Acoustics

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Lecture 1
Module 2

1-D wave equation, and it solution

Hello again, as we just starting our conversation in the last class on the wave propagation equation for sound, so that is what essentially we are going to do in today's lecture specifically what we will cover in today's lecture will be wave propagation in one-dimensional. Now in general if you have a source of sound and it travels through the medium or through the media then it could travel in a single dimension.

If it is travelling in specified geometry such as duct or tubes, so suppose you have a air-conditioning duct and there is noise being generated at one end by the compressor, these duct's are of uniform cross-section and as wave travels through this particular duct it travels in a way which can be fairly predicted by one-dimensional wave equation.

However in a more general case the propagation of wave is in all the 3 dimensions, so as I am speaking in this room there is sound the coming out from my throat and that is travelling in all the directions X, Y and Z directions. So in a more general sense we propagation is multi dimensional in all the 3 dimensions. In a large number of restricted cases however it can travel in one specific dimension and the laws of that is something we will be talking about today.

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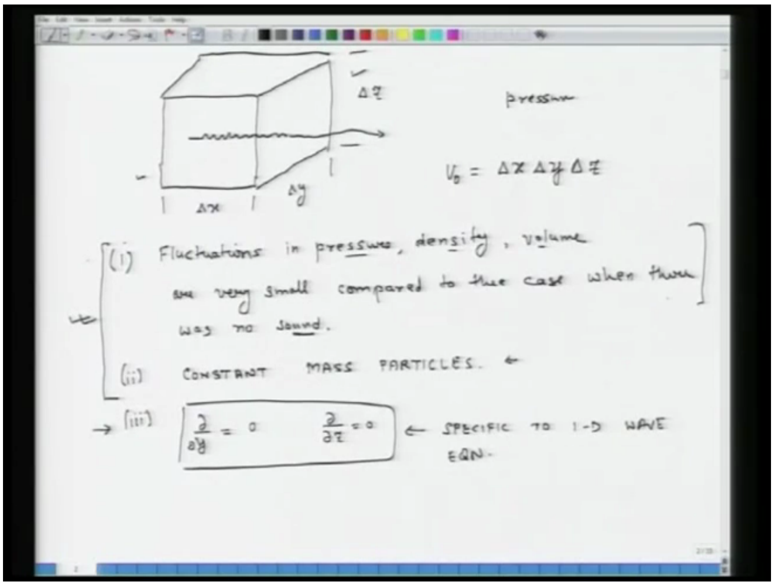

So what we will be doing is that essentially what we will be trying to understand is that if I have a small, call it a box of air, a small particle of air which has some finite dimensions Delta X, Delta Y and Delta Z then what we are interested in understanding is that, so suppose for start there is no sound in this box of air, small box of air.

And then because of some disturbance you have disturbance getting generated and that sound travels through this box and it travels out of the box. So what we are trying to understand is that how does pressure propagate through this box and then how does velocity changes? What impact does it have on volume of the box and what impact it has on the density of the air in the box? So that is what we are trying to understand and all those key understandings we will try to derive from the wave propagation equation.

So once again initially there is no sound in this box and then at a certain point we inject a pressure fluctuation and then we try to explore what are the consequences of that pressure fluctuation and how does that pressure wave call it travel through the medium. So my initial volume of this box is I will call V not which is the multiple of the 3 dimensions of the box Delta X, Delta Y and Delta Z.

Second thing is that before we start deriving all the formulae and the relations for wave propagation we will try to have some understanding of the assumptions we will be making as we walk through the derivation of 1-D or one-dimensional wave equation. So we will be essentially making 3 big assumptions and it is important to understand the basis of these assumptions which are being made.

So the first assumption we are going to make is that fluctuations in pressure, density and volume as this sound is travelling through the medium. So it is causing fluctuations in pressure, it is causing fluctuations in density and fluctuations in volume of the box. So these fluctuations are very small compared to the case when there was no sound or prior to the injection of sound fluctuation it is called sound pressure fluctuation there was some pressure, some volume and some density and the fluctuations in pressure density in volume are extremely small compared to the original or initial pressure density and volume and we will try to understand the basis of this prodigal statement a few minutes later.

The second assumption is that we have constant mass particles. In other words there is a mass contained in this box and that mass is neither coming out or neither going in into the, so that mass is neither coming out of the box and neither is the case that there is some mass entering into the box. So that is the other assumption because of the fundamental reason that when we have propagation of sound it is not because of mass transfer but rather we have energy transfer.

So as energy is getting transferred mass is not necessarily getting transferred at the same time. A good example would be that if you drop a piece of stone in a still Lake then you have all these ripples and small waves in the lake and as a consequence of these waves energy gets transferred from the point where the stone was or a pebble was dropped to the edge of the lake but that does not necessarily mean that actual mass is also that is water is getting transported from the point where that stone was dropped to the point which is there on the edge of the lake. So that is the basis for this assumption that we have constant mass particles.

And the final assumption we will make is that because this system is one-dimensional we are developing one dimensional wave equation. So variations along Y and Z directions they are exactly 0. So Del over del Y equals 0 and del over del Z equals 0. So once again first thing is fluctuations and pressure, density and volume they are extremely small compared to the case when there was no sound to begin with.

Second assumption we are going to make is that we have constant mass particles and these 2 assumptions are applicable even if we go to a 3-D wave equation but this particular the third one the third assumption where we are assuming del over del Y equals 0 and del over del z equals 0 that is specific to one dimensional wave equation. So this particular assumption is specific to 1-D wave equation while the other 2 are applicable to three-dimensional wave equations as well.

So now we have said that our first is that fluctuations and pressure density and volume they are extremely small compared to the case when there was no sound to begin with and now we will try to understand the actual bases underline this particular assumption. So earlier in one of our lectures we had shared some information about different pressure levels associated with different sound events.

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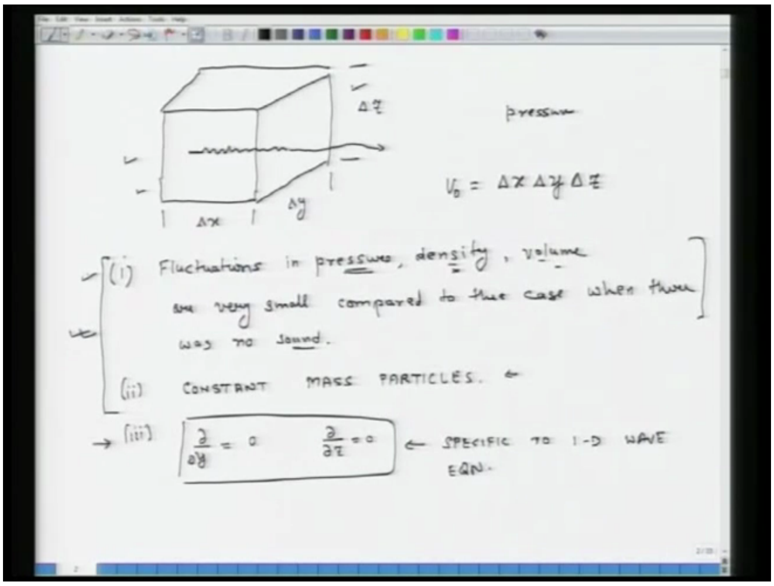

So let us have a relook at some of those numbers, so we have type of sound, okay. And then associated with it is some pressure level, pressure fluctuation, so the faintest possible sound which is just rarely audible to the human ear that we can call as sound at auditory threshold that is the faintest level of sound which the human ear can perceive and that is associated with a small extremely small ripple in the atmospheric pressure and that fluctuation and pressure is something like 2 into 10 to the power of minus 5 pascals (()) (11:57).

So if I have an extremely calm room, so there are no motors, no moving parts just a close room nothing is moving and because of some external disturbance there is some noise leaking in. So here the pressure fluctuations are something like 6.3 into 10 to the power of minus 4 pascals. If they are bunch of guys or if there is a single person and he is engaged in normal talking or speaking.

So normal talking and if I have a microphone or a listener 1 meter away from this individual, so at away 1 meter then the pressure fluctuation associated with this auditory event is 0.002 to 0.02 pascals, again extremely low pressure. Another example I could side would be if I have a car and it just passes by me and I am 10 meters away from the car.

So car at 10 meters then the associated pressure fluctuation is 0.02 to 0.20 pascals. Hearing damage, so if there is a device which is producing sound and the intensity of sound is such that the pressure near my ear is 0.356 pascals and I continue to listen to such sound for a long period of time then I could have potentially some hearing damage. Short term and then another even could be hearing damage.

Short term, so actually a more precise more precise thing would be almost instantaneous and that is associated with the pressure fluctuation of 20 pascals and then one more event jet engine at 30 meters and that is something like 632 pascals. So this is the range of different sounds we may listen. Of course you will not like to listen to this sound because that will cause significant damage to your hearing but this is the overall range of different types of sounds which we may encounter in our lives.

Now comparing this to atmospheric pressure and that is something like 1.013 into 10 to the power of 5 pascals. So if there is no sound there are no pressure fluctuations in a room then the rest pressure in the room would be approximately 10 to the power of 5 pascals and on top of that if there is some sound being generated then this pressure can go up and can go down by a number which may be in an extreme case something like 20 pascals or maybe an really extreme case may be 600 pascals.

So based on this observation that most of the sounds which we encounter with they are considerably smaller than atmospheric pressure we are able to make this assumption that the fluctuations in the pressure are extremely small compared to the condition when there is no sound in medium and if that is the case then I can also say that if these fluctuations are extremely small compared to P not which is the atmospheric pressure then I can also say that associated with this pressure fluctuations there are fluctuations in density and also fluctuations in volume which are also extremely small compared rest density and rest volume of this box of air which we are going to talk about and which we are going to analyze for wave propagation.

So that is the basis for the first assumption and it is important to understand this basis because we will be using a lot of approximations in the derivation of 1-D wave equation and also subsequently later as we start doing a lot of calculations for specific examples of 1-D wave equation, so we will be making these kinds of assumptions several times and it is important to understand the basis of these assumptions and the basis is obs data gathered from observations of reality and then plugging those, the conclusions derived from those observations into the derivation process of one dimensional wave equation.

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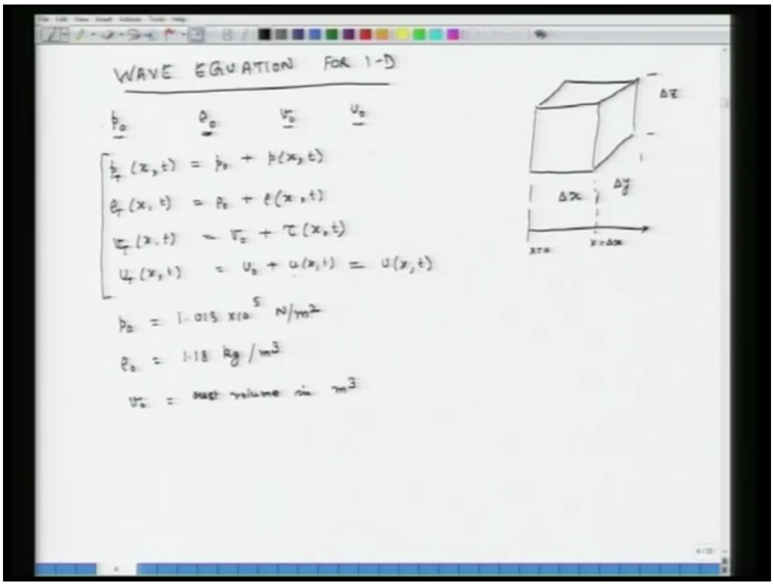

So now what we are going to do is, we will start deriving wave equation in one dimension or for one-dimension. Once again we draw a small box of fluid the dimensions here are Delta X, Delta Y and Delta Z such that the multiple of these 3 terms Delta X, Delta Y and Delta Z is the initial volume of the fluid. Now consider a case that when there is no sound then the pressure in this box is P not, the density is rho not, the volume is V not and the velocity is U not and typically we assume that U not is 0, so that there is no DC flow of air happening.

So this is pressure, this is density, this is volume and this is the initial velocity when there is no sound being propagated in this particular box. Now when sound gets injected into the box we are interested in how pressure changes in the box? How density is changing in the box? How velocity of particles is changing and how the volume is changing? So we write some specific equations. So my final pressure which is P underscore T it depends on x, so let us assume that this is my x0 here and at this point x equals Delta x. So my pressure will depend on X and it will also change with time, so it will depend on time, so that is nothing but initial pressure P not plus fluctuation in pressure times fluctuation in pressure which is a function of position x and time t.

Likewise I can write that density in the fluid will change with respect to x and t and that will be equal to initial density P not plus some fluctuation in the density which is rho and that is a function of time and x. Similarly volume of this piece of fluid is dependent on position and time and that is equal to initial volume V not plus change in volume which is Tau and then Tau is itself dependent on x and t.

And finally we write a relation for velocity UT which is dependent on position and time and that equals U not plus U, x and t and that because we assume that initially this piece of fluid is not moving, so my U not is 0, so I can call it, so this simplifies to just U being a function of x and t, okay. So let us write down the values of P not, rho not and so on and so forth. So P not is 1.013 into 10 to the power of 5 newtons per square meter, rho not equals 1.18 kilo grams per cubic meter and then of course V not is rest volume in meter cube. So these are the reference values for P not, rho not and V not.

So now we have to understand how P, rho, Tau, U can be calculated and how they are related to each other. So to do this what we have to use our 3 important relations, first relation as the relation for force equilibrium and that comes from newtons law. So essentially if I do a force equilibrium of this body of fluid I write down a relation and that relation is called the equation of momentum. So that will be one equation which will help me.

Then the second equation which I am going to derive is equation of continuity and there we will use the assumption that these particles are of constant mass and if that is the case then mass is being conserved and there will be a relation corresponding to that which is also called continuity equation and that will be the second equation.

And then the third equation we will use is the material constitutive equation which will relate changes in pressure to changes in volume. So we will have these 3 relations and these 3 relations then we will try to marry all of them together and try to figure out how changes in pressure, density, velocity and volume are happening and how they are inter playing with each other?

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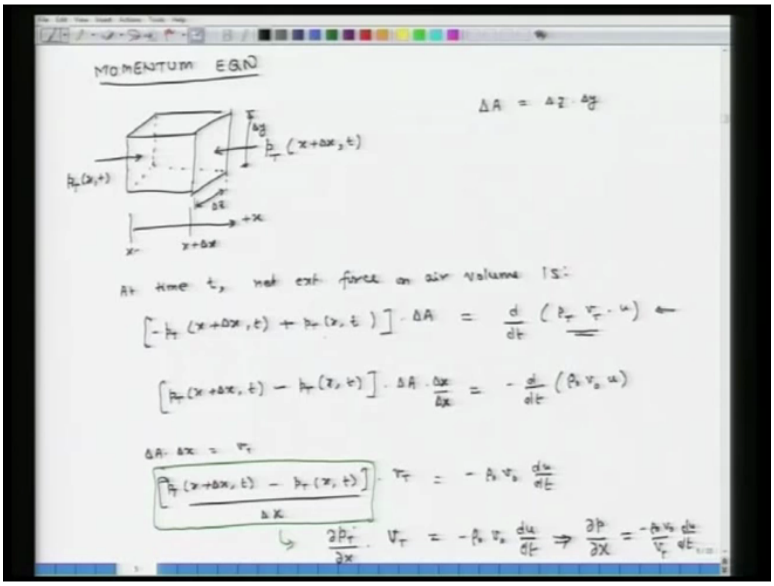

So we will start with momentum equation and this is basically nothing but newtons law. So once again I have this piece of fluid and let us say the position at this point is x, the position at this point is x plus Delta x, the pressure on this phase of the fluid is PT and that is a function of position and time. So the position at this phase is x x, so PT is dependent on x and t and here I have pressure and here the position is x plus Delta x and time is still t.

So now I write down the equation of momentum or do a force equilibrium on this piece of body. So at time t, net external force on air volume is Pt x plus delta x times t and so this is my positive x coordinate. So this is negative because it acting in the negative direction because it is pressure, so this is negative of PT and then on the other phase I have a positive number, so I put plus PT x delta x and since this is pressure I have to multiply it by area.

So the area is del A delta A which is nothing but delta x times Delta y where this dimension is it is delta y times Delta z and this dimension is delta z. So the area is delta z times Delta y, so I have to multiply this by area, so this is the overall net force which this body of fluid is experiencing and that newtons law tells us is nothing but rate of change of momentum.

So the momentum for this piece of material is density rho T times the volume VT, so that is my mass times velocity and then I have to take a differential of it with respect to time. So this is the newtons law as it applies to this piece of fluid, okay. So now we know that we made an assumption that the changes in volume and changes in density of this piece of fluid extremely small compared to the rest values that is rho not and V not.

So what I can do is I can replace this term by rho not V not and I will also put take this negative sign from here from the left side and move it to the right side. So from using this manipulation what I get is PT x plus delta x times t minus PT x times delta A which is the area on which this pressure is getting exerted equals minus d over dt rho not V not U . I can also multiply my my left hand side by Delta x and also divided by Delta x without changing the equation.

So I am just multiplying and dividing it by the same equation by the same number Delta x and realising that Delta x times delta A equals VT which is the volume of the fluid I get PT x plus delta x and time minus PT which is a function of x and time times VT and then I have to divide this whole thing by delta x equals minus and I will take rho not and V not out because they are constants. So minus rho not V not du over dt, okay.

Now this term which is there in the green box if I take its limit as x goes to 0 than this is essentially partial derivative of pressure which is PT with respect to x. So if I do that operation what I get is del Pt over del x times VT equals minus rho not V not du over dt minus rho not V not du over dt I can simplify this relation even further by plugging in the expansion for PT which we have developed here that PT is basically P not plus P and we know that the differential of P not with respect to x is exactly 0 because P not is constant. So I can simplify this relation further by saying that del PT over del x is same as del P over del x and then I take VT to the right side and then here I have still du over dt, okay.

So moving further, now the next simplification I am going to is on this green term and here I realized that Vnot over VT is approximately equal to 1 because as we saw earlier changes in volume are extremely small compared to the original volume because the fluctuations in pressure which we are talking about are extremely small compared to the original atmospheric pressure.

So this ratio of Vnot over VT is fairly close to one, so that is what we will do but I also wanted to share some details as to how small these changes are. So let us look at one case. So we know that, so if I get exposed to fluctuations and pressure which are of this order of magnitude about 20 newtons per square meter then there is a danger that I can get instant damage to my hearing abilities.

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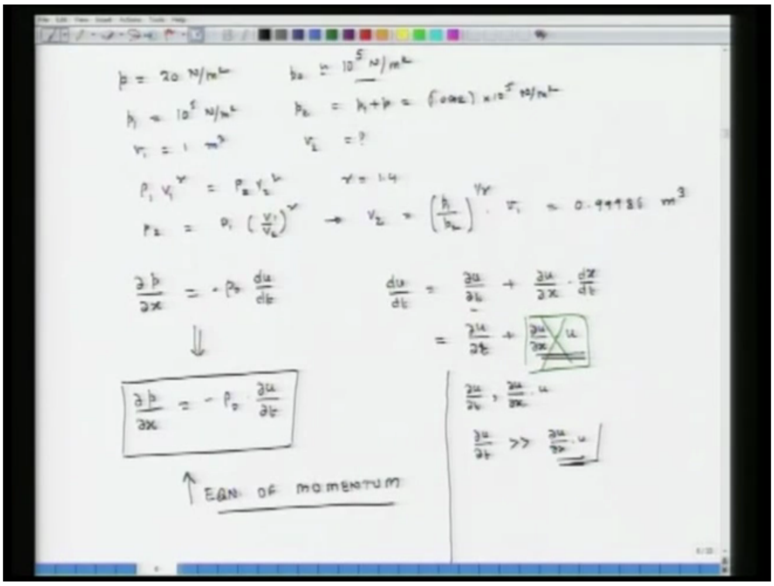

So let us see what kind of changes these kinds of pressures are associated with in terms of volume. So consider P equals 20 newtons per square meter and we know that P not is approximately equal to 10 to the power of 5 newtons per square meter 1.013 times 10 to the power of 5 which is approximately equal to this number.

So and we will see later that as sound travels in air it depends its propagation is governed by adiabatic laws for the gas and we will understand that a little later in the lecture. So let us say my initial condition for the air is P1 and P1 equals 10 to the power of 5 newtons per meter square P2 is P1 plus P and that is 1.002 times 10 to the power of 5 newtons per meter square and let us assume that my initial volume V1 is 1 meter cube, 1 cubic meter and I want to find what is V2?

So we know that the gas equation which will be used in this case will be adiabatic law, so P1, V1 gamma equals P2, V2 gamma and we assume that gamma is approximately equal to 1.4, so P2 equals P1 times V1 over V2 to the power of gamma. So this is how I can find P2 but I already know P2. So instead what I have to do is I have to find V2, so I can find V2 through this relation.

So V2 equals P1 over P2 the power of 1 over gamma times V1 and if I plug the values of P1 which is 10 to the power of 5 newtons per meters cube per meter square and P2 which is 1.0002 here and do this mathematical operation what I get is and V1 is one what I get is V2 equals 0.99986 cubic meters.

So essentially what it is telling us is that it is validating our claim that V not and VT are fairly close to one. So I can just eliminate this V not and VT and replace them by a constant which is 1. So using this understanding I rewrite this relation this particular relation, so what I get is del P over del x equals rho not du over dt and there is a negative sign here, so I maintain the negative sign.

Now I know that this is a this derivative du over dt can be expanded as partial derivative of u with respect to x that is position plus partial derivative of, so du over dt is partial derivative of u with respect to time plus partial derivative of u with respect to x times dx over dt.

So I can do this because I know that u is dependent on time and u is also dependent on position. So it is an addition of partial derivative with respect to time and then this term du over dx times dx over dt and I can rewrite this as del u over delt plus del u over del x times u because dx over dt is basically rate of change of position which is nothing but velocity.

Now once again we will make some approximation and we will drop this term using that thought process. So here we know that partial derivative of u with respect to time and partial derivative of u with respect to x. Partial derivative they change in a similar way, suppose you have a wave, so then after every x number of seconds after every specific number of second if there is a plane wave progressing in a medium then it will have its speak and it will have it has valley.

So the overall magnitude of that the variation in time and variation in x will be similar. In order in terms of order of magnitude but now if I am multiplying this by u then del u over del t is extremely large compact del u over del x times u because of this understanding. So as a consequence of this understanding I will be dropping this term out completely this non-linear term.

So this equation becomes even more simple and essentially what we get is partial derivative of pressure with respect to position is negative of rest density times partial derivative of velocity with respect to time. So this is my equation of momentum so this is the first equation.