# Thermal Engineering: Basic and Applied

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## Lecture - 31
## Critical Pressure Ratio and its Physical Significance

I welcome you all to the session of thermal engineering basic and applied. And today we shall continue our discussion on the critical pressure ratio and finally we shall try to discuss about the physical significance of critical pressure ratio essentially in the context of nozzle flow and the design of flow nozzles. So, if we try to recall in the last class, we could establish the expression of critical pressure ratio. And though we did not write the definition of critical pressure ratio but I have tried to mention that the critical pressure ratio is the ratio of exit pressure to the inlet pressure for which mass flow rate through the nozzle per unit area is maximum. So, let us try to write the expression of mass flow rate through the nozzle per unit area and then the expression of the critical pressure ratio.

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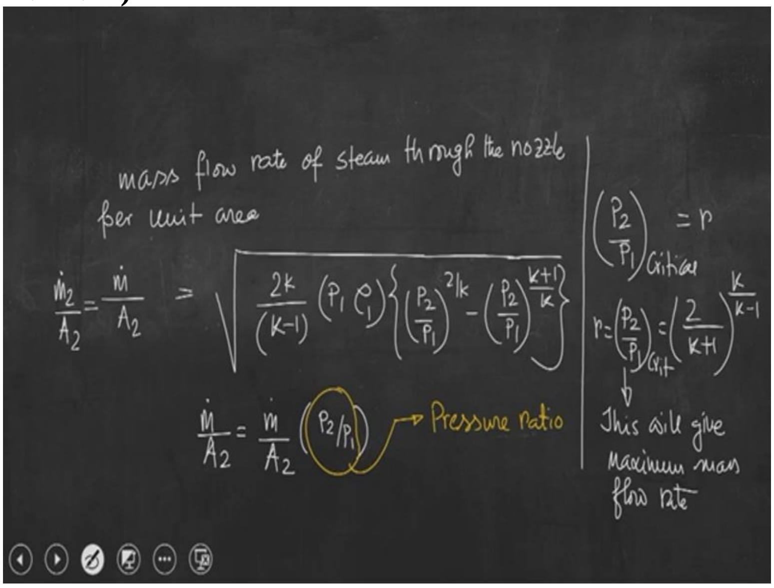


$$\text{So the mass flow rate of steam through the nozzle per unit area} = \frac{\dot{m}_2}{A_2} = \frac{\dot{m}}{A_2}$$

So why I am writing $\dot{m}$? Because from continuity we can write $\dot{m}_1 = \dot{m}_2 = \dot{m}$. So

$$\frac{\dot{m}_2}{A_2} = \frac{\dot{m}}{A_2} = \sqrt{\frac{2k}{k-1}(p_1\rho_1)\left\{\left(\frac{p_2}{p_1}\right)^{\frac{2}{k}} - \left(\frac{p_2}{p_1}\right)^{\frac{k+1}{k}}\right\}}$$

So this is the expression of the mass flow rate of steam through the nozzle per unit area. And we also have discussed that mass flow rate of steam per unit area or even multiplied with $A_2$ as $A_2$ is fixed, depending on the flow nozzle exit area, we can see that this quantity is function of $p_1, \rho_1$ & $k$ and also the ratio of $\frac{p_2}{p_1}$. So, for the constant $p_1, \rho_1$ & $k$ the quantity $\frac{\dot{m}}{A_2}$ is function of $\frac{p_2}{p_1}$.

If we consider that the pressure of stream which is entering to the nozzle is constant, we can assume the density at that particular cross section is constant, I mean at that cross section if the density variation is not there and *k* is also constant, then we can see this quantity is the function of $\frac{p_2}{p_1}$. In fact that is what we have discussed in the last class.

So this quantity $\frac{p_2}{p_1}$ is the pressure ratio that is the ratio of exit pressure to the inlet pressure when there is a flow of steam through the nozzle. We also could establish the expression of the critical value of this pressure ratio which will give the maximum mass flow rate per unit area through the nozzle and by doing a simple calculation we also could write the expression of critical value $\frac{p_2}{p_1}$.

$$\left(\frac{p_2}{p_1}\right)_{Cr} = r = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}$$

So this is the critical value of pressure ratio. Critical means this will give maximum mass flow rate through the nozzle. Now we can see that the critical pressure ratio depends on the value of *k* that is the index of expansion for the isentropic process. So when steam is passing through the nozzle, we assume that the flow can be modeled by an isentropic process. And following that process, we had tried to mathematically model the process. So that means we can write that when there is a flow of a compressible fluid through a convergent channel, the ratio of exit pressure to the inlet pressure of this nozzle or convergent channel, for which the mass flow rate through the channel or nozzle is maximum is known as the critical pressure and the corresponding exit pressure is known as the critical exit pressure.

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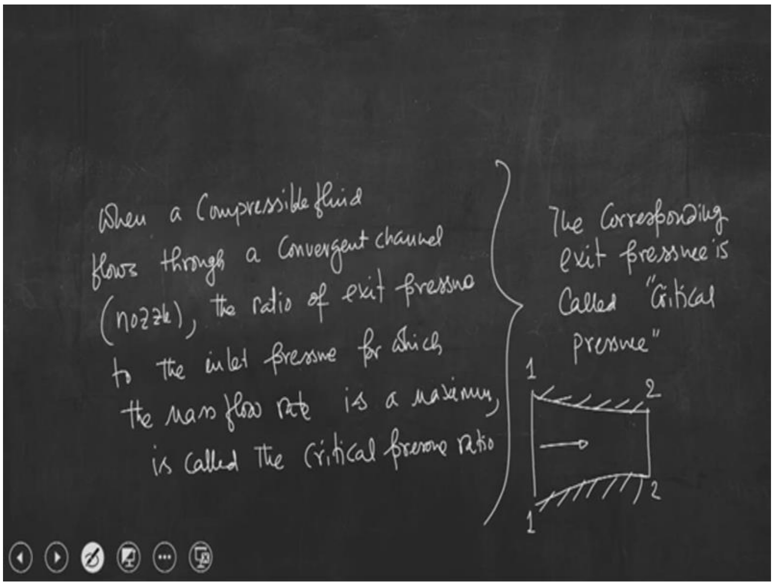


So when a compressible fluid flows through a convergent channel (nozzle), the ratio of exit pressure to the inlet pressure for which the mass flow rate is a maximum, is called the critical pressure ratio. And from this we also can write that the corresponding exit pressure is called critical pressure. So there is a convergent channel and a compressible flow and we are assuming that there is no heat loss. So when there is a flow through this convergent duct, we are assuming that the system is not having heat interaction with the surroundings and if the outlet is 2-2 and this is the inlet section 1-1 and this is the direction of flow, then the critical pressure ratio of $p_2/p_1$ for which the mass flow rate per unit area through the channel should be maximum and the corresponding pressure at the exit pressure is the critical pressure. So in this case 2 is the exit section, 1 is the inlet section. So the, corresponding pressure at the exit that is P2 is the critical pressure.

Now I would like to discuss a few issues in this context. So we have discussed about the critical pressure ratio that is nothing but the ratio of exit pressure to the inlet pressure for which the mass flow rate is maximum. So that means we are trying to have the maximum mass flow rate and we can see that mass flow rate will be maximum, if the velocity is maximum.

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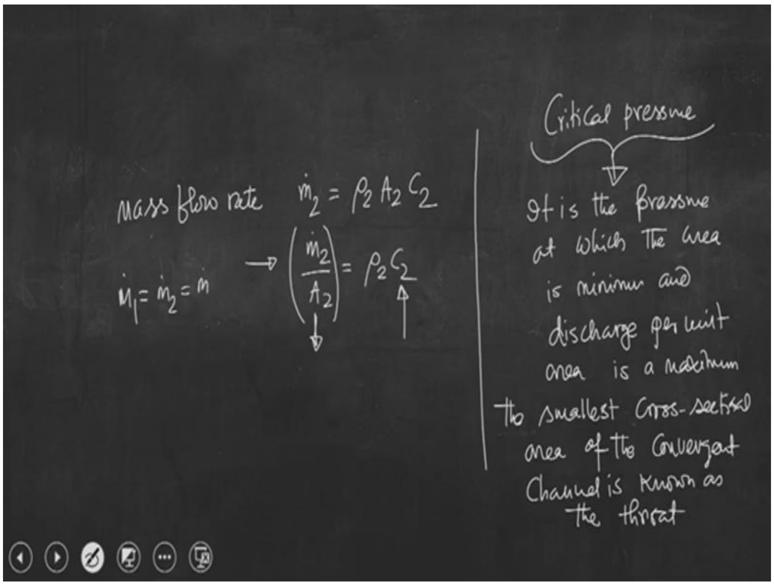


So if we try to go to the next slide so mass flow rate at section 2 is $\dot{m}_2$

$$\dot{m}_2 = \rho_2 A_2 c_2$$

$$\Rightarrow \frac{\dot{m}_2}{A_2} = \rho_2 c_2$$

Density will change with a drop in pressure and this is the modelled by the isentropic process equation. So we are in process of maximizing this quantity by having a suitable design of the nozzle. Essentially it indicates that we need to increase the velocity of steam at the exit of the nozzle. So when you are talking about critical pressure ratio for which mass flow rate is maximum that means the velocity of steam at the exit of the nozzle is becoming maximum. So velocity of steam at the exit of the nozzle becoming maximum and it indicates that the cross sectional area should be minimum.

$$\dot{m}_1 = \dot{m}_2 = \dot{m}$$

So this is fixed, density will definitely vary, with a change in pressure but if you would like to maximize this quantity, then $c_2$ has to increase. Since this quantity $\dot{m}_2$ is fixed, now if we need to increase $c_2$ definitely $A_2$ should reduce. So $\dot{m}_2$ is fixed that we can see from this quantity, because we cannot compromise with the mass flow rate. Our objective is to get higher $c_2$ and if we need to have it then $A_2$ will drop because our objective is to get maximum mass flow rate. So, basically when the nozzle is operated at this particular condition that is at this critical pressure condition and when the pressure at the exit of the nozzle is critical pressure cannot we say something else?

So that means we can say that basically the pressure at which the area is minimum and discharge per unit area is a maximum or the mass flow rate per unit area is maximum is the critical pressure. And the smallest cross section cross sectional area of the convergent channel is known as throat. So basically just we are trying to look at it from different perspective that is from this simple mathematical expression of the critical pressure. We have discussed that the corresponding pressure is critical pressure. So, our nozzle is said to be operating at the critical pressure condition when the mass flow rate is maximum. And area here is minimum because if we need to have mass flow rate is maximum, then we have seen that the velocity should be maximum. To obtain the higher velocity, the area should be reduced. So that means this is the section where the pressure is critical pressure and area is minimum and that is known as the throat of the convergent channel or nozzle. So this is very important. That means you should keep in mind that critical pressure is the pressure at which area is minimum and that minimum area of that particular convergent channel is known as the throat.

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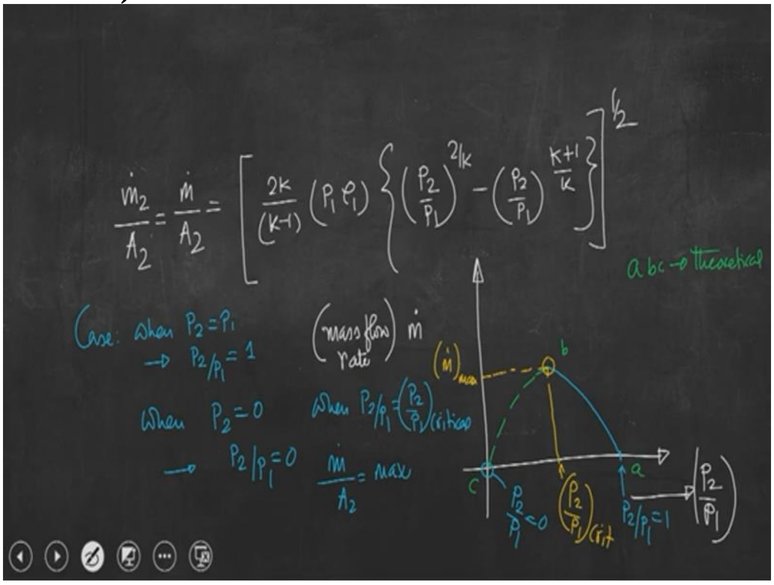


Now let us quickly write the expression of mass flow rate per unit area.

$$\frac{\dot{m}_2}{A_2} = \frac{\dot{m}}{A_2} = \left[\frac{2k}{k-1}(p_1\rho_1)\left\{\left(\frac{p_2}{p_1}\right)^{\frac{2}{k}} - \left(\frac{p_2}{p_1}\right)^{\frac{k+1}{k}}\right\}\right]^{\frac{1}{2}}$$

So this is the mathematical expression of the mass flow rate per unit area. Now we will try to plot the mass flow rate $\dot{m}$ vs. $p_2/p_1$. Why? If we go back, we have written that $\dot{m}/A_2$ is function of $p_2/p_1$. So let us try to have the graphical representation of $\dot{m}/A_2$ which is the function of $p_2/p_1$. Here I will be discussing three different cases.

Case-1 when $p_2 = p_1$ that means $\frac{p_2}{p_1} = 1$

So if we scale the axis from 0 to 1 then we can plot $\frac{p_2}{p_1} = 1$.

So when $\frac{p_2}{p_1} = 1$, we can see from this expression that mass flow rate is 0. So it will start from 0.

Case-2 when $p_2 = 0$ this means $\frac{p_2}{p_1} = 0$

You have studied in fluid mechanics that there will not be any flow, until and unless we are having the driving pressure difference. So maybe even if we look at this particular cross section of convergent channel then at exit it is $p_2$ and at entry section this is $p_1$. The driving pressure difference is $\Delta p = p_2 - p_1$or $\Delta p = p_1 - p_2$.

So if the pressure at exit equal to 0, then there will not be any flow. If there is no flow, we cannot think even to have mass flow rate, so it will be 0. So the point when $p_2/p_1$= 0 is plotted in the graph. That means here we can see that mass flow rate is equal to 0 and somewhere at the critical mass flow rate $p_2/p_1$it will be maximum.

So, when $\frac{p_2}{p_1}$= $\left(\frac{p_2}{p_1}\right)_{cr}$, $\frac{\dot{m}}{A_2}$ is maximum, so that means somewhere we will be getting maximum value. So try to look at why the curve will trend towards left. So when $p_2 = 0$ there is no flow. So when $p_2 = p_1$there is also no flow. When $p_2 = 0$, we can see from the mathematical expression that $\frac{\dot{m}}{A_2} = 0$. So if we try to go here, then $p_2$ should not be equal to $p_1$. If $p_2$ is less than equal to $p_1$ then there will be flow.

So, when $p_2 = 0$ then ideally we are supposed to get maximum flow rate; because if $p_2$ reduces then we will be having driving pressure difference. So when $p_2$ is becoming 0, we are supposed to get maximum mass flow rate. But if we look at the mathematical expression, when $p_2 = 0$ mass flow rate will be equal to 0. So let us now plot it then I will explain the physical issues later.

So let us give the name to the points say a, b, c. So this is the theoretical curve a b c that we have drawn. So now try to understand that reaching at the maximum value, if we further reduce $p_2$ and if we make $p_2 = 0$, though we are supposed to get maximum flow rate, but following this mathematical expression, we can see that the mass flow rate should be equal to 0.

But this is not the case that we will explain. So before going to this particular aspect though it is mathematically 0 that we can deny but you all have understood that we are bringing the ratio of $p_2/p_1$ from 1 to close to 0. So if we reduce $p_2$ then we are having more pressure difference, so we are supposed to get more flow rate. But that is not the case that we can see from this curve which is verified by this mathematical expression as well. So before going to discuss that particular part, let us now look into the physical significance of the critical pressure ratio.

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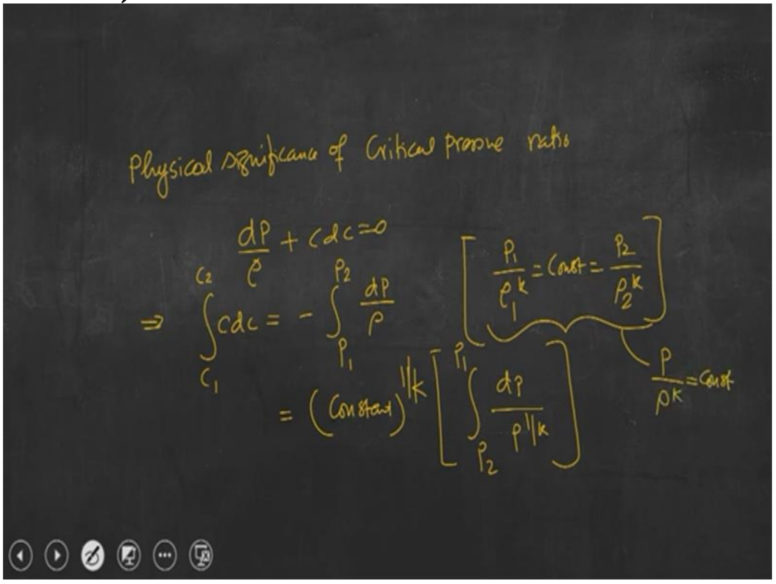


Physical significance of critical pressure ratio

So if we start our equation from that particular equation that is

$$\frac{dp}{\rho} + cdc = 0$$

This equation you are getting from $cdc + dh = 0$ and we are using the value of $dh$ rather we are substituting the value of $dh$ by $dp/\rho$ from the property relation $Tds = dh - vdp$. For an isentropic process $dh = 0$, so we can write $dh = vdp$ that is $dp/\rho$. Now in this equation we can write

$$\int_{c_1}^{c_2} cdc = -\int_{p_1}^{p_2} \frac{dp}{\rho}$$

To have the closure, we need to invoke the property relation that is

$$\frac{p_1}{\rho_1^k} = const = \frac{p_2}{\rho_2^k}$$

So now we can write

$$\int_{c_1}^{c_2} cdc = (const)^{\frac{1}{k}} \left[ \int_{p_2}^{p_1} \frac{dp}{p^{\frac{1}{k}}} \right]$$

So that is straight forward. So from here we can write that

$$\frac{p}{\rho^k} = const$$

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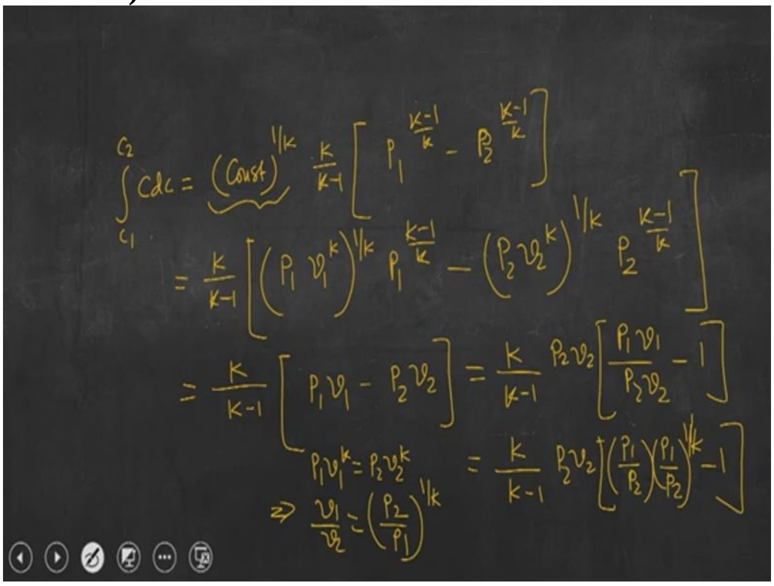

$$\int_{c_1}^{c_2} cdc = (const)^{\frac{1}{k}} \frac{k}{k-1}\left[p_1^{\frac{k-1}{k}} - p_2^{\frac{k-1}{k}}\right]$$

$$= \frac{k}{k-1}\left[\left(p_1 v_1^k\right)^{\frac{1}{k}}\ p_1^{\frac{k-1}{k}} - \left(p_2 v_2^k\right)^{\frac{1}{k}}\ p_2^{\frac{k-1}{k}}\right]$$

$$= \frac{k}{k-1}[p_1 v_1 - p_2 v_2] = \frac{k}{k-1}\ p_2 v_2 \left[\frac{p_1 v_1}{p_2 v_2} - 1\right]$$

Here $v_1$ and $v_2$ are the specific volume at section 1 and section 2 respectively of this steam that is flowing. So since

$$p_1 v_1^k = p_2 v_2^k$$

$$\Rightarrow \frac{v_1}{v_2} = \left(\frac{p_2}{p_1}\right)^{1/k}$$

So now we can write

$$\int_{c_1}^{c_2} cdc = \frac{k}{k-1}\ p_2 v_2 \left[\left(\frac{p_1}{p_2}\right)\left(\frac{p_2}{p_1}\right)^{\frac{1}{k}} - 1\right]$$

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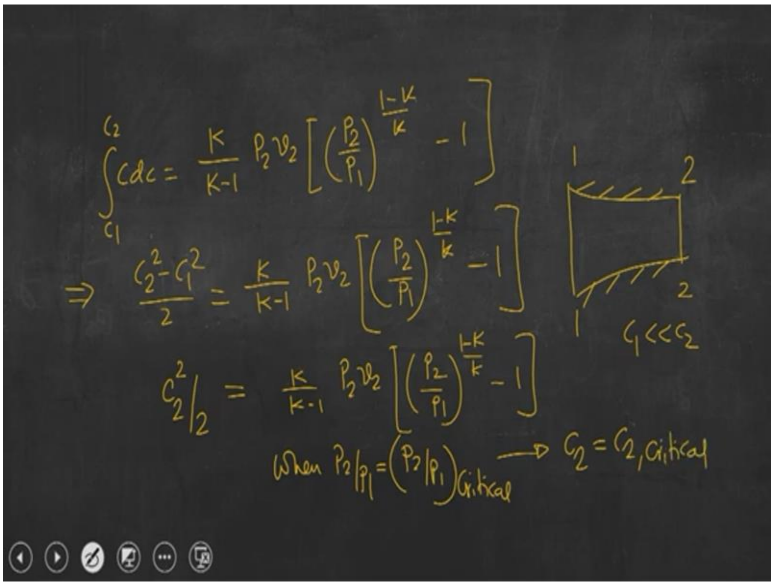


$$\int_{c_1}^{c_2} cdc = \frac{k}{k-1}\, p_2 v_2 \left[ \left(\frac{p_2}{p_1}\right)^{\frac{1-k}{k}} - 1 \right]$$

$$\Rightarrow \frac{c_1^2 - c_2^2}{2} = \frac{k}{k-1}\, p_2 v_2 \left[ \left(\frac{p_2}{p_1}\right)^{\frac{1-k}{k}} - 1 \right]$$

So we are trying to have the expression of $c_2$ in terms of the critical pressure ratio. Till now we have discussed about the critical pressure ratio and mass flow rate. So the critical pressure ratio is the pressure ratio at which area is minimum and mass flow rate per unit area is maximum and the corresponding pressure at the exit is the critical pressure. So now we would like to know the expression of critical velocity in terms of the critical pressure ratio. So we can have similar argument; see if there is a convergent channel with the cross section area 11 and 2 2 and we can see that $c_1 \ll c_2$, because area at 11 is much higher than the area 2 2. So definitely $c_2$ would be maximum and that is the objective of having this convergent nozzle. So we can write

$$\frac{c_2^2}{2} = \frac{k}{k-1}\, p_2 v_2 \left[ \left(\frac{p_2}{p_1}\right)^{\frac{1-k}{k}} - 1 \right]$$

$$\text{When } \left(\frac{p_2}{p_1}\right) = \left(\frac{p_2}{p_1}\right)_{cr} \rightarrow c_2 = c_{2_{cr}}$$

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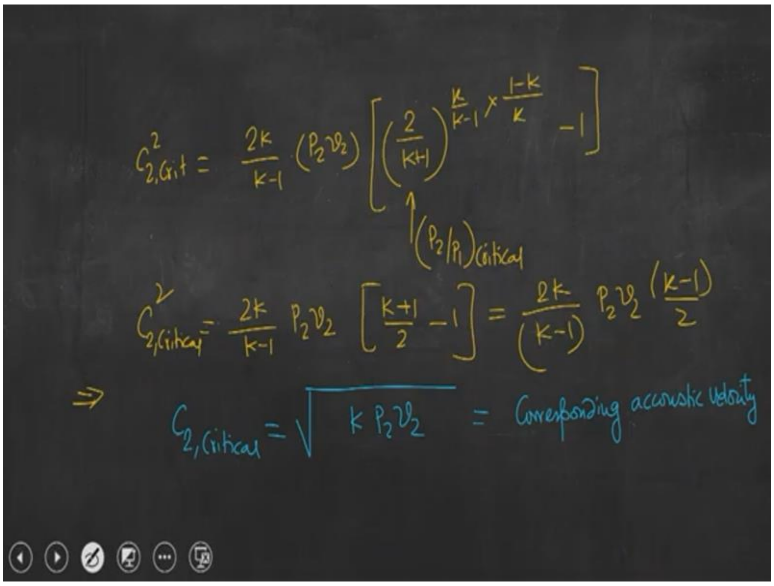


Now

$$c_{2_{cr}}^{\,2} = \frac{2k}{k-1}\,(p_2v_2)\left[\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}\times\frac{1-k}{k}} - 1\right]$$

$$\text{As}\left(\frac{p_2}{p_1}\right)_{cr} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}$$

$$c_{2_{cr}}^{\,2} = \frac{2k}{k-1}\,(p_2v_2)\left[\frac{k+1}{2} - 1\right] = \frac{2k}{(k-1)}\,(p_2v_2)\,\frac{(k-1)}{2}$$

$$\Rightarrow c_{2_{cr}}^{\,2} = kp_2v_2$$

$$\Rightarrow c_{2_{cr}} = \sqrt{kp_2v_2} = \text{corresponding acoustic velocity}$$

So this is the sonic velocity at the exit. So basically we can write that it is equal to corresponding acoustic velocity. So from this particular mathematical exercise, we can discuss that when the ratio of exit pressure to the inlet pressure is the critical pressure ratio, the corresponding velocity at the exit of the nozzle is the critical velocity and it is the sonic velocity. So, now let me tell you that when exit pressure is the corresponding critical pressure of the nozzle, exit velocity also is the critical velocity and that is nothing but the acoustic velocity at the same temperature and pressure of the flow. So this is very important.

So now if we go back to the particular expression, we have mentioned that when the pressure ratio has reached the critical pressure ratio flow rate per unit area is maximum and the corresponding exit pressure is the critical pressure. We have agreed upon this. Now from this mathematical exercise we have established that when the pressure is the exit pressure corresponding exit velocity is the critical velocity which is nothing but the sonic velocity of the flow at the same temperature and pressure of the medium. So now question is you have seen if we go to this particular slide so when the velocity at the exit is the sonic velocity, the velocity is $c_{2_{cr}}$ = sonic velocity of the media. So if you reduce further pressure that pressure is to be sensed by the inlet section of the nozzle. When the pressure ratio is critical pressure ratio, exit pressure is the critical pressure and from the last expression we have seen that the critical velocity is the sonic velocity. so if you try to reduce the pressure further at the exit of the nozzle that information is to be sensed by the inlet section of the nozzle to increase the mass flow rate.

Whenever there is a reduction in pressure that means the nozzle is operating at the critical pressure ratio, if we try to have further reduction in pressure that pressure wave should be moved to the inlet section. That means that information should be sensed by the inlet section of the nozzle which indicates a further reduction in pressure. That pressure wave should move to the inlet section because inlet section should know that there is a further reduction in pressure at the exit section of the nozzle.

When a pressure wave is moving through an elastic media that moves with a sonic velocity. Now if the flow velocity itself is the sonic velocity, then with further reduction in pressure that wave will not be able to reach at the inlet section of the nozzle and at that case whatever flow rate we have seen, it will remain constant. So this is $\dot{m}_{max}$. So this point b is basically $\left(\frac{p_2}{p_1}\right)_{cr}$. So nozzle has reached at the critical condition and a further reduction in pressure at the exit needs to be sensed by the inlet section to increase the flow rate. But that information has to be carried through the medium itself. Since the nozzle exit velocity is the sonic velocity that we have established few minutes back, hence that further reduction in pressure that information will not be reached at the inlet section of the nozzle to increase the flow rate. And as a result the mass flow rate will remain constant and this is the experimentally measured data. So this curve a b d is experimental curve. So try to understand that if you reduce pressure at the exit beyond the critical pressure, mathematically you can see that the mass flow rate should be equal to 0. But the physics is that for further reduction in pressure that information should be sensed by the inlet section and to get that information, a pressure wave should move through the elastic media with a sonic velocity. But already the velocity is sonic velocity. So that information will not be sensed by the inlet section. As a result of which the mass flow rate remains constant and that is the experimentally observed phenomenon. So I would like to write 2-3 important points now.

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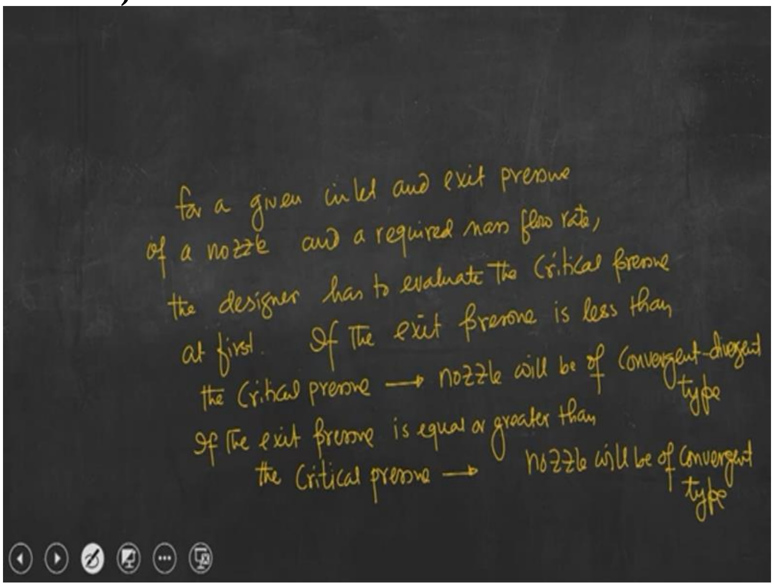


So for a given inlet and exit pressure of a nozzle and a required mass flow rate, the nozzle designer has to evaluate the critical pressure at first. If the exit pressure is less than the critical pressure, then the nozzle will be of convergent divergent type. If the exit pressure is equal or greater than the critical pressure, then the nozzle will be of convergent type. So this is very important. So this is the conclusion of this analysis that whenever someone is designing the nozzle for a given inlet and exit pressure and required mass flow rate, the designer should first calculate the critical pressure ratio. If the exit pressure is less than the critical pressure ratio, then the type of the nozzle should be the convergent divergent type. If the exit pressure is equal or greater than the critical pressure ratio, then it should be the convergent type.

One important thing is that when the pressure is the critical pressure ratio, mass flow rate through the nozzle is maximum and that we have established. And at that condition our nozzle is said to be choked. So when the nozzle is operating at the critical pressure ratio, exit pressure is the critical pressure, velocity of steam at the exit is the sonic velocity; at that condition mass flow rate through the nozzle is maximum and the nozzle is said to be choked. So, this is very important.

So this is very important question that when nozzle is said to be choked. When the exit pressure of the nozzle is the critical pressure, the maximum mass flow rate through the nozzle is obtained or achieved then the nozzle is said to be choked.

So if we try to summarize, then we have discussed about the physical significance of the critical pressure ratio. From there, you have seen that although if we further reduce the pressure below the critical pressure at the exit of the nozzle, mathematically mass flow rate is becoming 0. But in experimental observation, it is seen that the mass flow rate remains constant and the underlying physics that we have tried to explain is that the sonic velocity is reached when the nozzle is operating at the critical pressure ratio. So with this I stop here today and in the next class we shall try to solve one numerical problem on the flow analysis of the nozzle. Thank you.