Mechanics of Fiber Reinforced Polymer Composite Structures
Prof. Debabrata Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology-Guwahati

Module-04
Macromechanics of Lamina-II
Lecture-08
Strength Failure Criteria-Part II

Hello and welcome to the second lecture of this fourth module.

(Refer Slide Time: 00:43)

FOCUS of Module 4

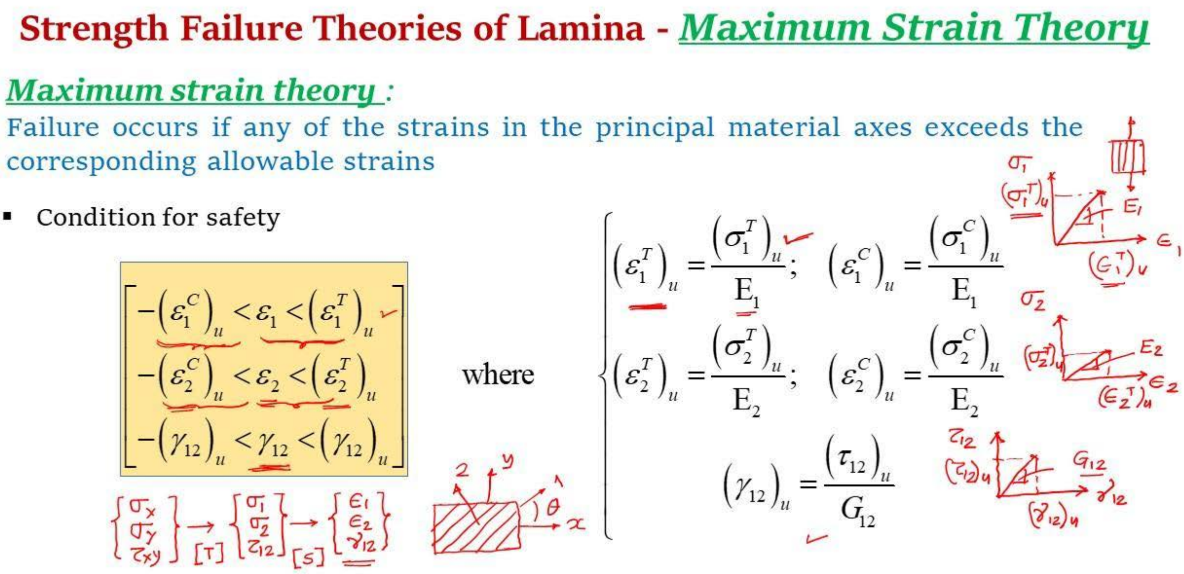
Macromechanics of Lamina

- Strength Failure Theories/Criteria of Lamina
  - Independent Theories/Criteria
    - Maximum Stress Criteria
    - Maximum Strain Criteria
  - Interactive Theories/Criteria
    - Tsai-Hill Criterion
    - Hoffman Criterion
    - Tsai-Wu Criterion
  - Comparisons of different Theories
- Hygrothermal Stresses in Lamina

In this module, we have been discussing the strength failure theories of lamina and in the last lecture, we discussed that the philosophy of strength failure theories in lamina is same as those for isotropic materials. But the fact that in orthotropic lamina the strength and stiffnesses are direction dependent are actually taken into account in developing the failure theories in orthotropic lamina. A major difference in applying failure theories in orthotropic lamina compared to the failure theories in isotropic materials is that, in orthotropic lamina, instead of finding out the principal stresses and maximum shear stress, the stresses with reference to the principal material directions are determined since the strengths and stiffnesses of orthotropic lamina are specified with reference to the principal material axis.

Therefore, for any state of stress the stresses with reference to the principal material axis are determined and those are compared with the corresponding strengths to assess the safety or failure of a lamina. There are two types of failure criteria viz. independent criteria and interactive criteria. In the last lecture the maximum stress criteria was discussed. In today's lecture we shall discuss the maximum strain criteria and then few of the interactive criteria for the strength failure theories of a lamina.

(Refer Slide Time: 02:43)



Maximum Strain Theory

This is also an independent or non-interactive theory. Analogous to the maximum stress criteria, in maximum strain criteria, the material axes strains are determined and are compared with the corresponding ultimate strains and the failure is said to have occurred if any of the strains in the principal material axis exceeds the corresponding allowable strains. Mathematically,

$$\begin{bmatrix} -\left(\varepsilon_1^C\right)_u < \varepsilon_1 < \left(\varepsilon_1^T\right)_u \\ -\left(\varepsilon_1^C\right)_u < \varepsilon_1 < \left(\varepsilon_1^T\right)_u \\ -\left(\gamma_{12}\right)_u < \gamma_{12} < \left(\gamma_{12}\right)_u \end{bmatrix} \textit{where} \left\langle \begin{matrix} \left(\varepsilon_1^T\right)_u = \dfrac{\left(\sigma_1^T\right)_u}{\mathrm{E}_1}; & \left(\varepsilon_1^C\right)_u = \dfrac{\left(\sigma_1^C\right)_u}{\mathrm{E}_1} \\ \left(\varepsilon_2^T\right)_u = \dfrac{\left(\sigma_2^T\right)_u}{\mathrm{E}_2}; & \left(\varepsilon_2^C\right)_u = \dfrac{\left(\sigma_2^C\right)_u}{\mathrm{E}_2} \\ \multicolumn{2}{c}{\left(\gamma_{12}\right)_u = \dfrac{\left(\tau_{12}\right)_u}{G_{12}}} \end{matrix} \right\rangle$$

Given the state of stress with respect to the x-y axes, σ_x, σ_y and τ_{xy}, the state of stress with respect to the material axes (1-2) σ_1, σ_2 and τ_{12} are determined as discussed earlier using the stress transformation as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

And then corresponding material axes strains are determined by multiplying the material axes stresses by the compliance matrix as

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

These material axes strains are then compared with the corresponding allowable strains as the conditions of safety

$$\begin{bmatrix} -\left(\varepsilon_1^C\right)_u < \varepsilon_1 < \left(\varepsilon_1^T\right)_u \\ -\left(\varepsilon_1^C\right)_u < \varepsilon_1 < \left(\varepsilon_1^T\right)_u \\ -\left(\gamma_{12}\right)_u < \gamma_{12} < \left(\gamma_{12}\right)_u \end{bmatrix}$$

As shown in the Fig., from the stress strain curves of the lamina loaded in along its material axes an noting the ultimate stresses (failure point) and the modulus (slope), the allowable strains are obtained by dividing the strengths by the corresponding moduli as follows, with the assumptions that the lamina is linearly elastic till its failure.

$$\begin{cases} \left(\varepsilon_1^T\right)_u = \dfrac{\left(\sigma_1^T\right)_u}{\mathrm{E}_1}; \quad \left(\varepsilon_1^C\right)_u = \dfrac{\left(\sigma_1^C\right)_u}{\mathrm{E}_1} \\ \left(\varepsilon_2^T\right)_u = \dfrac{\left(\sigma_2^T\right)_u}{\mathrm{E}_2}; \quad \left(\varepsilon_2^C\right)_u = \dfrac{\left(\sigma_2^C\right)_u}{\mathrm{E}_2} \\ \left(\gamma_{12}\right)_u = \dfrac{\left(\tau_{12}\right)_u}{G_{12}} \end{cases}$$

Again like maximum stress theory, while comparing the strains, proper care should be taken to decide the corresponding allowable strains depending upon the sign. **F**or shear stress it is independent of sign in the material axis.

A lamina is loaded in tension along 1 and the allowable longitudinal tensile strain $(\varepsilon_1^T)_u$ is nothing but the ultimate longitudinal tensile stress $(\sigma_1^T)_u$ divided by E_1. Similarly, a lamina is loaded in tension along 2 and the allowable longitudinal tensile strain $(\varepsilon_2^T)_u$ is nothing but the ultimate longitudinal tensile stress $(\sigma_2^T)_u$ divided by E_2. In the same way loading the lamina in compression $(\varepsilon_1^C)_u$ and $(\varepsilon_2^C)_u$ could also be obtained from the stress strain plot. From the shear loading of the lamina in the 1-2 plane from the stress strain plot, noting the failure point and the slope, the corresponding allowable shear strain is $(\gamma_{12})_u$ could be obtained by dividing the ultimate shear stress $(\tau_{12})_u$ by G_{12}

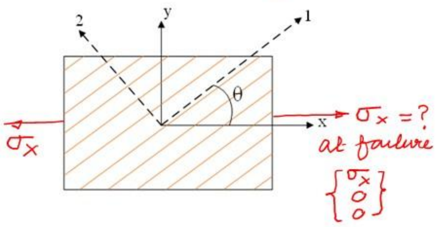
Again, like the maximum stress theory, maximum strain theory also consists of five different criteria. To understand the maximum strain theory, we shall solve the same problem of determination of off-axis tensile strength of a lamina maximum strain theory.

(Refer Slide Time: 09:37)

Strength Failure Theories of Lamina -
Off-axis tensile strength using *Maximum Strain Theory*

$\boxed{\sigma_x \neq 0, \sigma_y = \tau_{xy} = 0}$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \sigma_x c^2 \\ \sigma_x s^2 \\ -\sigma_x sc \end{Bmatrix}$$



$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \underbrace{\begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix}}\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_x c^2 \\ \sigma_x s^2 \\ -\sigma_x sc \end{Bmatrix} = \begin{Bmatrix} \frac{1}{E_1}\left(c^2 - \nu_{12} s^2\right)\sigma_x \\ \frac{1}{E_2}\left(s^2 - \nu_{21} c^2\right)\sigma_x \\ -\frac{1}{G_{12}}(sc)\sigma_x \end{Bmatrix} \begin{matrix} \leftarrow \epsilon_1 \\ \leftarrow \epsilon_2 \\ \leftarrow \gamma_{12} \end{matrix}$$

We have already determined the off-axis tensile strength using maximum stress theory, so we would like to do the same exercise using maximum strain theory.

As shown in the Fig., the state of stress with reference to x-y is $\begin{Bmatrix} \sigma_x \\ 0 \\ 0 \end{Bmatrix}$. Therefore, with reference to the material axes the stresses are obtained using the transformation as

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix}=\begin{bmatrix}c^2 & s^2 & 2sc\\ s^2 & c^2 & -2sc\\ -sc & sc & c^2-s^2\end{bmatrix}\begin{Bmatrix}\sigma_x\\ 0\\ 0\end{Bmatrix}=\begin{Bmatrix}\sigma_x c^2\\ \sigma_x s^2\\ -\sigma_x sc\end{Bmatrix}$$

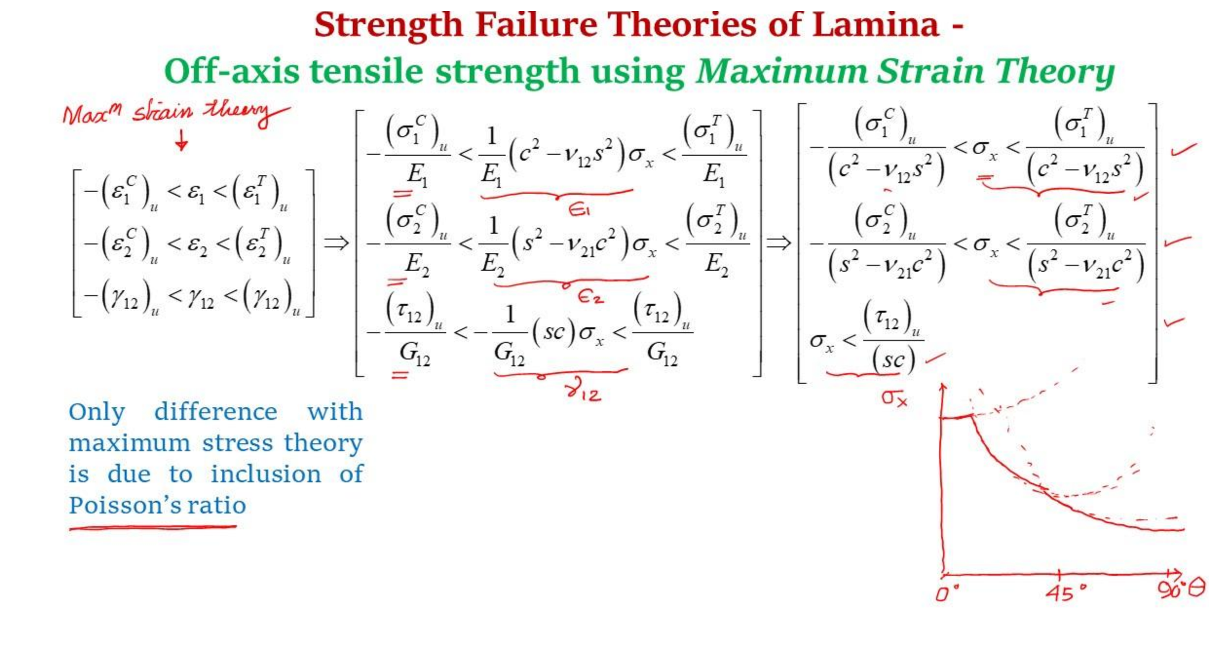
Then the strains with reference to the material axes are determined as

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}=\begin{bmatrix}S_{11} & S_{12} & 0\\ S_{12} & S_{22} & 0\\ 0 & 0 & S_{66}\end{bmatrix}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix}=\begin{bmatrix}\dfrac{1}{E_1} & -\dfrac{\nu_{12}}{E_1} & 0\\ -\dfrac{\nu_{21}}{E_2} & \dfrac{1}{E_2} & 0\\ 0 & 0 & \dfrac{1}{G_{12}}\end{bmatrix}\begin{Bmatrix}\sigma_x c^2\\ \sigma_x s^2\\ -\sigma_x sc\end{Bmatrix}=\begin{Bmatrix}\dfrac{1}{E_1}\left(c^2-\nu_{12}s^2\right)\sigma_x\\ \dfrac{1}{E_2}\left(s^2-\nu_{21}c^2\right)\sigma_x\\ -\dfrac{1}{G_{12}}(sc)\sigma_x\end{Bmatrix}$$

So, once we have these strains in the material axis, now we can apply the maximum strain theory. Using the maximum strain theory the conditions for safety are

$$\begin{bmatrix}-\left(\varepsilon_1^C\right)_u<\varepsilon_1<\left(\varepsilon_1^T\right)_u\\ -\left(\varepsilon_2^C\right)_u<\varepsilon_2<\left(\varepsilon_2^T\right)_u\\ -\left(\gamma_{12}\right)_u<\gamma_{12}<\left(\gamma_{12}\right)_u\end{bmatrix}\Rightarrow\begin{bmatrix}-\dfrac{\left(\sigma_1^C\right)_u}{E_1}<\dfrac{1}{E_1}\left(c^2-\nu_{12}s^2\right)\sigma_x<\dfrac{\left(\sigma_1^T\right)_u}{E_1}\\ -\dfrac{\left(\sigma_2^C\right)_u}{E_2}<\dfrac{1}{E_2}\left(s^2-\nu_{21}c^2\right)\sigma_x<\dfrac{\left(\sigma_2^T\right)_u}{E_2}\\ -\dfrac{\left(\tau_{12}\right)_u}{G_{12}}<-\dfrac{1}{G_{12}}(sc)\sigma_x<\dfrac{\left(\tau_{12}\right)_u}{G_{12}}\end{bmatrix}\Rightarrow\begin{bmatrix}-\dfrac{\left(\sigma_1^C\right)_u}{\left(c^2-\nu_{12}s^2\right)}<\sigma_x<\dfrac{\left(\sigma_1^T\right)_u}{\left(c^2-\nu_{12}s^2\right)}\\ -\dfrac{\left(\sigma_2^C\right)_u}{\left(s^2-\nu_{21}c^2\right)}<\sigma_x<\dfrac{\left(\sigma_2^T\right)_u}{\left(s^2-\nu_{21}c^2\right)}\\ \sigma_x<\dfrac{\left(\tau_{12}\right)_u}{(sc)}\end{bmatrix}$$

(Refer Slide Time: 11:58)

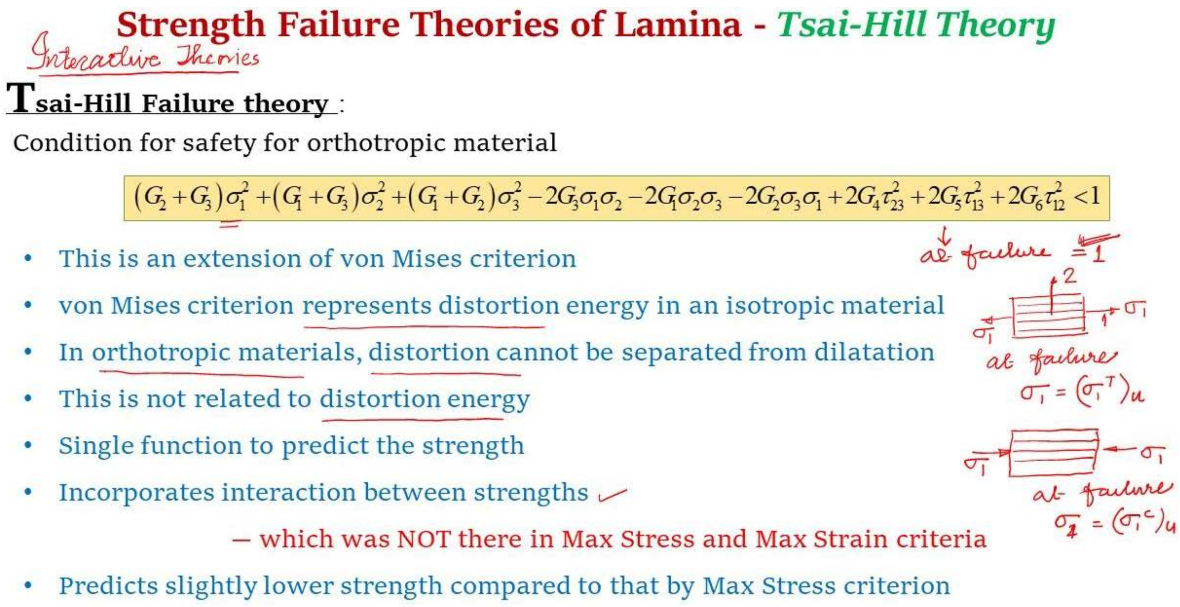


Now in this case since we are considering tensile stress, therefore σ_x is positive. Therefore, there are three conditions for safety instead of five (conditions corresponding to negative ε_1 and ε_2 are omitted) So, the conditions are

$$\begin{bmatrix} \sigma_x < \dfrac{\left(\sigma_1^T\right)_u}{\left(c^2 - \nu_{12} s^2\right)} \\ \sigma_x < \dfrac{\left(\sigma_2^T\right)_u}{\left(s^2 - \nu_{21} c^2\right)} \\ \sigma_x < \dfrac{\left(\tau_{12}\right)_u}{\left(sc\right)} \end{bmatrix}$$

Similar to what we did in maximum stress theory, here also we could plot σ_x with θ to understand the influence of θ on the off-axis tensile strength of a lamina as shown in the Fig. and we could see the different modes of failure (longitudinal tensile near $\theta = 0^o$, shear away from $\theta = 0^o$ till certain angle and transverse tensile beyond 45^o and near 90^o). Only difference with maximum stress theory is due to the Poisson's effect as could be seen from the plot. If we put Poisson's ratio as zero, there is no difference between the maximum stress theory and the maximum strain theory.

(Refer Slide Time: 15:20)



After discussin the non-interactive theories, we shall now discuss some of the interactive theories.

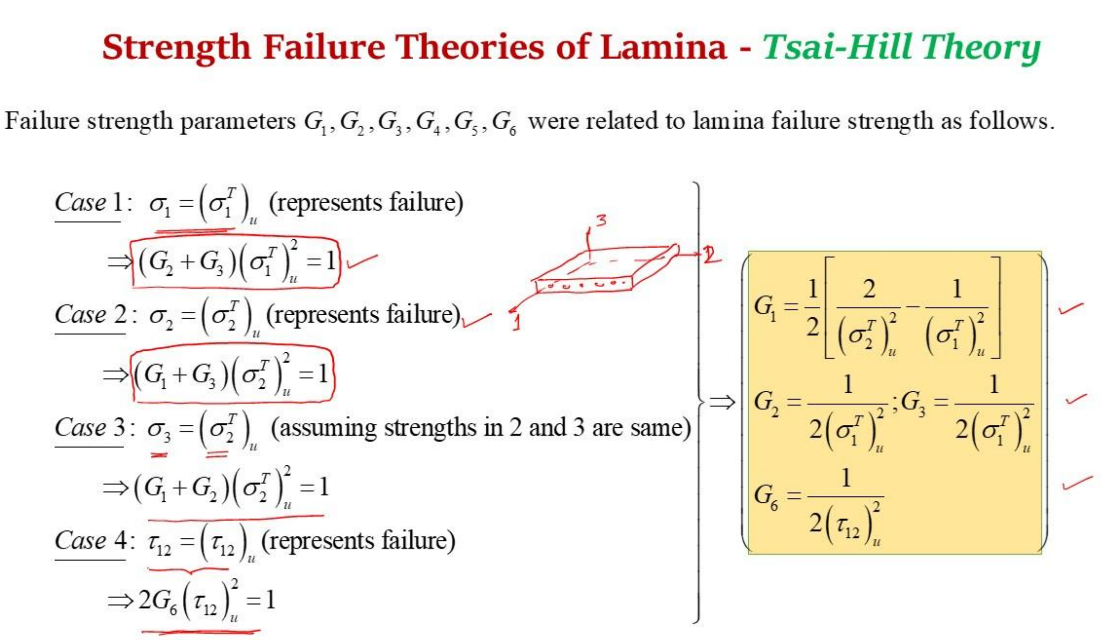
Tsai-Hill failure theory

Hill proposed a kind of distortion energy theory or the von Misses criterion for anisotropic material and Tsai adopted that for an orthotropic lamina therefore it is Tsai-Hill theory. This states that the condition for safety for an orthotropic material is given by this.

$$\left(G_2+G_3\right)\sigma_1^2+\left(G_1+G_3\right)\sigma_2^2+\left(G_1+G_2\right)\sigma_3^2-2G_3\sigma_1\sigma_2-2G_1\sigma_2\sigma_3-2G_2\sigma_3\sigma_1+2G_4\tau_{23}^2+2G_5\tau_{13}^2+2G_6\tau_{12}^2<1$$

This is a kind of extension of the von-Misses criterion. We know that von Misses criterion actually represents the distortion energy in an isotropic material. Now, for any given state of stress the state of stress and the corresponding strains are split into hydrostatic and deviatoric parts. Hydrostatic is actually responsible for the volume change and the deviatoric part is actually responsible for distortion and the energy corresponding to distortion is the distortion energy. Using the distortion energy the von Misses equivalent stresses are obtained. Therefore, this actually represents the distortion energy. But unlike isotropic materials, in orthotropic materials, distortion cannot be separated from dilatation. While in an isotropic materials we can actually separate the two stresses and the corresponding strains but in orthotropic it may not be possible because of the existence of shear extension coupling. Since it is not possible to separate the distortion energy from the dilatation energy and therefore, in true sense it is not related to distortion even though it is adapted in the same line but this is not distortion energy theory. Unlike the non-interactive criteria like maximum stress and maximum strain criteria where there are five different sub criteria, here a single function predicts the strength. More importantly it incorporates interactions between the strengths which were not there in maximum stress and maximum strain criteria. However, the predicted strength is slightly lower compared to that by maximum stress and strain criterion.

(Refer Slide Time: 18:32)



So, let us see that how these parameters G_1, G_2, G_3, G_4, G_5, G_6 in this are calculated? These are kind of parameters related to the strengths. These are determined by loading the lamina along the material axes till failure and then obtaining relations for determination of these parameters as follows.

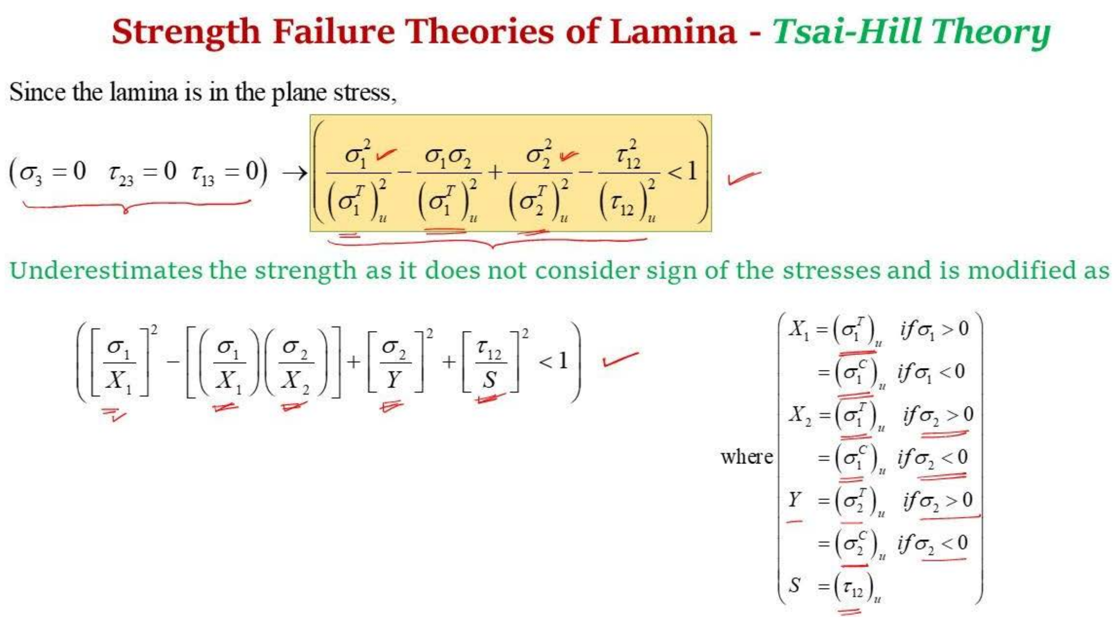
$$\left.\begin{array}{l}
\underline{Case\ 1}:\ \sigma_1 = \left(\sigma_1^T\right)_u \text{ (represents failure)} \\
\qquad \Rightarrow \left(G_2+G_3\right)\left(\sigma_1^T\right)_u^2 = 1 \\
\underline{Case\ 2}:\ \sigma_2 = \left(\sigma_2^T\right)_u \text{ (represents failure)} \\
\qquad \Rightarrow \left(G_1+G_3\right)\left(\sigma_2^T\right)_u^2 = 1 \\
\underline{Case\ 3}:\ \sigma_3 = \left(\sigma_2^T\right)_u \text{ (assuming strengths in 2 and 3 are same)} \\
\qquad \Rightarrow \left(G_1+G_2\right)\left(\sigma_2^T\right)_u^2 = 1 \\
\underline{Case\ 4}:\ \tau_{12} = \left(\tau_{12}\right)_u \text{ (represents failure)} \\
\qquad \Rightarrow 2G_6\left(\tau_{12}\right)_u^2 = 1
\end{array}\right\} \Rightarrow \left(\begin{array}{l}
G_1 = \dfrac{1}{2}\left[\dfrac{2}{\left(\sigma_2^T\right)_u^2} - \dfrac{1}{\left(\sigma_1^T\right)_u^2}\right] \\
G_2 = \dfrac{1}{2\left(\sigma_1^T\right)_u^2}; G_3 = \dfrac{1}{2\left(\sigma_1^T\right)_u^2} \\
G_6 = \dfrac{1}{2\left(\tau_{12}\right)_u^2}
\end{array}\right)$$

For example, if we put only $\sigma_1 = \left(\sigma_1^T\right)_u$ and all other stress are 0, and this represents a failure condition and putting the expression

$$\left(G_2+G_3\right)\sigma_1^2 + \left(G_1+G_3\right)\sigma_2^2 + \left(G_1+G_2\right)\sigma_3^2 - 2G_3\sigma_1\sigma_2 - 2G_1\sigma_2\sigma_3 - 2G_2\sigma_3\sigma_1 + 2G_4\tau_{23}^2 + 2G_5\tau_{13}^2 + 2G_6\tau_{12}^2 < 1$$

equal to 1, we get $\left(G_2+G_3\right)\left(\sigma_1^T\right)_u^2 = 1$. Similarly by putting other failure conditions, we get three more relations as shown and we could solve for G_1, G_2, G_3, G_6 as shown.

(Refer Slide Time: 22:42)



Considering a lamina in plane stress $(\sigma_3 = 0 \quad \tau_{23} = 0 \;\; \tau_{13} = 0)$ and putting these values of G_1, G_2, G_3, G_6 the Tsai-Hill Theory for a 2D lamina becomes-

Since the lamina is in the plane stress,

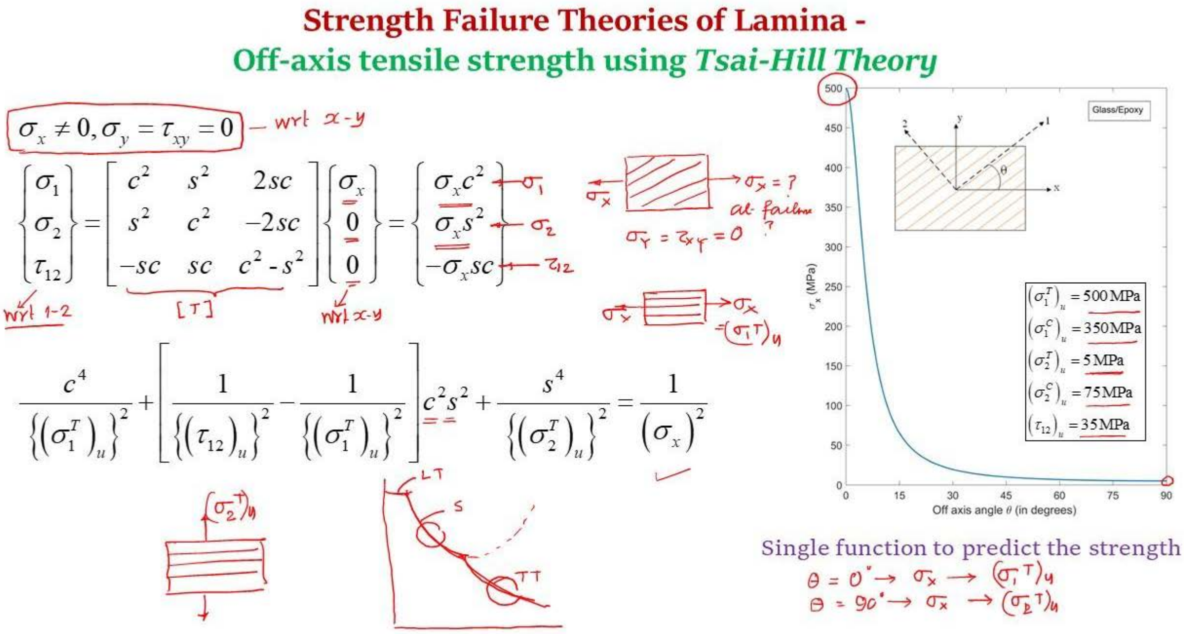
$$(\sigma_3 = 0 \quad \tau_{23} = 0 \;\; \tau_{13} = 0) \rightarrow \left(\frac{\sigma_1^2}{\left(\sigma_1^T\right)_u^2} - \frac{\sigma_1\sigma_2}{\left(\sigma_1^T\right)_u^2} + \frac{\sigma_2^2}{\left(\sigma_2^T\right)_u^2} + \frac{\tau_{12}^2}{\left(\tau_{12}\right)_u^2} < 1\right)$$

However, since it does not consider the sign of the stresses, it actually underestimates the strength. It could be clearly seen that since σ_1 and σ_2 are squared, the influence of +ve or –ve stress is lost and we know that the strengths are different depending upon whether the normal stress is +ve or –ve. This is modified to take care of the signs as follows..

$$\left(\left[\frac{\sigma_1}{X_1}\right]^2 - \left[\left(\frac{\sigma_1}{X_2}\right)\left(\frac{\sigma_2}{X_2}\right)\right] + \left[\frac{\sigma_2}{Y}\right]^2 + \left[\frac{\tau_{12}}{S}\right]^2 < 1\right) \qquad \text{where} \left(\begin{aligned} X_1 &= \left(\sigma_1^T\right)_u \quad if\,\sigma_1 > 0 \\ &= \left(\sigma_1^C\right)_u \quad if\,\sigma_1 < 0 \\ X_2 &= \left(\sigma_1^T\right)_u \quad if\,\sigma_2 > 0 \\ &= \left(\sigma_1^C\right)_u \quad if\,\sigma_2 < 0 \\ Y &= \left(\sigma_2^T\right)_u \quad if\,\sigma_2 > 0 \\ &= \left(\sigma_2^C\right)_u \quad if\,\sigma_2 < 0 \\ S &= \left(\tau_{12}\right)_u \end{aligned}\right)$$

So, this is how the Tsai-Hill theory is slightly modified to take care of the sign of the stresses.

(Refer Slide Time: 25:14)



So, now suppose we want to do the same exercise what we have done for maximum stress and maximum strain theory. That means we would like to determine the off-axis tensile strength of a lamina using Tsai-Hill theory. As shown in the Fig., the lamina is subjected to tensile strength σ_x, and we would like to know what is the σ_x at failure? From the given state of stress with respect to the x-y axes, we determine the material axes stresses as follows.

$$\sigma_x \neq 0, \sigma_y = \tau_{xy} = 0$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \sigma_x c^2 \\ \sigma_x s^2 \\ -\sigma_x sc \end{Bmatrix}$$

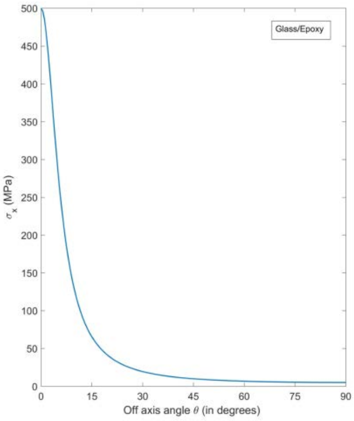
Here, c stands for *cosθ,* s stands for *sinθ.*

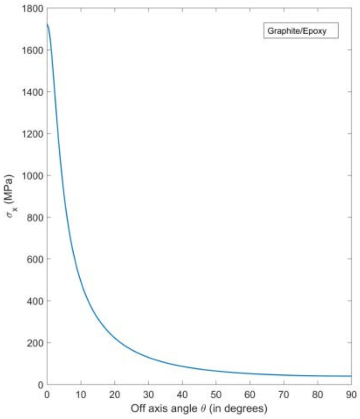
By putting these expressions of material axes stresses in the Tsai-Hill theory, the condition for failure is

$$\frac{c^4}{\left\{\left(\sigma_1^T\right)_u\right\}^2} + \left[\frac{1}{\left\{\left(\tau_{12}\right)_u\right\}^2} - \frac{1}{\left\{\left(\sigma_1^T\right)_u\right\}^2}\right] c^2 s^2 + \frac{s^4}{\left\{\left(\sigma_2^T\right)_u\right\}^2} = \frac{1}{\left(\sigma_x\right)^2}$$

So, if we plot σ_x with θ say for a typical glass/epoxy and graphite/epoxy lamina we get the plots as shown for the following properties

$(\sigma_1^T)_u = 500\,\text{MPa}$
$(\sigma_1^C)_u = 350\,\text{MPa}$
$(\sigma_2^T)_u = 5\,\text{MPa}$
$(\sigma_2^C)_u = 75\,\text{MPa}$
$(\tau_{12})_u = 35\,\text{MPa}$





$(\sigma_1^T)_u = 1725MPa$
$(\sigma_1^C)_u = 1350MPa$
$(\sigma_2^T)_u = 40MPa$
$(\sigma_2^C)_u = 275MPa$
$(\tau_{12})_u = 95MPa$

As could be seen, it is a continuous curve and a single function predicts the strength. Unlike in the case of maximum stress and maximum strain, there were actually three curves.

Here also, at $\theta = 0°$, at failure $\sigma_x = (\sigma_1^T)_u$, and at $\theta = 90°$, at failure, $\sigma_x = (\sigma_2^T)_u$, as expected for the cases.

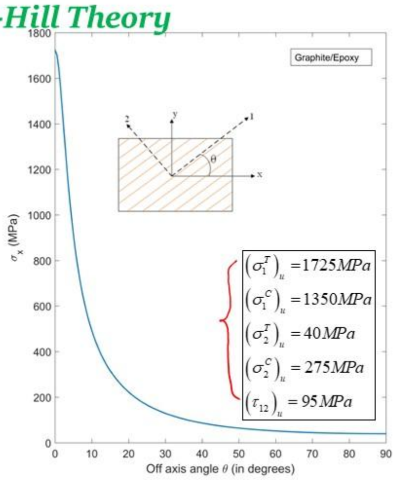
(Refer Slide Time: 30:52)

Strength Failure Theories of Lamina -
Off-axis tensile strength using *Tsai-Hill Theory*

$$\sigma_x \neq 0, \sigma_y = \tau_{xy} = 0$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \sigma_x c^2 \\ \sigma_x s^2 \\ -\sigma_x sc \end{Bmatrix}$$

$$\frac{c^4}{\left\{(\sigma_1^T)_u\right\}^2} + \left[\frac{1}{\left\{(\tau_{12})_u\right\}^2} - \frac{1}{\left\{(\sigma_1^T)_u\right\}^2}\right] c^2 s^2 + \frac{s^4}{\left\{(\sigma_2^T)_u\right\}^2} = \frac{1}{(\sigma_x)^2}$$



Single function to predict the strength

Here, the mode of failure cannot be determined. We can of course have a deeper look into all the terms and make out but it is not visible looking at the expression straight away.

(Refer Slide Time: 31:50)

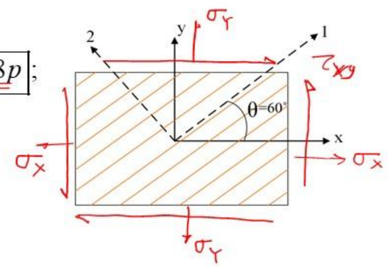
Strength Failure Theories of Lamina - *Tsai-Hill Theory*

Example:

A 60° GR/E lamina subjected to $\boxed{\sigma_x = 4p, \sigma_y = -6p, \tau_{xy} = 8p}$;

Using Tsai-Hill theory, determine max. value of $p > 0$,
so that the lamina is safe.

Given, $\boxed{\begin{matrix} \left(\sigma_1^T\right)_u = 1725MPa, \left(\sigma_1^C\right)_u = 1350MPa \\ \left(\sigma_2^T\right)_u = 40MPa, \quad \left(\sigma_2^C\right)_u = 275MPa \\ \left(\tau_{12}\right)_u = 95MPa \end{matrix}}$



Now let us take the same example we have solved using maximum stress theory, just to have a comparison of the strength prediction by maximum stress theory and by Tsai-Hill theory.

Example:

A 60° GR/E lamina subjected to $\boxed{\sigma_x = 4p, \sigma_y = -6p, \tau_{xy} = 8p}$;

Using Tsai-Hill theory, determine max. value of $p > 0$,
so that the lamina is safe.

Given, $\boxed{\begin{matrix} \left(\sigma_1^T\right)_u = 1725MPa, \left(\sigma_1^C\right)_u = 1350MPa \\ \left(\sigma_2^T\right)_u = 40MPa, \quad \left(\sigma_2^C\right)_u = 275MPa \\ \left(\tau_{12}\right)_u = 95MPa \end{matrix}}$

Solution:

We follow the same steps as discussed earlier that is we determine the material axes stresses and then put those material axes stresses in the failure theory (in this case the Tsai-Hill theory) to obtain the value of p at failure.

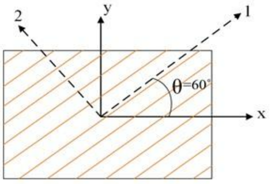
$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} 4p \\ -6p \\ 8p \end{Bmatrix} \rightarrow \boxed{\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{Bmatrix} 3.4 \\ -5.4 \\ -8.3 \end{Bmatrix} p}$$

(Refer Slide Time: 33:22)

Strength Failure Theories of Lamina - *Tsai-Hill Theory*

Solution:

$$\text{Condition for safety}: \begin{pmatrix} -1350\times10^6 < \sigma_1 < 1725\times10^6 \\ -275\times10^6 < \sigma_2 < 40\times10^6 \\ -95\times10^6 < \tau_{12} < 95\times10^6 \end{pmatrix}$$



$$\left\{\begin{matrix}\sigma_x \\ \sigma_Y \\ \tau_{xY}\end{matrix}\right\} \rightarrow \text{wrt } x\text{-}y$$

$$\underbrace{\left\{\begin{matrix}\sigma_1 \\ \sigma_2 \\ \tau_{12}\end{matrix}\right\}}_{\text{w.r.t. } 1\cdot2} = \underbrace{\begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ -sc & sc & c^2-s^2 \end{bmatrix}}_{[T]\ \theta=60^\circ} \left\{\begin{matrix}4p \\ -6p \\ 8p\end{matrix}\right\} \rightarrow \boxed{\left\{\begin{matrix}\sigma_1 \\ \sigma_2 \\ \tau_{12}\end{matrix}\right\} = \left\{\begin{matrix}3.4 \\ -5.4 \\ -8.3\end{matrix}\right\} p}$$

Now given the strength properties as

$$\left(\sigma_1^T\right)_u = 1725MPa,\ \left(\sigma_1^C\right)_u = 1350MPa$$
$$\left(\sigma_2^T\right)_u = 40MPa,\quad \left(\sigma_2^C\right)_u = 275MPa$$
$$\left(\tau_{12}\right)_u = 95MPa$$

The material axes stresses as

$$\boxed{\left\{\begin{matrix}\sigma_1 \\ \sigma_2 \\ \tau_{12}\end{matrix}\right\} = \left\{\begin{matrix}3.4 \\ -5.4 \\ -8.3\end{matrix}\right\} p}$$

Depending upon the sign of the stresses, we get the following as the strength parameters

$$\begin{pmatrix} \left.\begin{aligned} X_1 &= \left(\sigma_1^T\right)_u \quad if\,\sigma_1 > 0 \\ &= \left(\sigma_1^C\right)_u \quad if\,\sigma_1 < 0 \end{aligned}\right\} \Rightarrow X_1 = 1725MPa \\ \left.\begin{aligned} X_2 &= \left(\sigma_1^T\right)_u \quad if\,\sigma_2 > 0 \\ &= \left(\sigma_1^C\right)_u \quad if\,\sigma_2 < 0 \end{aligned}\right\} \Rightarrow X_2 = 1350MPa \\ \left.\begin{aligned} Y &= \left(\sigma_2^T\right)_u \quad if\,\sigma_2 > 0 \\ &= \left(\sigma_2^C\right)_u \quad if\,\sigma_2 < 0 \end{aligned}\right\} \Rightarrow Y = 275MPa \\ S = \left(\tau_{12}\right)_u = 95MPa \end{pmatrix}$$

And putting in the Tsai-Hill theory as

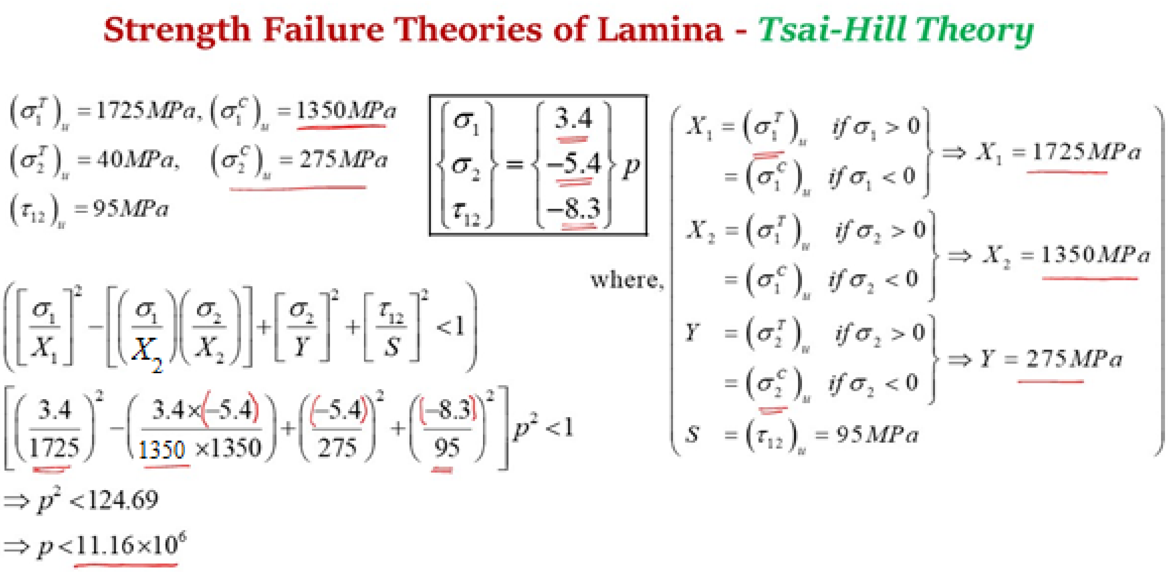
$$\left(\left[\frac{\sigma_1}{X_1}\right]^2 - \left[\left(\frac{\sigma_1}{X_2}\right)\left(\frac{\sigma_2}{X_2}\right)\right] + \left[\frac{\sigma_2}{Y}\right]^2 + \left[\frac{\tau_{12}}{S}\right]^2 < 1\right)$$

$$\left[\left(\frac{3.4}{1725}\right)^2 - \left(\frac{3.4\times -5.4}{1350\times 1350}\right) + \left(\frac{-5.4}{275}\right)^2 + \left(\frac{-8.3}{95}\right)^2\right] p^2 < 1$$

$$\Rightarrow p^2 < 124.69$$

$$\Rightarrow p < 11.16\times 10^6$$

(Refer Slide Time: 34:35)



We get that p should be less than 11.16×10^6 and using maximum stress theory we got something like $p = 11.4\times 10^6$. So, this predicts slightly less than that what is predicted by maximum stress theory.

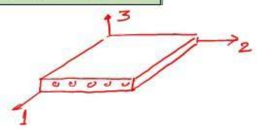
(Refer Slide Time: 36:17)

Strength Failure Theories of Lamina - *Hoffman Failure Theory*

Hoffman Failure theory :

To account for different strengths in tension and compression—
Hoffman added linear terms to the Hill's equation as

$$\boxed{C_1(\sigma_2-\sigma_3)^2+C_2(\sigma_3-\sigma_1)^2+C_3(\sigma_1-\sigma_2)^2+C_4\sigma_1+C_5\sigma_2+C_6\sigma_3+C_7\tau_{23}^2+C_8\tau_{31}^2+C_9\tau_{12}^2<1}$$



$$\text{in plane } 1-2 \rightarrow (\sigma_3=0 \quad \tau_{23}=0 \;\; \tau_{13}=0)$$

$$\text{transverse isotropy in } 2-3 \rightarrow (\sigma_3^T)_u=(\sigma_2^T)_u, (\sigma_3^C)_u=(\sigma_2^C)_u, S_{31}=S_{12}$$

C_i s (i=1,2,3,...,9) could be determined from 9 strengths wrt principal material axes

- Interaction between failure modes
- Takes care of the sign of stresses

$\sigma_1 \Rightarrow (\sigma_1^T)_u$, $\sigma_1=(\sigma_1^C)_u$
$\sigma_2=(\sigma_2^T)_u$ $\sigma_2=(\sigma_2^C)_u$
$\sigma_3=(\sigma_3^T)_u$ $\sigma_3=(\sigma_3^C)_u$
$\tau_{12}=(\tau_{12})_u$, $\tau_{23}=(\tau_{23})_u$, $\tau_{13}=(\tau_{13})_u$

Hoffman theory

One of the drawbacks of the Tsai-Hill theory in its general form is that it does not take care of the sign of the stresses. So, to account for different strengths in tension and compression, Hoffman actually added few linear terms in the Hill's equation and put forward the Hoffman theory as

$$\boxed{C_1(\sigma_2-\sigma_3)^2+C_2(\sigma_3-\sigma_1)^2+C_3(\sigma_1-\sigma_2)^2+C_4\sigma_1+C_5\sigma_2+C_6\sigma_3+C_7\tau_{23}^2+C_8\tau_{31}^2+C_9\tau_{12}^2<1}$$

$$\text{in plane } 1-2 \rightarrow (\sigma_3=0 \quad \tau_{23}=0 \;\; \tau_{13}=0)$$
$$\text{transverse isotropy in } 2-3 \rightarrow (\sigma_3^T)_u=(\sigma_2^T)_u, (\sigma_3^C)_u=(\sigma_2^C)_u, S_{31}=S_{12}$$

This takes care of the sign in addition to the interactions among different stresses. The parameters C_1, C_2 , etc are determined by putting certain failure conditions as follows

$$\left(\begin{array}{l} Case1:\sigma_1=(\sigma_1^T)_u\,;Case2:\sigma_1=(\sigma_1^C)_u \\ Case3:\sigma_2=(\sigma_2^T)_u\,;Case4:\sigma_2=(\sigma_2^C)_u \\ Case5:\sigma_3=(\sigma_3^T)_u\,;Case6:\sigma_3=(\sigma_3^C)_u \\ Case7:\tau_{23}=(\tau_{23})_u\,;Case8:\tau_{13}=(\tau_{13})_u\,; \\ Case9:\tau_{12}=(\tau_{12})_u \end{array}\right)$$

For plane stress in 1-2 plane ie. $\sigma_3=\tau_{23}=\tau_{13}=0$ and with 2-3 as the plane of transverse isotropy i.e $(\sigma_2^T)_u=(\sigma_3^T)_u$ and $(\sigma_2^C)_u=(\sigma_3^C)_u, (\tau_{13})_u=(\tau_{12})_u$

Now we obtain these C_1, C_2 all these are actually in terms of this 5 strength parameters of the lamina and using these and putting these conditions we get the Hoffman's failure theory as the condition for safety as

$$\Rightarrow \boxed{-\frac{\sigma_1^2}{\left(\sigma_1^C\right)_u\left(\sigma_1^T\right)_u}+\frac{\sigma_1\sigma_2}{\left(\sigma_1^C\right)_u\left(\sigma_1^T\right)_u}-\frac{\sigma_2^2}{\left(\sigma_2^C\right)_u\left(\sigma_2^T\right)_u}+\frac{\left(\sigma_1^C\right)_u+\left(\sigma_1^T\right)_u}{\left(\sigma_1^C\right)_u\left(\sigma_1^T\right)_u}\sigma_1+\frac{\left(\sigma_2^C\right)_u+\left(\sigma_2^T\right)_u}{\left(\sigma_2^C\right)_u\left(\sigma_2^T\right)_u}\sigma_2+\frac{\tau_{12}^2}{S_{12}^2}<1}$$

So, here it is important to note that there are interactions between the stresses and it actually takes care of the sign. Because there is a linear term it actually recognizes the sign of the stress unlike in Tsai-Hill theory.

(Refer Slide Time: 40:01)

Strength Failure Theories of Lamina - *Hoffman Failure Theory*

$$\begin{pmatrix} Case1:\ \sigma_1=\left(\sigma_1^T\right)_u; & Case2:\ \sigma_1=\left(\sigma_1^C\right)_u \\ Case3:\ \sigma_2=\left(\sigma_2^T\right)_u; & Case4:\ \sigma_2=\left(\sigma_2^C\right)_u \\ Case5:\ \sigma_3=\left(\sigma_3^T\right)_u; & Case6:\ \sigma_3=\left(\sigma_3^C\right)_u \\ Case7:\ \tau_{23}=\left(\tau_{23}\right)_u; & Case8:\ \tau_{13}=\left(\tau_{13}\right)_u \\ Case9:\ \tau_{12}=\left(\tau_{12}\right)_u & \end{pmatrix}$$

$(\sigma_1^T)_u = (\sigma_1^C)_u$

For plane stress in 1-2 plane i.e. $\sigma_3=\tau_{23}=\tau_{13}=0$ and with 2-3 as the plane of transverse isotropy i.e. $\left(\sigma_2^T\right)_u=\left(\sigma_3^T\right)_u$ and $\left(\sigma_2^C\right)_u=\left(\sigma_3^C\right)_u$, $\left(\tau_{13}\right)_u=\left(\tau_{12}\right)_u$

$$\Rightarrow \boxed{-\frac{\sigma_1^2}{\left(\sigma_1^C\right)_u\left(\sigma_1^T\right)_u}+\frac{\sigma_1\sigma_2}{\left(\sigma_1^C\right)_u\left(\sigma_1^T\right)_u}-\frac{\sigma_2^2}{\left(\sigma_2^C\right)_u\left(\sigma_2^T\right)_u}+\frac{\left(\sigma_1^C\right)_u+\left(\sigma_1^T\right)_u}{\left(\sigma_1^C\right)_u\left(\sigma_1^T\right)_u}\sigma_1+\frac{\left(\sigma_2^C\right)_u+\left(\sigma_2^T\right)_u}{\left(\sigma_2^C\right)_u\left(\sigma_2^T\right)_u}\sigma_2+\frac{\tau_{12}^2}{S_{12}^2}<1}$$

(Refer Slide Time: 41:31)

Strength Failure Theories of Lamina - *Tsai-Wu Failure Criterion*

Tsai-Wu tensor failure Criterion

- All the criteria discussed – inadequate in representing experimental data
- To improve the correlation — increase the number of terms in the prediction equation
  — better curve fitting
  — more interaction
- Postulated that the condition for safety is

$$\boxed{F_i\sigma_i + F_{ij}\sigma_i\sigma_j < 1} \qquad i, j = 1, 2, 6$$

For a 2D lamina with plane stress in 1-2 and transverse isotropy in 2-3,

$$\boxed{F_1\sigma_1 + F_2\sigma_2 + F_6\tau_{12} + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + 2F_{12}\sigma_1\sigma_2 < 1} = 1 \text{ at failure}$$

Every time a criterion is put forward, they are actually correlated with the experimental observations and it was observed that all these are actually inadequate in representing the experimental data. Therefore, to obtain a better correlation with the experimental data Tsai-Wu failure theory was proposed with increasing number of terms in the equation so that it gives better fit with the experimental data and there will be more interactions among the stress parameters.

So, Tsai-Wu actually postulated the condition for safety as

$$\boxed{F_i\sigma_i + F_{ij}\sigma_i\sigma_j < 1} \qquad i, j = 1, 2, 6,$$

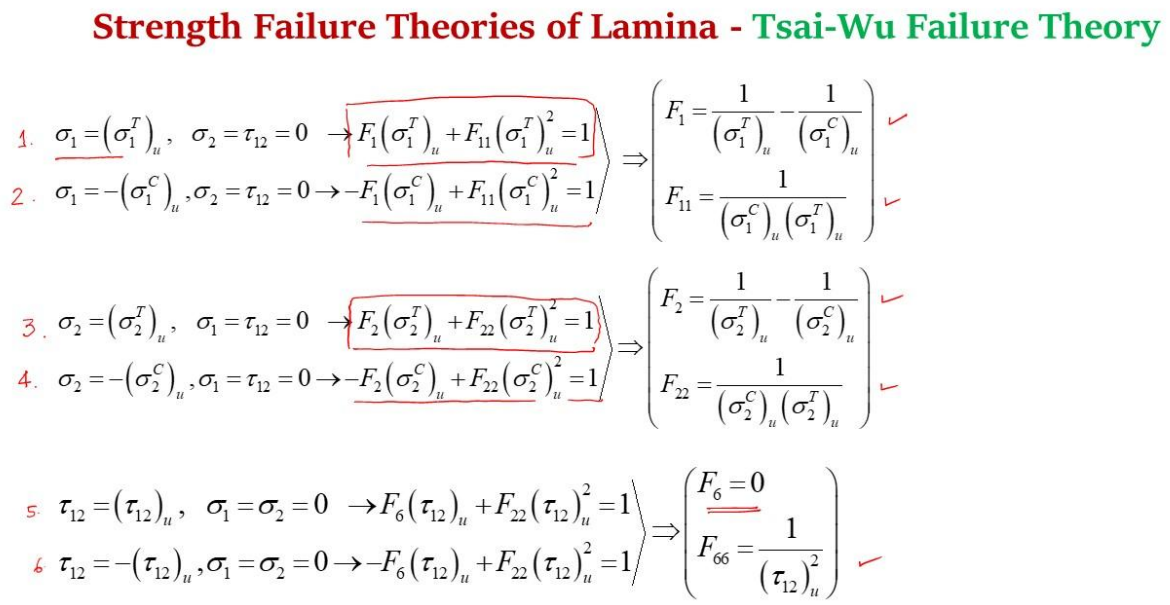
where i, j = 1, 2, 6 means that among the 6 strengths, σ_1, σ_2, σ_3, τ_{12}, τ_{23}, τ_{13}, all the possible interactions are taken an on expanding this, will have 36 terms. However this is simplified for a two-dimensional lamina with plane stresses 1-2 and transverse isotropy in 2-3 and

For a 2D lamina with plane stress in 1-2 and transverse isotropy in 2-3,

$$\boxed{F_1\sigma_1 + F_2\sigma_2 + F_6\tau_{12} + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + 2F_{12}\sigma_1\sigma_2 < 1}$$

Still we have seven constants which need to be determined in terms of the strength parameters.

(Refer Slide Time: 43:43)



Putting different failure conditions of the lamina in the material axes 1-2, we get some relations as follows from which these constants F_1, F_{11}, F_2, F_{22}, F_6, F_{66} are determined as shown.

$$\left.\begin{aligned}
&\sigma_1=(\sigma_1^T)_u,\quad \sigma_2=\tau_{12}=0 \rightarrow F_1(\sigma_1^T)_u+F_{11}(\sigma_1^T)_u^2=1\\
&\sigma_1=-(\sigma_1^C)_u,\sigma_2=\tau_{12}=0\rightarrow -F_1(\sigma_1^C)_u+F_{11}(\sigma_1^C)_u^2=1
\end{aligned}\right\rangle
\left(\begin{aligned}
&F_1=\frac{1}{(\sigma_1^T)_u}-\frac{1}{(\sigma_1^C)_u}\\
&F_{11}=\frac{1}{(\sigma_1^C)_u(\sigma_1^T)_u}
\end{aligned}\right)$$

$$\left.\begin{aligned}
&\sigma_2=(\sigma_2^T)_u,\quad \sigma_1=\tau_{12}=0 \rightarrow F_2(\sigma_2^T)_u+F_{22}(\sigma_2^T)_u^2=1\\
&\sigma_2=-(\sigma_2^C)_u,\sigma_1=\tau_{12}=0\rightarrow -F_2(\sigma_2^C)_u+F_{22}(\sigma_2^C)_u^2=1
\end{aligned}\right\rangle
\left(\begin{aligned}
&F_2=\frac{1}{(\sigma_2^T)_u}-\frac{1}{(\sigma_2^C)_u}\\
&F_{22}=\frac{1}{(\sigma_2^C)_u(\sigma_2^T)_u}
\end{aligned}\right)$$

$$\left.\begin{aligned}
&\tau_{12}=(\tau_{12})_u,\quad \sigma_1=\sigma_2=0 \rightarrow F_6(\tau_{12})_u+F_{22}(\tau_{12})_u^2=1\\
&\tau_{12}=-(\tau_{12})_u,\sigma_1=\sigma_2=0\rightarrow -F_6(\tau_{12})_u+F_{22}(\tau_{12})_u^2=1
\end{aligned}\right\rangle
\left(\begin{aligned}
&F_6=0\\
&F_{66}=\frac{1}{(\tau_{12})_u^2}
\end{aligned}\right)$$

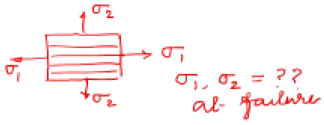
(Refer Slide Time: 45:33)

Strength Failure Theories of Lamina—Tsai-Wu Failure Theory

For equal strength in tension and compression $\rightarrow F_1 = 0, F_2 = 0, F_{11} = \dfrac{1}{\left(\sigma_1^T\right)_u^2}, F_{22} = \dfrac{1}{\left(\sigma_2^T\right)_u^2}$

$$\Rightarrow \boxed{\frac{\sigma_1^{\ 2}}{\left(\sigma_1^T\right)_u^2} + 2F_{12}\sigma_1\sigma_2 + \frac{\sigma_2^{\ 2}}{\left(\sigma_2^T\right)_u^2} + \frac{\tau_{12}^{\ 2}}{\left(\tau_{12}\right)_u^2} = 1}$$

This equation is similar to Tsai-Hill failure criterion except F_{12} for which a biaxial test needs to be conducted.



For equal strength in tension and compression $\rightarrow F_1 = 0, F_2 = 0, F_{11} = \dfrac{1}{\left(\sigma_1^T\right)_u^2}, F_{22} = \dfrac{1}{\left(\sigma_2^T\right)_u^2}$

$$\Rightarrow \frac{\sigma_1^{\ 2}}{\left(\sigma_1^T\right)_u^2} + 2F_{12}\sigma_1\sigma_2 + \frac{\sigma_2^{\ 2}}{\left(\sigma_2^T\right)_u^2} + \frac{\tau_{12}^{\ 2}}{\left(\tau_{12}\right)_u^2} = 1$$

Now still we did not get F_{12} which is associated with $\sigma_1\sigma_2$ and to determine F_{12} we need to apply biaxial stress and need to observe where it fails. That means we take a lamina and apply σ_1 and σ_2 and find out what is the combination of σ_1 and σ_2 at failure. And from that we can get F_{12} but there may be large number of combinations. One condition is only σ_1 and no σ_2, another condition is only σ_2, and no σ_1. So, in between there could be infinite combinations between σ_1 and σ_2.

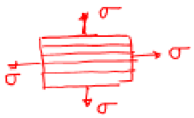
(Refer Slide Time: 47:03)

Strength Failure Theories of Lamina—Tsai-Wu Failure Theory

Thus, by conducting a biaxial test considering $\underbrace{\sigma_1 = \sigma_2 = \sigma}$ and all other stresses are zero

$$(F_1 + F_2)\sigma + (F_{11} + F_{22} + 2F_{12})\sigma^2 = 1$$

$$\Rightarrow \boxed{F_{12} = \frac{1}{2\sigma^2}\left[1 - \left(\frac{1}{(\sigma_1^T)_u} + \frac{1}{(\sigma_1^C)_u} + \frac{1}{(\sigma_2^T)_u} + \frac{1}{(\sigma_2^C)_u}\right)\sigma + \left(\frac{1}{(\sigma_1^T)_u(\sigma_1^C)_u} + \frac{1}{(\sigma_2^T)_u(\sigma_2^C)_u}\right)\sigma^2\right]}$$



$\Rightarrow \sigma = ?$ at failure

For that a biaxial test is conducted with equal stress $\sigma_1 = \sigma_2 = \sigma$ and all other stresses are zero and when putting this we get

$$(F_1 + F_2)\sigma + (F_{11} + F_{22} + 2F_{12})\sigma^2 = 1$$

And when we put the expressions for F_1, F_{11}, F_2, F_{22}, we get

$$\boxed{F_{12} = \frac{1}{2\sigma^2}\left[1 - \left(\frac{1}{(\sigma_1^T)_u} + \frac{1}{(\sigma_1^C)_u} + \frac{1}{(\sigma_2^T)_u} + \frac{1}{(\sigma_2^C)_u}\right)\sigma + \left(\frac{1}{(\sigma_1^T)_u(\sigma_1^C)_u} + \frac{1}{(\sigma_2^T)_u(\sigma_2^C)_u}\right)\sigma^2\right]}$$

Here σ is the stress at failure for the conducted biaxial stress $\sigma_1 = \sigma_2 = \sigma$. So, once we know this F_{12} then we can write the complete the Tsai-Wu theory. It is seen that especially for glass epoxy, it correlates better compared to other interactive theories.

In summary like we have discussed the failure theories for orthotropic lamina. First, we understood how they are different compared to the isotropic materials. The main difference is that here the stresses in the material direction is more important as the strengths of orthotropic lamina are defined with reference to material axis. Therefore, unlike isotropic material we do not determine the principal stresses but we find out the material axis stresses.

Then those material axis stresses are compared to the corresponding strengths. In case of independent non-interactive theories each of these strengths is compared independently and then the failure or safety is assessed. In case of interactive theories, the combined effects of the strengths are actually taken into account by the interaction terms and a single equation predicts whether the failure will occur or not.