Acoustics
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Lecture 2
Module 2
Solution for 1-D Wave Equation

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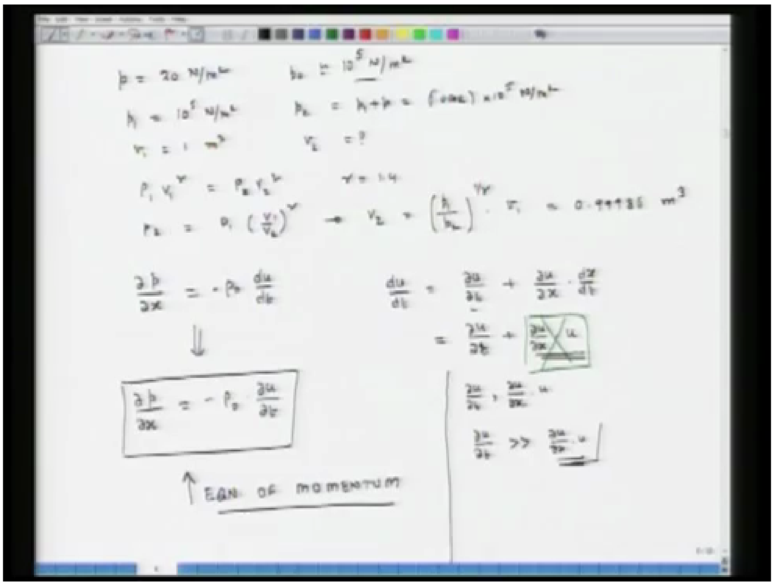

So the next equation we are going to develop is from the material constitutive behaviour, so how and that will explain or help us to understand how is volume or change in volume related to change in pressure and so on and so forth? This piece of fluid which we are considering it could behave in range of (()) (0:41) and we have to figure out which is the most appropriate model which predict its behaviour.

So this could be behaving in adiabatic way that could be one model that could be one model or the fluid here could be behaving in an isothermal way or it could be behaving in an isobaric way and so on and so forth. So the question is how is gas behaving in this piece of material volume? Is the behaviour adiabatic? Is the behaviour isothermal? Is the behaviour isobaric or is it some other thing?

And how do we know what is the exact behaviour of this piece of fluid? So this question was also addressed by Newton himself and he made an assumption that as sound propagates in air the fluctuations and pressure and fluctuations in volume associated with this propagation of sound they are related in a way which mimics an isothermal behaviour.

So in that case he said that P times V is constant and if I use this relation then I can figure out how the wave is propagating but it turns out that Newton was not correct in this particular assumption that propagation of sound is an isothermal phenomena. Later (()) (2:30), he also tried to understand this question and answer this question and he made an assumption that the propagation of sound is essentially and isentropic happens in isentropic conditions that is my P times V to the power of gamma is constant.

Or in other words it is an adiabatic behaviour and subsequent measurement showed that he was indeed correct and Newton was in error. So the question is that, why is that the case what was erroneous about Newton's assumption of isothermal behaviour and what was correct about (()) (3:27) assumption of adiabatic or isentropic behaviour.

So to understand this we have to see how heat propagates in fluid medium and how sound travels and then we see how are these 2 phenomena interlinked and how do they influence each other at the physical level? So if I compress a piece of gas it will generate heat. Now that heat could leak out from the gas and go away get dissipated or it could remain contained in it, in the gas itself.

So the question is, is it leaking out in such a way that the temperature of the gas remains constant then that would be an isothermal behaviour or is it contained in the fluid volume in some sense and that is the case then that would be saying that the behaviour of gases is something close to an isentropic process or adiabatic process. So to understand this let us look at some observe data.

So let us assume that I have a sound wave and it is propagating at 10,000 hertz. So my auditory range is from 20 hertz to 20,000 hertz. So I am assuming that there is a particular tone which is propagating in air at 10,000 hertz. Now as it is propagating in air, in some parts air is getting compressed and in some parts air is getting expanded, where it is getting compressed heat is getting generated, where the air is experiencing rarefaction heat is being absorbed or cooling is happening.

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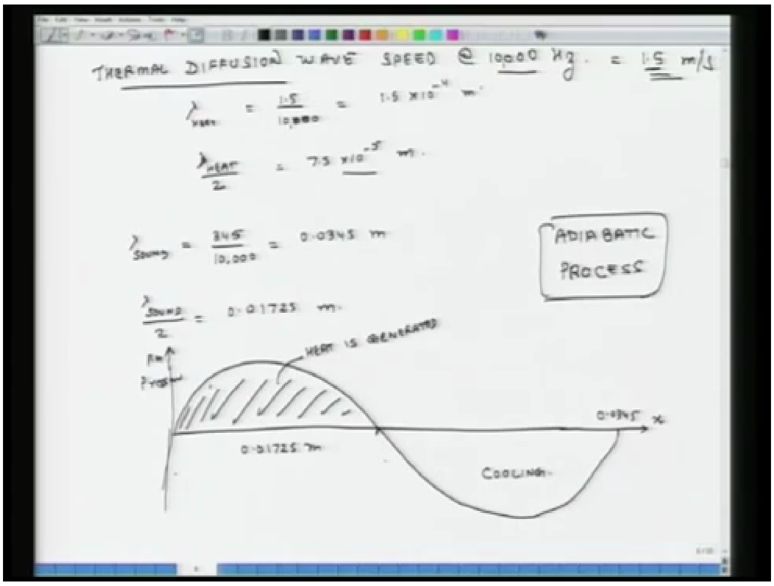

So there is some thermal process also associated with this acoustic process and the speed of travel of this heat wave which is also called thermal diffusion wave. So thermal diffusion wave speed had 10,000 hertz is 1.5 meters per second this is based on some observed data is available in engineering literature which means that the wavelength of this heat wave at 10,000 hertz is velocity of heat wave which is 1.5 over frequency, so that is 10,000 and that comes to 1.5 times 10 to the power of minus 4 meters, okay.

1 half of this wave length is essentially this number divided by 2. So that is 7.5 times 10 to the power of minus 5 meters. When sound travels the speed of sound as measured at room temperature conditions is about 345 meters per second. So lambda for sound at 10,000 hertz is velocity of sound which is 345 divided by frequency. So that comes to 0.0345 meters and half of this wavelength is 0.01725 meters, okay.

So now consider a medium and what I will do is, I will plot the sound wave, so let us say this is as this is a pure tone it maybe it will be having some sinusoidal shape. So this is my x, so we have is travelling along the x direction and this is its amplitude of pressure, this number is 0.01725 meters and here x is 0.0345 meters. So in the first phase of this sound wave, first part of this sound wave pressure is rising above atmospheric pressure and what that means is, heat is getting generated heating is happening and in this case cooling is happening.

Now when heat is getting generated and that happens in this 0.01725 meter of a range and then in the next 0.01725 you are having cooling and the first thing as heating is getting generated, heat travels at an extremely slow speed which is 1.5 meters. So the travel of heat as it is getting generated it does not travel far enough to enter into the cooling range and by that time you have the next cycle of the sound coming up.

So what is happening is, that heat is getting generated in a small place of fluid it moves a little bit because of its propagation at this speed 1.5 meters per second and an once that happens sound starts experiencing in that area you have rarefaction of the pressure and once that happens you start having cooling, so heat is getting generated moves a little bit and it has not moved sufficiently enough to leak out of the system and then you have cooling happening because of rarefaction.

Because the sound velocity is extremely high 345 meters per second, so heat is not able to leak out from the system in an appreciable sense and essentially what that means is that we have an adiabatic process, adiabatic process is something which captures this kind of behaviour fairly accurately. So it is for this reason (()) (11:24) when he made the adiabatic assumption he was correct and Newton when he assumed that the gas is behaving in an isothermal way he was in error in context of sound waves.

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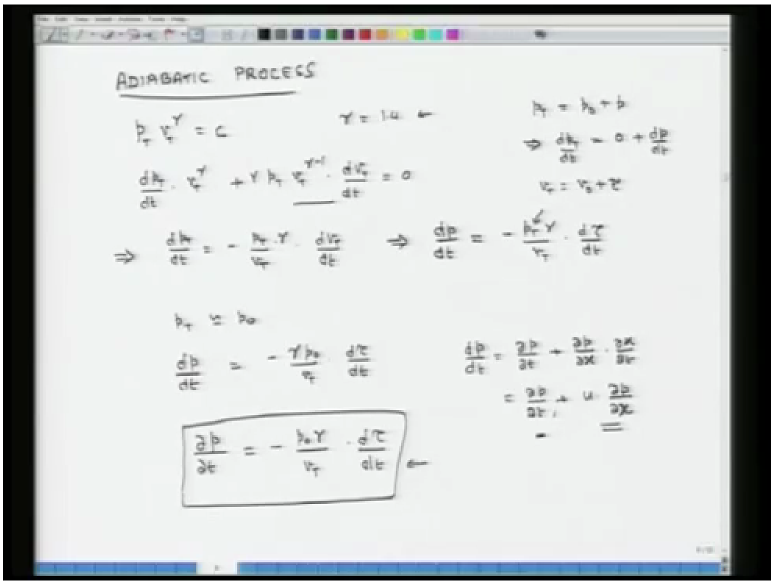

So using this understanding we will develop the gas law for sound. So we know for adiabatic process adiabatic process PT VT to the power of gamma equals constant where gamma equals 1.4 it is a gas constant and the value is 1.4 for gases which are diatomic in nature whose molecules have 2atoms I now differentiate this.

So essentially what I get is dPT over dt times PT to the power of gamma plus PT VT to the power of gamma minus 1 times d VT over dt equals 0. So I am differentiating this equation in time and then if I rearrange this essentially what I get is dPT over dt equals, so I move this on the other side of the equation, so what I get is minus PT over VT and times gamma times dVT over dt.

Now we know that PT equals Pnot plus P, so essentially what that means is the differentials of PT are same as differentials of P not which is 0 plus dP over dt. So using this understanding I can replace dPT over dt by dP over dt. So I get dP over dt equals minus PT gamma over VT times all also I know that VT equals V not plus Tau. So differential of volume VT with respect to time is differential of Vnot which is 0 because it is a constant plus differential of Tau with respect to time. So this thing I can rewrite as d Tau over dt.

Further I know that PT is approximately equal to P not, so I make that change here and my final relation is, so my next relation is dP over dt equals minus gamma P not over VT times dTau over dt making one final simplification we know that dP over dt is partial derivative of pressure with respect to time plus partial derivative of pressure with respect to x times del x over delt and that is u times partial derivative with respect to x and we have seen earlier that this term is extremely small this non-linear term compared to this one.

So I can replace dP over dt by just its partial derivative, so my final equation in differential form for the adiabatic processes, partial derivative of pressure with respect to time equals negative of atmospheric pressure times gamma over VT times d Tau over dt, this is the second equation and this is essentially equation for adiabatic process but in a differential form with the assumptions embedded into this equation as they relate to propagation of sound. So I have developed an equation for momentum and now I have developed an equation for adiabatic gas process.

So the third and final equation which I am going to develop and after that I will start synthesising all these 3 equations. So third and final equation it relates to conservation of mass and it is also called continuity equation.

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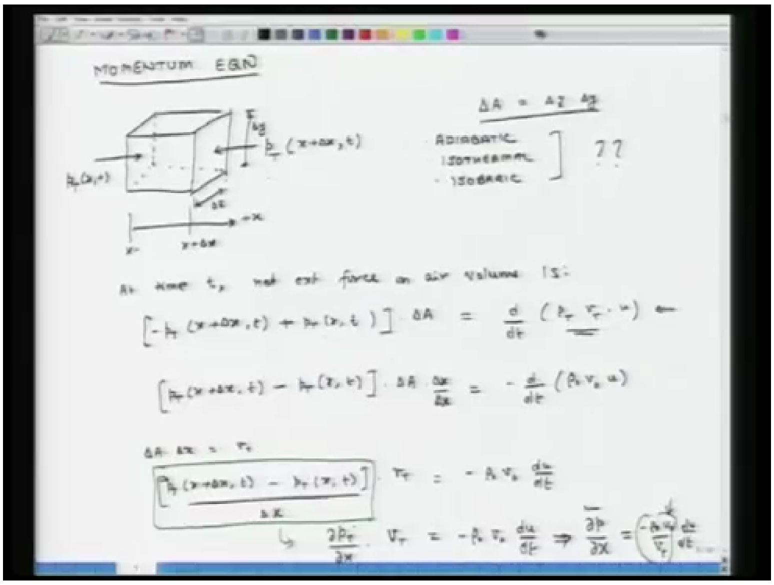

So the basis of continuity equation is that there may be some inflow from this side of the surface in this fluid and there is some outflow happening from this and if I add these 2 numbers up that the total amount of variation in mass should remain 0 because these are constant mass particles.

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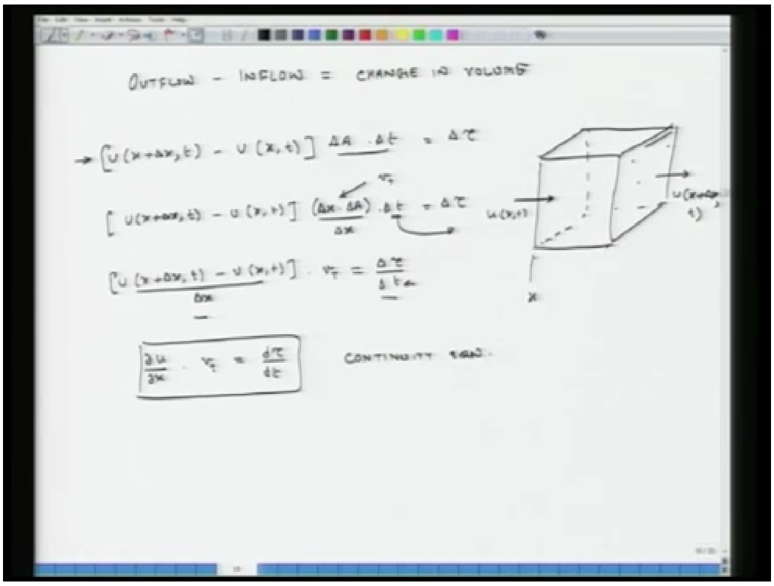

My outflow minus inflow equals change in volume, now once again I will draw the same picture, this is inflow your position is x and u is a function of x and time I have an outflow here u is a function of x plus delta x and time. So my outflow is u which is a function of x delta x and time minus outflow velocity minus inflow velocity and then that I have to multiply this area.

So that area is Delta A and as this is velocity I have to also multiply it by dt and this equals change in volume because I am assuming that changes in density are negligible. So this is the equation I am getting. So I can rewrite this equation as u x plus delta x times minus u being a function of x and t and then here I am dividing this by delta x and multiplying it by delta x. So Delta x times Delta A, so I have multiplied with Delta x and then I am dividing it by delta x times dt equals change in volume.

This term delta x time delta A is VT, so I can replace that by VT. So I can rewrite this as u x plus delta x, t minus u is a function of x and time and in this case what I am going to do is, I am going to take time and move it on the right-hand side of the equation. So I get delta Tau over dt. So actually these are all finite quantities, small but finite quantities, so I am going to replace d by delta.

Now if I assume that the size of this fluid volume is extremely small such that its approaching 0 that is one assumption and I am also assuming that the amount of time which is getting elapsed which is Delta t is also extremely small then in that case I take the limits for this expression and the limit for this expression and what I get is, here I get a derivative of u with respect to x times VT equals a derivative of volume with respect to time. So this is my third equation and this is continuity equation. This is the third equation and the continuity equation.

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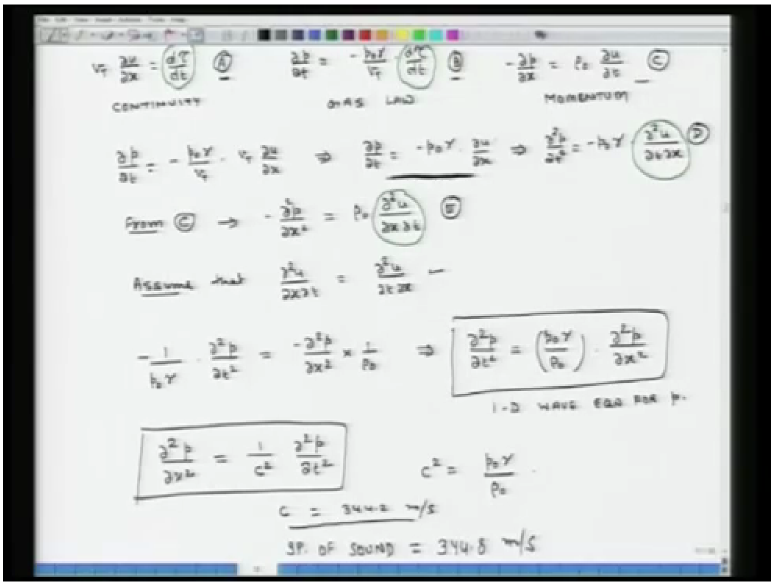

So next I synthesise the continuity equation, gas equation and Newton's law and try to develop one single equation for pressure. So let us rewrite these equations, so continuity equation as VT times del u over del x equals d Tau over dt I call this A this is my continuity then the gas law is del P over del t equals minus P not gamma over VT times d Tau over dt this is gas law adiabatic gas law, so this is equation B.

And the Newton's law or the momentum equation is negative of rate of change of pressure in with respect x equals rho not times rate of change of velocity in time and this is equation C and this is momentum, this is a moment equation. So what we are trying to do here is that we have 3 equations and we are trying to eliminate other variables such that we get one final equation for pressure or we can do something same for velocity also. So in this case all what we are trying to do is developing an equation for pressure. As to how pressure is changing in x and in time.

So if I use A and B these 2 equations they have this d Tau over dt term and if I use these 2 equations to eliminate the Tau variable then what I get is, so del P over del t equals minus Pnot gamma over VT and here I have d Tau over dt which equals this term, so I get VT times del u over del x. So I get partial derivative of pressure with respect to time equals minus P not gamma times partial derivative of u with respect to x.

And now if I differentiate this whole equation, so again my aim is now to eliminate u from this equation and equation C if I eliminate then I get an equation in pressure and see I can figure out how it is changing in x and time. So that is what I am trying to do, so the way I am going to do is that I will differentiate this equation with respect to time and I am going to differentiate my momentum equation with respect to x and then eliminate u in that way.

So I differentiate this equation with respect to time, what I get is second derivative of pressure with respect to time equals minus P not gamma times second derivative of u with respect to del t del x, so I call this equation D and then from C I get if I differentiate this whole equation in x what I get is second derivative of pressure with respect to x equals rho not second derivative of u with respect x and t.

So here if we assume that del 2 u with respect to second derivative u with respect x and t is same as cross derivative of u with respect to t and x because if we are assuming here that u and its subsequent derivatives in x and t at least (()) (27:05) to the second level they are continuous and if that is the case then this equation this relationship will hold 2, in that case I can eliminate the terms encircled in green colour from equations d and e and I can get the final equation.

So that is what I do and finally what I get is by eliminating you from d and e what I get is negative of 1 over P not gamma times second derivative of pressure with respect to time equals negative second derivative of pressure with respect to x into 1 over rho not or I get del 2 P over del t square equals P not gamma over rho not times second derivative of pressure with respect to x. So this is 1-D wave equation for pressure.

Now if I replace this term in parenthesis by a constant and this constant is a positive number because P not is positive, rho not is positive and gamma is positive. So it is a positive number, so if I replace this by a constant called C square and we will understand this meaning of C square later than the final equation which I get is this where C square is P not gamma over rho not.

The value of C if I compute through this relation is C equals 344.2 meters per second. The value of C comes to 344.2 meters per second, if I measure the speed of sound in air then what measure data tells me is that speed of sound equals 344.8 meters per second, so this again from published literature. So what appears here is that C is fully close to the measured value of speed of sound.

And what we will try to find out is the how speed of sound and C are related and we will find out that in few minutes from now that they are actually exactly the same, C is actually the speed of sound but before we get there let us try to solve this equation and develop some understanding of this equation. So what we are going to do is first we will solve this equation and after that we will try to understand the implications of this equation and how they relate to propagation of sound in one-dimension, okay.

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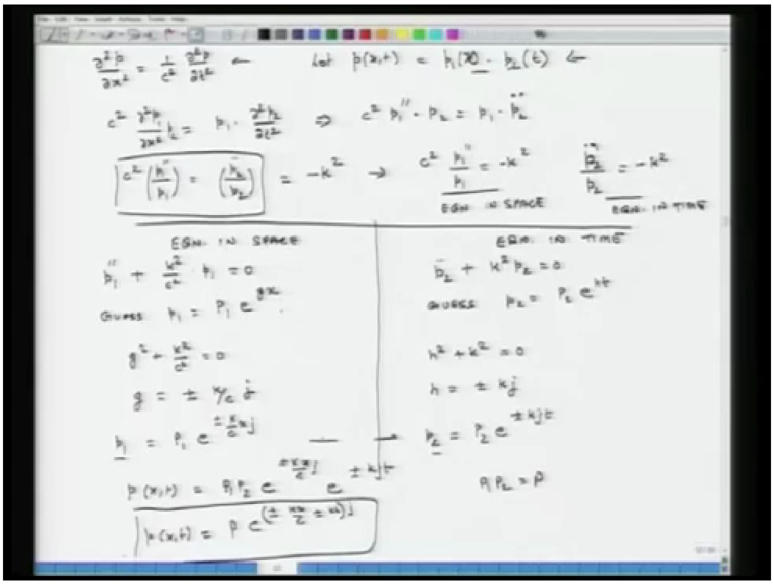

So what we will try to do is, we will first solve this equation in a general sense and then try to understand what is the meaning of this equation in context of wave propagation in 1 dimension? So you have this equation where you have on one side differentials in x and on other side differentials with respect to time only and what that immediately tells you is that if we use a variables separable approach where pressure is dependent on x and time in a variable separable form of sort of way then that kind of a format may work out in terms of solution for this equation.

So we assume that let P x, t be a product of P1 which is dependent only on time or actually let us say x times P2 which is dependent only on time and now we plug this form of pressure into this and what we get is c square del second derivative of pressure not pressure P1 with respect to x equals P1 times del 2 P2 over del t square or we can also write the same thing in shorter format as c square P1 prime prime times P2 equals P1 times P2 double dot.

So now I separate the variables I bring everything dependent on x on one side and everything dependent on time to the other side of the equation. So what I get is c square P1 double prime over P1 equals P2 double dot over P2 and this equation can be only true if the terms, this term in parenthesis and this term in parenthesis they are both constant, they do not depend on time or in x because this equation has to be valid for all values of time and all values of x.

So we assume, so based on that we say that P1 prime prime over P1 and the term on the right side is equal to a constant and this constant we call it as k square and this k can be complex. So then we have to columns, so what that gives me is c square P1 prime prime over P1 equals minus k square this is the equation in space and the other one is P2 double dot over P2 equals k square this is equation in time.

So now we solve both these 2 equations separately for P1 and P2 and once we get the relationship for P1 and P2 we multiply those 2 using this and we get the actual value of P. So we have a column for equation in space and then we have another one for equation in time. So for the equation in space I get P1 second derivative plus k square over c square times P1 equals 0 and on the time side I have P2 double dot plus k square P2 equals 0.

So now we will guess, so let us guess that P1 equals upper case P1 times e to the power of gx and here we guess P2 equals upper case P2 e to the power of H times t because it is dependent on time on that side. So we plug this in the respective ordinary differential equations and what we get is g square plus k square over c square equals 0 and here we get h square plus k square equals 0.

So here g equals plus minus k over c j and here h equals plus minus k times j and again P and g and h all these things can be complex P1, P2, h and j all of them they can be complex. So my P1 is upper case P1 e to the power of minus k over c x times j and P2 is upper case P2 e to the power of plus minus k j t. So finally I now synthesise these multiply these 2 and I get the pressure.

So pressure which is a function of x and time is P1 times P2, so I get P1 times P2 e to the power of minus k x over c times j and actually this should be plus minus and again it is plus minus kjt or if I simplify, so I can write P1 times P2 is P then this is essentially P e to the power of plus minus kx over c plus minus k t j. So this is the solution for the wave.

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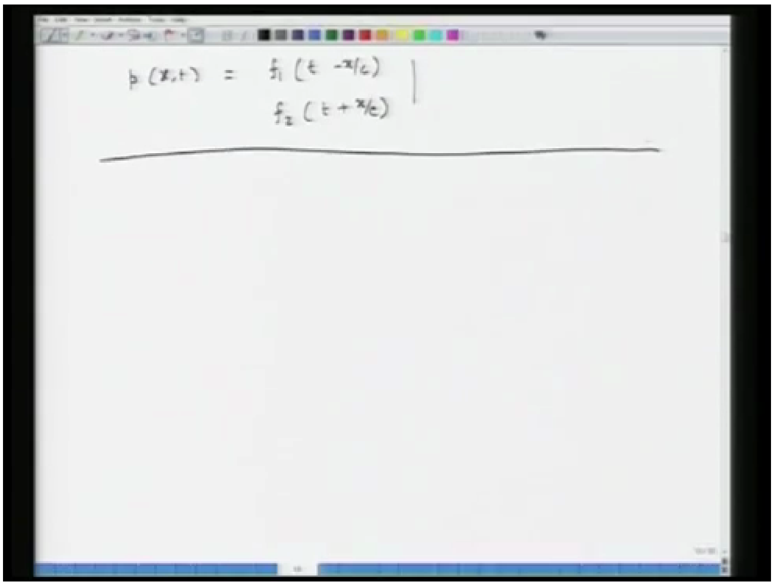

So this is one approach but in a more general sense I can also say I can also say that P x, t is nothing but a function f1 which depends on t minus x over c or it could also be a function of another function f2 which is dependent on t plus x over c and this relationship also we see holds true in this case because if I take the k out from the parenthesis essentially I get x plus c plus minus t and so on and so forth.

So we will finish today's lecture at this point and in the next lecture what we will do is we will try to understand the implication of this form of a solution and it will help us understand some very fundamental concepts as they relate to propagation of wave in one-dimension, what is a forward travelling wave? What is a backward going wave? Why is c the same as speed of sound which is the speed at which sound wave propagates and so on and so forth?

So what we have done in today's class is we have developed the wave equation using momentum equation , continuity equation and adiabatic gas law, we synthesise them and we developed a wave equation for pressuring in one-dimension and we have developed a solution for this wave equation.

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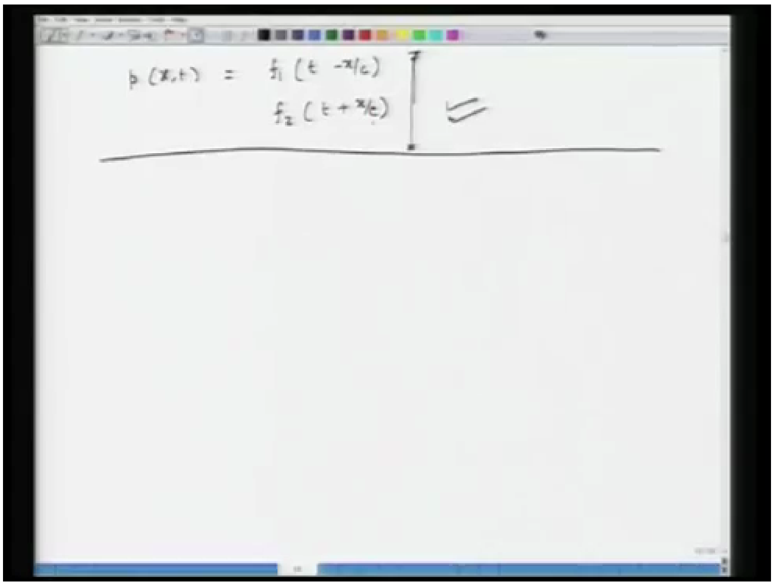

And in the next class we will elaborate on this particular format of the solution and see what it tells us in terms of characteristics of 1-D wave, so with that we conclude today's lecture, thank you very much.