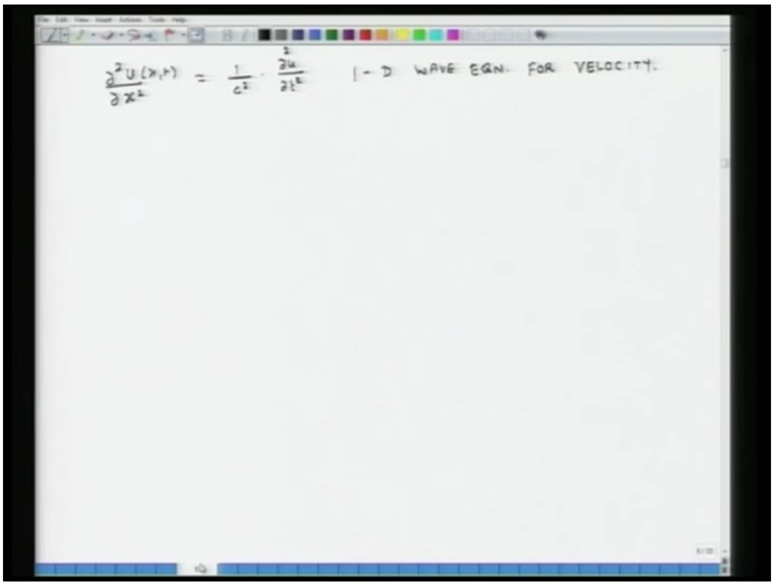


The derivation for this velocity equation is extremely similar to the derivation for the pressure vacation which we have gone through earlier. So it will be helpful for you as a listener and as a student of this course that you go back and actually derive this wave equation using similar approach which we used for deriving the pressure wave equation.

Now just as a solution for a wave equation for pressure was or could be written as f1 t minus x over c plus f2 t plus x over c, we can by analogy also develop a solution for this velocity wave equation for this velocity wave equation. So, let us write that down, so by analogy u of x, t I can write it as f1 t minus x over c plus f2 t minus x over c and so on and so forth. Plus fa1 t minus x over c plus fa2 t minus x over c excuse me should be these should be positive and so on and so forth.

Or alternatively as we had shown that this could be rewritten as a real component of some of complex variables we can write similar relation for velocity as u of x, t is the real component of u plus e st minus x over c. So again u plus could depend on s plus u minus e st plus x over c and once again u plus and u minus could depend on frequency as was the case where P plus and P minus could depend on frequency.

So u plus is actually a function of s and so is u minus is a function of s as well. So now as I did for the case of a pressure wave equation I can rewrite this as real of and I take est out u plus, so let me put a different style of Brackets here u plus e minus sx over c plus u minus e plus xs over c and I can rewrite this term in curly brackets as u of x and s. So it will get u which is a function of x and s times e to the power of st, okay.

So, till so far what we are seeing is that the mathematics for developing relations, for velocity and for pressure are extremely similar. I am just replacing u with P and so on and so forth. So now what I am going to do is I am going to connect this u with the pressure and see how pressure and velocity are related and for this I have to use the momentum equation which I developed in the last class and it is that momentum equation which connects u with pressure, velocity with pressure.

So from the momentum equation we know that partial derivative of pressure with respect to x is nothing but negative rho not partial derivative of velocity with respect to time and once again here P is a function of x and t as well. So let us label these equations, so let us call this equation 1, equation 2 and this is equation 3. So if I combined these equations what I get is from the right side.

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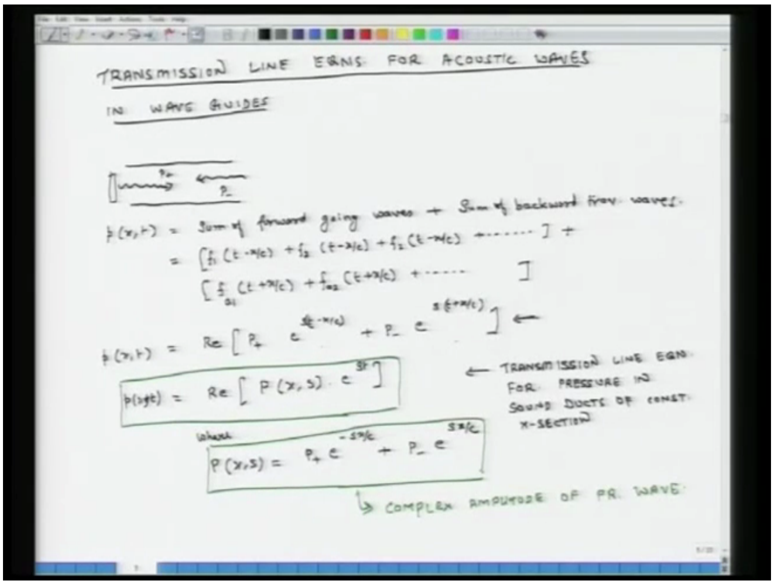


So before I start combining let me just rewrite the equation for pressure as well which we develop here’s equation.

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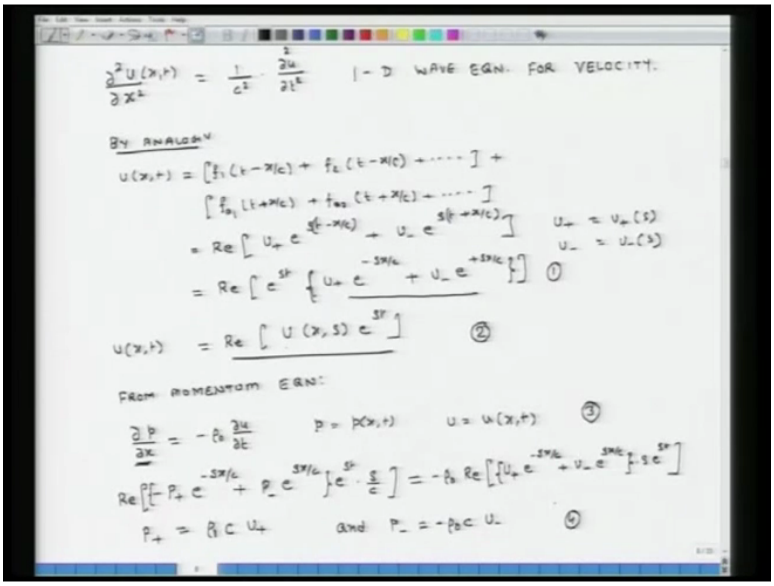

So I will rewrite this and I use that equation and I combine with these equations and what I get is from the right side I differentiate the equation for pressure and what I get is real of minus P plus which again depends on s times e minus s x over c plus P minus e to the power of sx over c this entire thing multiplied by e st times s over c.

So what I just did here is that I took this relation (()) (8:47) for pressure differentiated this relation with respect to x and plugged it into the left hand side of the momentum equation. So that is what variation of pressure with respect to x looks like and this is equal to minus rho not and I am going to do the something similar to velocity. So I am going to differentiate this equation, equation 1 with respect to time partially differentiate it with respect to time.

So what I get is minus rho not times real of u plus e minus x over c plus u minus e sx over c this entire thing times s e st the whole thing into rectangular brackets. So as this and that equation is valid for all values of time and all valid values of x this equation can hold true only if the terms related to minus sx over c on left side are exactly the coefficients of e minus sx over c are same as coefficients of e minus sx over c on the right side and so on and so forth.

From this understanding we conclude by equating appropriate terms from left side to right side and so on and so forth that P plus is equal to rho not c u plus and P minus equals rho not c u minus but there is a negative sign before that. So from this I can calculate u plus in terms of P plus or P plus in terms of u plus and same thing I can do with u negative. So this is my equation number 4.

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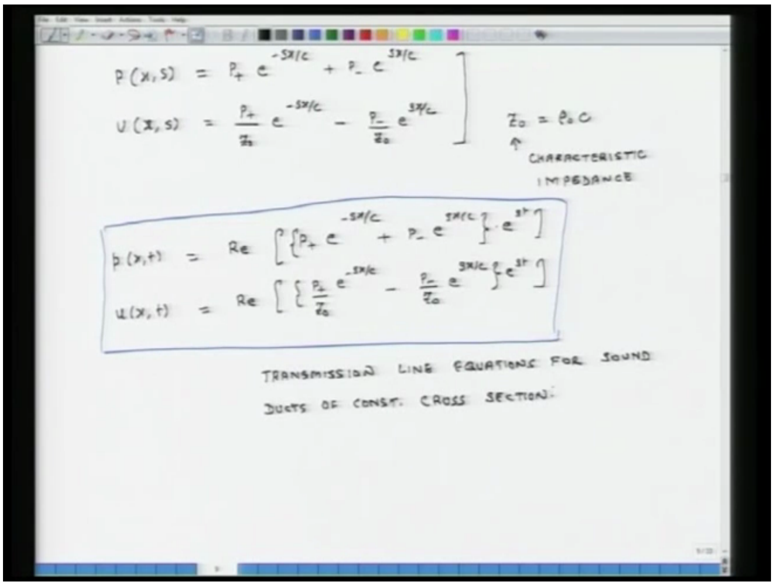


Thus I can rewrite my transmission line equations the following form, this is the complex amplitude for pressure and that is P plus which depends on frequency times e minus sx over c plus P minus e to the power of sx over c similarly u of x, s which is complex amplitude for velocity is P plus over I am introducing a new term z not which I will define later times e minus sx over c minus P minus over z not e to the power of s, x over c where z not is same as rho not c. This term z not which is equal to rho not over c is called characteristic impedance.

So from these equations I can now write the full form for pressure and velocity. So P is a function of x and time equals real of P plus e minus sx over c plus P minus e sx over c times e to the power of st and u which depends on again x and time is real of P plus e minus sx over c over z not minus P negative over z not e sx over c e to the power of st, these 2 equations are called transmission line equations for sound ducts of constant cross-section. So from these 2 equations you can calculate pressure and you can also calculate velocity of sound as it travels through a waveguide which has a constant cross-section.

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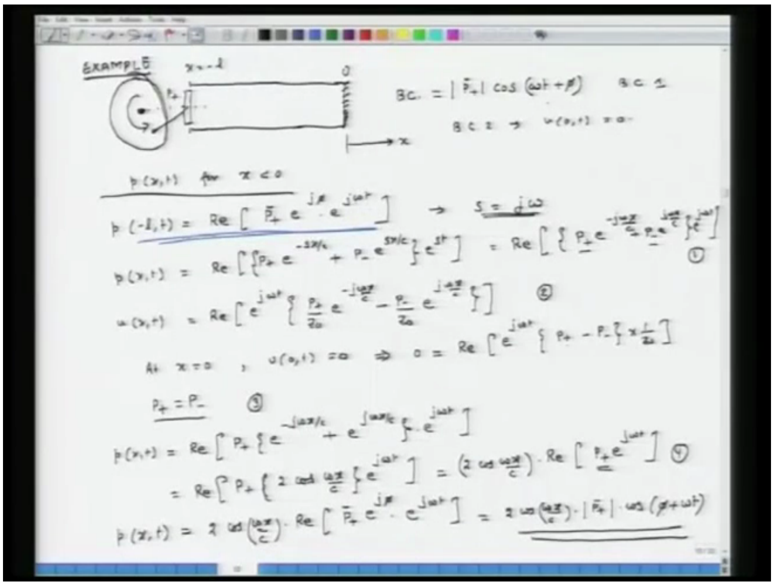

So now what we will do is we will do a couple of examples, so that things become clearer to us. So let us do an example, so I have again a tube but unlike the last case here the tube is of a finite length and also the tube is closed at the extreme end. So let us put a coordinate system here I am measuring x from this location and the value of x is 0 here, the open and of the tube is where x equals minus l.

I have a piston and this piston is generating pressure wave, so let us this circle is rotating like this and as this circle is rotating this piston is moving back and forth because of this particular linkage and let us say it generates a positive pressure wave of P plus such that at x equals l the strength of the wave is P plus bar the magnitude of it times cosine omega t plus phi. So this is my boundary condition.

This is one boundary condition, the other boundary condition is that at x equals 0 I have a very rigid wall. So no wave cuts across this wall and the wall does not move at all, so essentially what that means is that the velocity at x equals 0 is 0. So given these 2 boundary conditions what we have to find is the pressure.

So this is boundary condition 1 BC 2 is u 0, t equals 0, so these are the 2 boundary conditions and now we have to figure out what is the value of P x, t for x less than 0, so the length of the tube is l, so for this entire length how is pressure changing with respect to time and x. So let us rewrite this first boundary condition. So P minus l, t equals real of P plus bar which is a constant number times e j phi times e j omega t.

So all what I have done is I have rewritten the same expression in this format and when I take its real value I get the same thing back. So from this we find that s equals j omega we get this that s equals j omega. So now we plug this and put it into the equation for pressure. So P x, t is real of P plus e to the power of minus x over c plus P minus P to the power of sx over c e st.

And now I know that s equals j omega, so I get real of P plus j omega x over c plus P minus e j omega x over c e j omega t the whole thing in rectangular brackets, okay. So let us call this equation 1. Similarly I can write the equation for velocity u of x, t is real of e j omega t times P plus over z not e minus j omega x over c minus P minus over z not e j omega x over c. Now I know that at x equals 0 u of 0, t equals 0 because this is a rigid wall.

So the wall is not moving, so whatever is the wall the velocity of the wall is going to be the same as velocity of fluid particles, so velocity of fluid articles is also 0 at x equals 0. So imposing this condition what I get is 0 equals real of e j omega t P plus minus P minus and then I can take z not out. So from this particular boundary condition this can be true only if P plus minus P minus is by itself 0 because it has to hold valid for all values of time.

So that gives me P plus equals P minus, so the implication of a rigid boundary condition in a waveguide is that P plus is equal to P minus. So let us label again some equations, so that is equation2 and equation 3. So I am putting equation 3 back into the first equation, so what I get is P x, t equals real of P plus, so P plus here is same as P minus, so I take P plus out e minus j omega x over c plus e j omega x over c times e j omega t.

Now I know from my understanding of complex variables that the term within the curly bracket which is minus j omega x over c and e plus j omega x over c when I add these 2 terms up I essentially get situation where sine terms gets cancelled out and the cosine term itself remains. So what I get is real of P plus times 2 cosine omega x over c times e j omega t. So I take cosine omega x over c out of the rectangular brackets.

So 2 cosine omega x over c times real of P plus times e j omega t. so let us call this equation 4. Now if I put the value of x at minus l or if I try to figure out what is the value of P plus from this relation then what I get is my final solution for pressure is P x of t equals 2 cosine omega x over c times real of P plus par e j phi times e j omega t, okay. And when I take the real value of what I get is 2 cosine omega x over c times P plus bar times cosine of phi plus omega t. So this is the final expression for pressure in this tube as sound is travelling along this tube and this is the case for a standing wave.

Likewise excuse me I can use this understanding and the fact that P plus is equal to P minus and I can also develop a relation for velocity using equation 2. So I can now I figured out what is the relation for pressure for all values of x and time and I can also figure out what is the value of u for all values of x and time.

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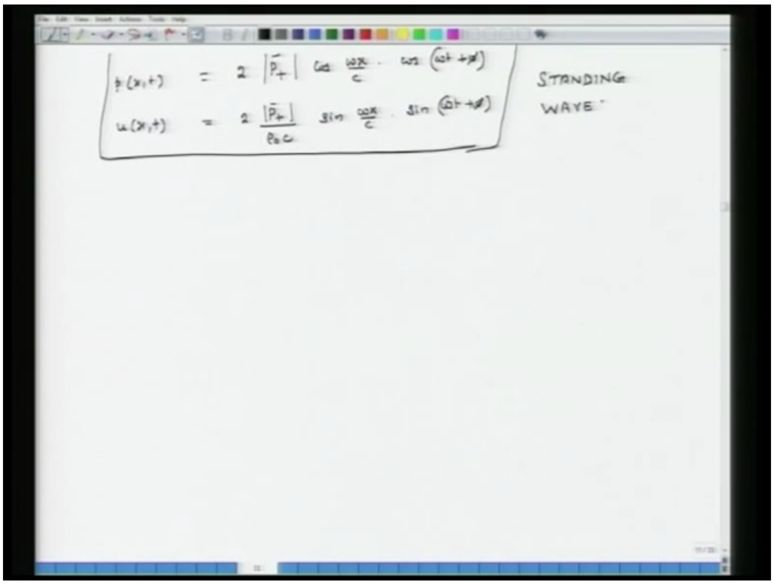


So let us summarize these 2 relations, so my final answer is P of x, t equals 2 P plus bar this thing has to have its magnitude times cosine omega x over x times cosine omega t plus phi and velocity is equal to 2 P plus bar over rho not c, sine omega x over c times sine omega t plus phi. It turns out that this is also the relation for standing waves.

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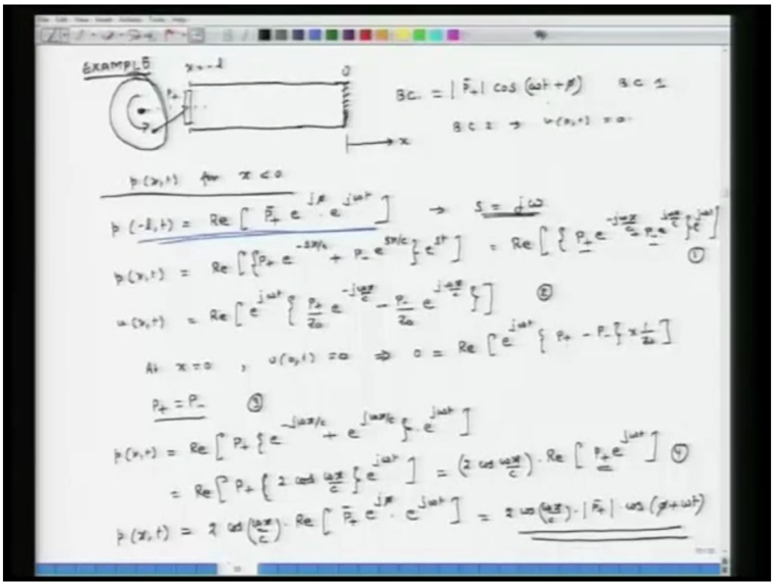


So what is happening here is that you are having pressure being generated at one end, sound is travelling it hits the other wall and it hits the wall at x equals 0 gets reflected and once things have stabilised in this tube because the steady state solution which this whole the pressure wave equation is giving us your get standing waves for pressure and also for velocity.

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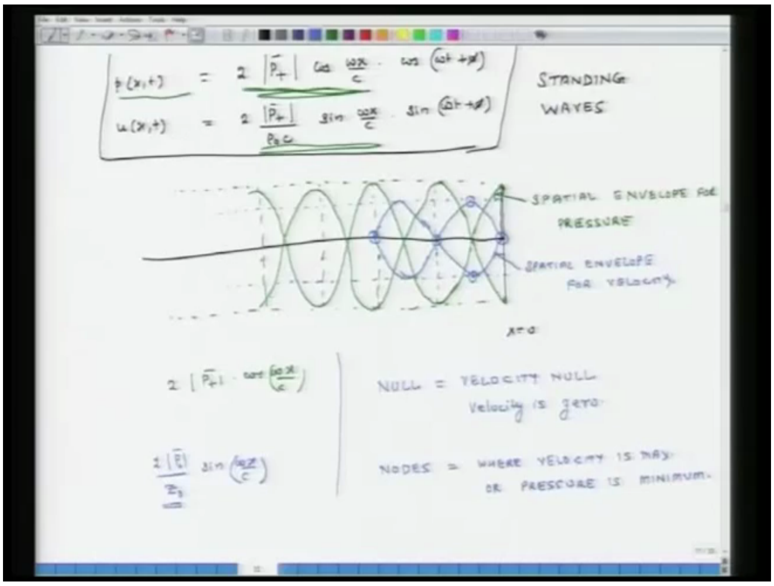


And the way these waves vary with respect to x we will draw it here, so here my x equals 0. Let us say I am going to plot in the negative direction because my open end of the tube is at x equals minus l and I will draw 2 envelopes. So at x equals 0 my pressure is maximum, right now what I am going to plot is only this portion of pressure and only this portion of velocity.

So at x equals 0 my pressure is maximum and let us say that maximum, now that maximum could vary with because of time fluctuation. So it could vary between these 2 limits then at at x equals lambda over 2 that pressure goes down to its negative value and then after at lambda it again comes down to its positive maximum. So it varies something like this.

So let us say here x equals lambda over 2, so it varies like this and the other mirror image of this envelope is something like this. So this is called spatial envelope for pressure and this spatial envelope for pressure is depicted by the expression 2 P plus bar times cosine omega x over c. For velocity at x equals 0, the velocity is 0 at this point.

So wherever you have velocity its minimum, pressure is maximum and wherever pressure is minimum, velocity is maximum, so this spatial envelope for velocity could be depicted by the relation 2 P plus bar over z not sine omega x over c wherever I have a minima of pressure I will have a maximum or maxima for velocity and so on and so forth. So if I have to plot this spatial envelope for velocity the magnitude of that spatial envelope for velocity is going to be 2 P plus over z not.

So it is going to be because z not is more than one, so it is going to be lesser than the spatial envelope for pressure. So let us say that envelopes are limited by this blue dotted line. So it is limited by this blue dotted line, so it is going to be maximum here, so it is going to vary like this and I can keep on extending it backwards as x grows on the negative side.

So this look curve is once again spatial envelope for velocity and we see that it is the envelopes magnitude is 0 at x equals 0 because we have a rigid wall there and that essentially injects a rigid wall boundary condition which implies that velocity of particles of fluid is 0 at that specific location, with that I wanted to close this lecture but before I do that I will like to introduce 2-3 more terms which we will be using in later lectures.

So in context of standing waves we found that there are places or there are locations where you have a null that is, so the first I am introducing is Null and more specifically you can also call it as a Velocity Null and it is a place where velocity is 0. So these points these are all velocity nulls.

Corresponding to velocity nulls you have locations, corresponding to velocity nulls you have pressure maximum. So pressure is maximum where you have a velocity null, now these points where you have velocity as maximum there the pressure is minimum, so these points are called Nodes and these are where velocity is Max or pressure is minimum. In case of a tube which terminates with a rigid wall the value of minimum pressure is exactly 0 and the Valley of velocity is also exactly 0.

But if you have a termination condition where it is not an absolute case that is the termination condition is such that the wall is not absolutely rigid then the pressure maybe still at a minimum but it may not necessarily be exactly at 0 value and same thing holds true for velocity as well. So the reason these nulls or minima values for velocity and pressure are exactly 0 is because a, we have a rigid wall termination condition and the second reason is that there is no damping happening in the system.

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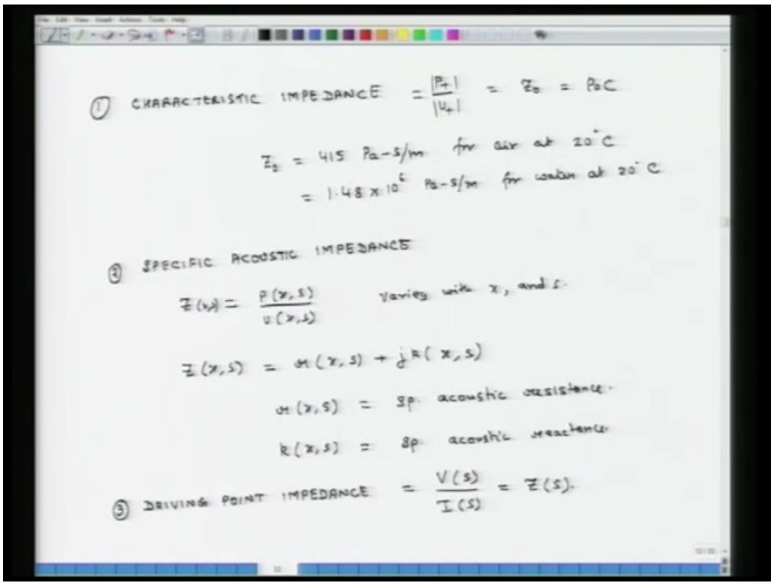


I also wanted to introduce 3 couple of more terms, so we have talked about characteristic impedance and characteristic impedance we defined as P plus over U minus the magnitudes of these and that is essentially Z not and this is equal to rho not C and the value of characteristic impedance is equal to 415 pascal second per meter for air at 20 degrees centigrade and 1 atmospheres.

If we have sound propagation happening in water than the value for that situation is 1.48 times 10 to the power of 6 pascal second per meter for water at 20 degrees centigrade and this is freshwater we are talking about, if I go to supply water than these numbers change and so it is important to understand the exact nature of the medium which we are using as sound is travelling to it.

So that is the first definition, the second one is specific acoustic impedance and that equals, so this is equal to P of x and s divided by u of x and s. So unlike characteristic impedance which is a pure number, specific acoustic impedance can change with x and it can also change with s which is frequency. So this varies with x and s, okay. And this is designated as z and z could be a function of the already mentioned x and s.

Now z of x and s could be written as a real part, so designated by the letter r and that again can depend on x and s and also an imaginary part, so I have a j times k, r which again is a function of x and s is called specific acoustic resistance and it depends on the damping parameters of the system. So z, x and s is a property of the system which includes the medium and all the devices working in the medium that specific in acoustic impedance can vary with respect x and s.

And it can be broken up into a real part which relates to damping phenomena in the system and the real part is called specific acoustic resistance and the imaginary component is called specific acoustic reactance and that relates to compliance and inertial related parameters of the system.

The final definition I wanted to introduce here is called driving point impedance and this is more used in electrical domain and this is essentially same as V which is a function of s, V standing for voltage divided by I which again depends on s and this is nothing but z of s. So what we have covered today is a continuation of 1-D with propagation of sound we have understood that its solution would be represented in very general form as f1 t minus x over c plus f2 t plus x over c and we have understood the physical significance of f1 and f2.

And then after that we have talked about waveguides and transmission line equations as in the case of sound propagating through ducts of uniform cross-section and then we have done a couple of examples and we have also understood, how standing waves gets created in a tube which is of a finite length and which terminates rigidly. So with this I close today’s lecture and we will continue our journey of understanding 1-D sound propagation in the next lecture, thank you very much.