Mechanics of Fiber Reinforced Polymer Composite Structures

Prof. Debabrata Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology Guwahati

Module - 3
Macromechanics of Lamina - I
Lecture - 05
Hooke's Law for 2D Lamina

Welcome to the third module of this course Mechanics of Fiber Reinforced Polymer Composite Structures. Till now, we have finished first and second module. In the first module, we had a brief introduction to the composite materials in general and then, we had specific discussions on different terminologies as well as different aspects of the course on fiber reinforced polymer composites like, micromechanics and macromechanics of lamina, laminate macromechanics etc. We understood one important aspect of fiber reinforced polymer matrix composites is that these are anisotropic and therefore analysis of such composite structures needs anisotropic elasticity.

Therefore, in the second module, we had a brief introduction to anisotropic elasticity where we understood that for a fully anisotropic material, 21 independent elastic constants are required to characterize the material. Then we had discussions on the planes of material property symmetry and we understood that in many materials, there exists plane of material property symmetry and the existence of planes of material property symmetry leads to the reduction in required number of independent elastic constants. Existence of one plane of materials property symmetry (monoclinic materials) leads to reduction in independendent elastic constants from 21 to 13. In the case of orthotropic materials where there are 3 mutually perpendicular planes of material property symmetry, only 9 independent elastic constants are required. For isotropic materials, only 2 independent elastic constants are required.

So, with this background understanding, in the third module we will actually discuss macromechanics of lamina. Why macromechanics of lamina? Because, in fiber reinforced polymer composites, actually, the structural components are in the form of laminate which is made by stacking number of laminae. Therefore, in order to understand the mechanics of such composite structures, it is important that we understand the mechanics of lamina. Therefore, in the third module, we will discuss the macromechanics of lamina.

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FOCUS of Module 3

- Macromechanics of Lamina
  - Hooke's law for 2D unidirectional lamina
  - Stiffness and Compliance matrices in terms of Engineering Constants
  - Influence of fiber angle on elastic constants and invariant forms

Macromechanics of Lamina 1

In macromechanics of lamina, we will first discuss Hooke's law for 2-dimensional unidirectional lamina thus establishing the stress-strain relationship for a unidirectional lamina in terms of stiffness and compliance matrix for a lamina.

Next, we will discuss how the elements of these stiffness and compliance matrices actually related to the measurable engineering constants for lamina. In our last lecture, we have seen that in general for an orthotropic material, there are 9 independent elastic constants which are actually related to the measurable engineering constants. Here we will specifically focus our attention in the case of an orthotropic lamina. We will also try to understand what is the influence of fiber angles on the behaviour of a lamina.

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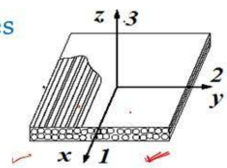
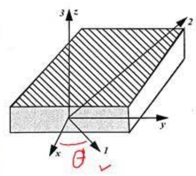
Macromechanics of Lamina

- Lamina – basic form of continuous fiber reinforced composites where a large number of fibers are impregnated into matrix
- – Thin and its thickness << other dimensions



- – Heterogeneous— the properties are location dependent
  eg. Stress strain relation at a point on the matrix is different than at a point on the fiber
- – Anisotropic— the properties are direction dependent
- eg. Properties wrt an axis are dependent on the orientation of the axis



- Unidirectional Lamina are Orthotropic– three mutually perpendicular planes of material property symmetry– Principal Material Axes (1-2-3)
- Specially Orthotropic Lamina - Analysis axes (x-y-z) coincide with the material axes (1-2-3)
- Generally Orthotropic Lamina - Analysis axes (x-y-z) do not coincide with the material axes (1-2-3)

Macromechanics of Lamina 2

Before we actually start macromechanics of a lamina, let us quickly revisit that some of the important features of a lamina. A lamina, as we know, is the basic form of continuous fiber reinforced composite structures. This is nothing but a large number of fibers which are impregnated into matrix. As shown in the figure, a lamina is very thin and its thickness is much smaller compared to its other two dimensions.

In addition, a lamina is heterogeneous, because it consists of two constituents, (viz. the fiber and the matrix) whose material properties are different. Heterogeneous material is a material where the properties are location dependent. Suppose in a lamina, if I randomly put the tip of my pen, if it is on the fiber, then the material properties are different compared to if it is on the matrix, then the material properties are different. Thus the stress-strain relation at a point on the matrix is different than that at a point on the fiber and hence, it is heterogeneous. More importantly, a lamina is anisotropic (orthotropic), unidirectional lamina is orthotropic. Anisotropic materials are those materials where the properties are direction dependent. In a lamina, the stiffness along say, longitudinal direction is different than that along the transverse direction and therefore, it is direction dependent.

However, in a unidirectional lamina, because it is orthotropic, there are 3 mutually perpendicular planes of material property symmetry and is not fully anisotropic. It requires 9 independent elastic constants. A unidirectional lamina could be either specially orthotropic where, the analysis axes, coincide with the material axes (refer to the Figure, 1-2-3 are the planes of orthotropy, where 1 means the 2-3 plane).

A plane is actually characterised by a surface normal. So, 1, 2 and 3 are the directions of orthotropy or the planes of material property symmetry. Now, there are cases when the loading axes or the analysis axes do not coincide with the axes of orthotropy (material axes) and the lamina is called a generally orthotropic lamina. As shown in the Fig, for the specially orthotropic, x-y-z (analysis axes or global axes) coincides with 1-2-3 (material axes or local axes). On the other hand, x-y-z (analysis axes or global axes) does not coincide with 1-2-3 (material axes or local axes) in the case of a generally orthotropic lamina and the direction / orientation of fiber (direction 1) makes an angle θ with the x-axes. Therefore, depending upon what is the fiber orientation angle, it may be a specially orthotropic or a generally orthotropic lamina. If if x-y-z coincides 1-2-3 with , then the angle between 1 and x will be 0^{o} or 90^{o} and it will be actually a specially orthotropic lamina.

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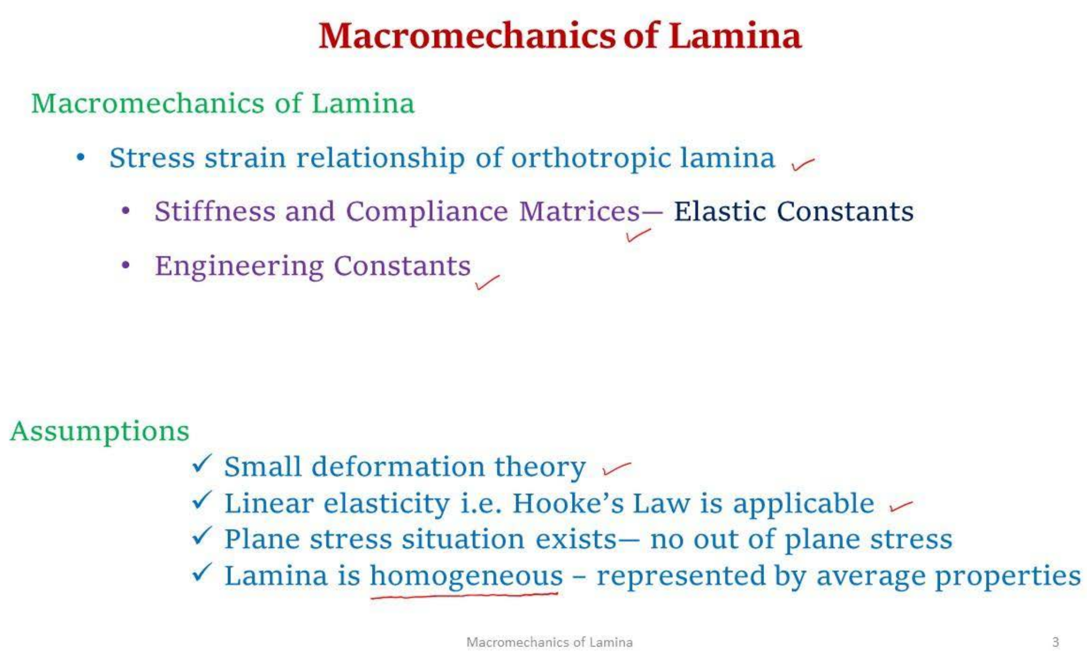

With this background understanding, the objective of this lecture will be to develop the stress-strain relationship of a lamina, keeping in mind that lamina is very thin, and we will try to establish the stiffness and compliance matrix for a lamina. Then, analogous to what we have done for an orthotropic material we will try to look at the engineering constants for an orthotropic unidirectional lamina.

In studying macromechanics of lamina, the following assumptions are made

- Small deformation
- Linearly elastic following Hooke's law
- Thin lamina and hence plane stress situation and no out-of-plane stresses
- Lamina is homogenous and represented by average properties.

The fact that it is macromechanics analysis, the lamina is considered to be homogeneous even though a lamina is actually heterogeneous (consists of fiber and matrix) and the properties are actually location dependent. But in order to understand the behaviour of the lamina, we will consider this to be macroscopically homogeneous meaning that they are represented by their average properties.

How these average properties are obtained? We take a lamina and load it in UTM and try to draw its stress-strain diagram, we can get its Young's modulus in a particular direction. So, even though that Young's modulus in that direction is actually decided by the Young's modulus of the fiber, Young's modulus of the matrix and relative proportion of the fiber and the and the matrix, but in macromechanical analysis, we will go ahead with those average properties.

When we will study the micromechanics analysis, we will see in details how the constituents properties and the relative presence of the constituent actually dictate the effective property of a lamina.

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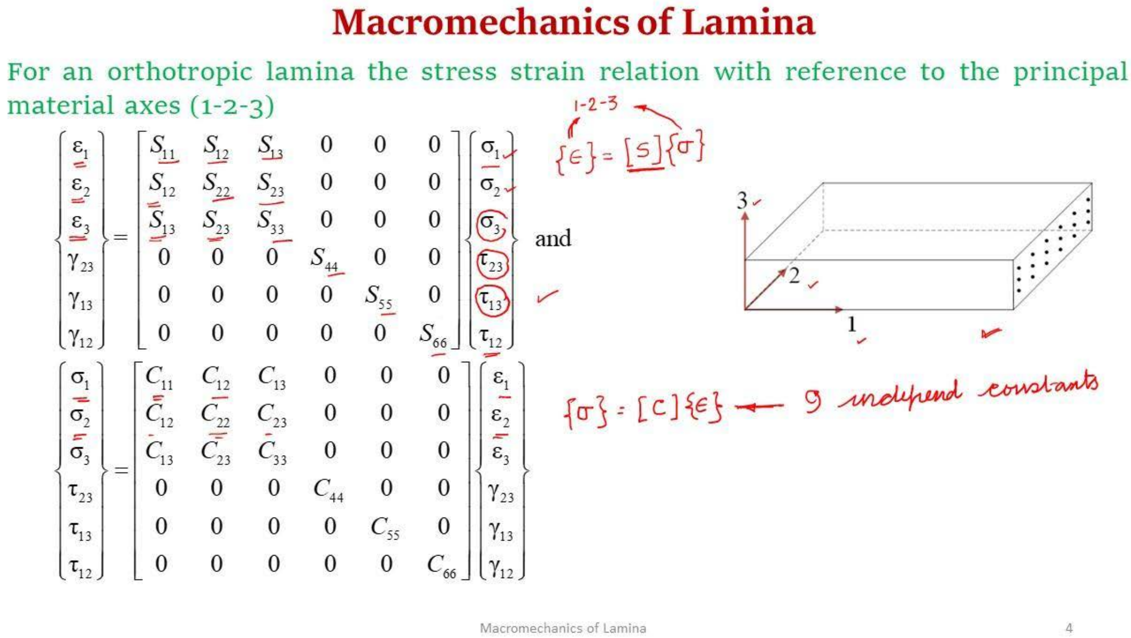

We have seen in the last module that for an orthotropic material, the six components of strains are related to the corresponding six components of stresses by means of the compliance matrix [S] as

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

Here, the stresses and strains are actually with respect to 1-2-3 axes, which are nothing but the direction of orthotropy. Similarly, the stresses are related to the strains by the stiffness matrix [C] (which is the inverse of the compliance matrix [S]) as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

The lamina is thin and the thickness is much less compared to its lateral dimensions, though the thickness is exaggerated in the Fig.

We have already discussed that both [S] and [C] are symmetric and we need actually 9 independent elastic constants. So, this is for any 3-dimensional orthotropic body.

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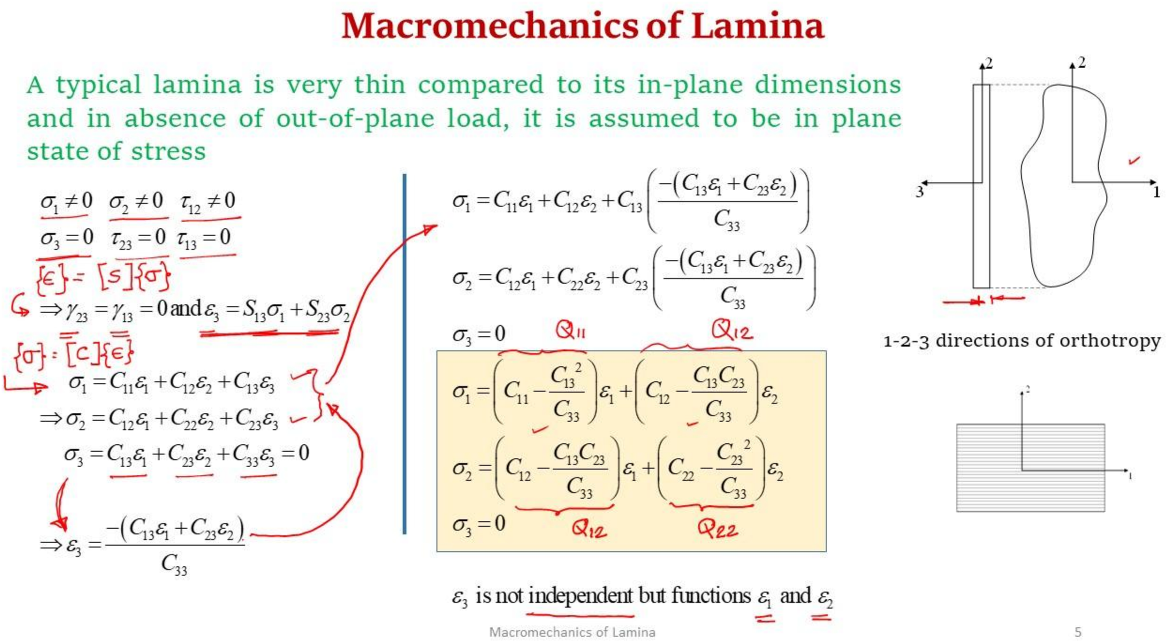

So, now, if a typical lamina, because it is very thin and suppose no out-of-plane load is applied, it could be assumed to be in the state of plane stress ie. the load is only in the 1-2 plane and there is no variation of the load distribution along the thickness direction.

In a 2D lamina, therefore, only 3 components of stresses exist, two in-plane normal stresses, σ_1, σ_2 and one in-plane shear stress τ_{12} and out-of-plane stresses like σ_3, τ_{23}, τ_{13} are 0. Now, suppose if we put this in this in Eqn

$$\sigma_1 \neq 0 \quad \sigma_2 \neq 0 \quad \tau_{12} \neq 0$$
$$\sigma_3 = 0 \quad \tau_{23} = 0 \quad \tau_{13} = 0$$
$$\Rightarrow \gamma_{23} = \gamma_{13} = 0 \text{ and } \varepsilon_3 = S_{13}\sigma_1 + S_{23}\sigma_2$$

$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3$$
$$\sigma_2 = C_{12}\varepsilon_1 + C_{22}\varepsilon_2 + C_{23}\varepsilon_3$$
$$\sigma_3 = C_{13}\varepsilon_1 + C_{23}\varepsilon_2 + C_{33}\varepsilon_3 = 0 \Rightarrow \boxed{\varepsilon_3 = \frac{-(C_{13}\varepsilon_1 + C_{23}\varepsilon_2)}{C_{33}}}$$

And putting this $\varepsilon_3 = S_{13}\sigma_1 + S_{23}\sigma_2$ in the expressions for , σ_1, σ_2

$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\left(\frac{-(C_{13}\varepsilon_1 + C_{23}\varepsilon_2)}{C_{33}}\right)$$

$$\sigma_2 = C_{12}\varepsilon_1 + C_{22}\varepsilon_2 + C_{23}\left(\frac{-(C_{13}\varepsilon_1 + C_{23}\varepsilon_2)}{C_{33}}\right)$$

$$\sigma_3 = 0$$

$$\sigma_1 = \left(C_{11} - \frac{C_{13}{}^2}{C_{33}}\right)\varepsilon_1 + \left(C_{12} - \frac{C_{13}C_{23}}{C_{33}}\right)\varepsilon_2$$

$$\sigma_2 = \left(C_{12} - \frac{C_{13}C_{23}}{C_{33}}\right)\varepsilon_1 + \left(C_{22} - \frac{C_{23}{}^2}{C_{33}}\right)\varepsilon_2$$

$$\sigma_3 = 0$$

So, we get the relationship between σ_1, σ_2 and ε_1, ε_2 by this.

$$\sigma_1 = \left(C_{11} - \frac{C_{13}{}^2}{C_{33}}\right)\varepsilon_1 + \left(C_{12} - \frac{C_{13}C_{23}}{C_{33}}\right)\varepsilon_2 = Q_{11}\varepsilon_1 + Q_{12}\varepsilon_2$$

$$\sigma_2 = \left(C_{12} - \frac{C_{13}C_{23}}{C_{33}}\right)\varepsilon_1 + \left(C_{22} - \frac{C_{23}{}^2}{C_{33}}\right)\varepsilon_2 = Q_{21}\varepsilon_1 + Q_{22}\varepsilon_2$$

$$\sigma_3 = 0$$

Of course, there is a third normal strain ε_3, but this is not independent and is a function of ε_1 and ε_2. Therefore, this is dropped from the stress-strain relationship.

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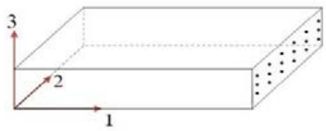
Hooke's Law for Specially Orthotropic Lamina

Stress strain relation for an orthotropic lamina w.r.t. the axes of orthotropy

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \text{ and } \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$



[Q] is the Reduced Stiffness matrix and these are $\boxed{Q_{ij} = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}} \quad i, j = 1, 2, 6}$

4 independent constants

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2}; \; Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2}$$

$$Q_{12} = \frac{S_{12}}{S_{11}S_{22} - S_{12}^2}; Q_{66} = \frac{1}{S_{66}}$$

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So, we could write the relationship between the in-plane stresses to the corresponding in-plane strain in 1-2 axes as

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \text{ and } \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

Please note again that $\varepsilon_3 \neq 0$, but this is not independent either. This is actually dependent on ε_1 and ε_2, therefore, it is not put in the stress-strain matrix. So, the stress-strain relationship finally goes to relating the 2 in-plane normal stresses σ_1, σ_2 and one shear stress τ_{12} to the corresponding normal strains and shear stress by means of this [Q] matrix or taking inverse with, by means of [S] matrix.

This [Q] is known as reduced stiffness matrix. Reduced means, we have actually reduced it to 2-dimensional stress-strain relationship from the coefficient of the stiffness matrix in 3D. Therefore, it is reduced stiffness matrix. And because [Q] could be written in terms of [S] by the relation $[Q]=[S]^{-1}$, therefore, we can also write the relationship between the terms of the reduced stiffness matrix with this, the compliance matrix as

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2}\ ;\ Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2}$$

$$Q_{12} = \frac{S_{12}}{S_{11}S_{22} - S_{12}^2}\ ;\ Q_{66} = \frac{1}{S_{66}}$$

So, in short form [Q] matrix in terms of C is $\boxed{Q_{ij} = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}}}$ $i, j = 1, 2, 6$ Why 1 to 6? You can see here; because these are the in-plane components, σ_1, σ_2 and τ_{12}. So, we have established the reduced stiffness matrix and the compliance matrix for an orthotropic unidirectional lamina where of course the stresses and strains are related, with respect to the 1-2-3 or the directions of orthotropy.

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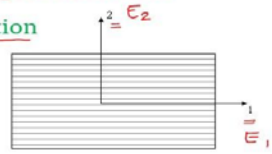
Engineering Constants for Specially Orthotropic Lamina

Considering a UD lamina with 1-2 as the principal material direction



E_1 is the Longitudinal Young's Modulus (along 1)

E_2 is the Transverse Young's Modulus (along 2)

ν_{12} is the major Poisson's Ratio defined as $\nu_{12} = -\dfrac{\varepsilon_2}{\varepsilon_1}$ when only σ_1 is acting and

ν_{21} is the ~~minor~~ minor Poisson's Ratio defined as $\nu_{21} = -\dfrac{\varepsilon_1}{\varepsilon_2}$ when only σ_2 is acting

G_{12} is the in-plane (1-2) shear modulus

Experimentally, E_1, E_2, ν_{12}, G_{12} could be measured and related to four (4) elements of compliance matrix

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Now, let us see what are the corresponding engineering constants? We can easily represent the stress-strain relationship in terms of the coefficients of the compliance or stiffness matrix. However, these are not measurable and the measurable quantities are the engineering constants like Young's modulus, Poisson's ratio and shear modulus. We have discussed already that for isotropic material, we have only 2 measurable engineering constants E and ν. But in the case of orthotropic materials, we need Young's modulus with a suffix, because it is direction dependent, therefore we need E_1, E_2, E_3.

Similarly, we need Poisson's ratios as ν_{12}, ν_{23}, ν_{31}. In the same way, we need shear moduli as G_{12}, G_{23}, G_{31}, because they are direction dependent. Now, let us consider a unidirectional lamina with 1-2 as the principal material direction. What is principal material direction? 1 is the longitudinal direction the fiber, 2 is the transverse direction of the fiber. So, E_1 is called the longitudinal Young's modulus, means Young's modulus along 1. E_2 is the Young's modulus along direction 2, which is also known as the transverse Young's modulus. ν_{12} is the major Poisson's ratio. It is major, because by definition ν_{12} is the ratio of the lateral strain to the longitudinal strain when only longitudinal stress is applied.

Therefore, $\nu_{12} = -\varepsilon_2/\varepsilon_1$, when σ_1 is acting, and this is called major Poisson's ratio. The minor Poisson's ratio $\nu_{21} = -\varepsilon_1/\varepsilon_2$ is defined as the ratio of the strain along 1 to the strain along 2 when only stress is applied along direction 2. G_{12} is the in-plane shear modulus. Conducting experiments, if we load a unidirectional lamina in the longitudinal direction and look at the stress-strain curve, the slope will be E_1.

Similarly, if we load in the transverse direction, we can find out E_2. From either of these, we can find out ν_{12} and ν_{21}. So, experimentally, these could be measured and related to the 4 elements of compliance matrix.

Similarly, the four elements of the reduced stiffness matrix viz. Q_{11}, Q_{12}, Q_{22}, Q_{66} could be related to the engineering properties of a unidirectional lamina. Now, let us see how we do that.

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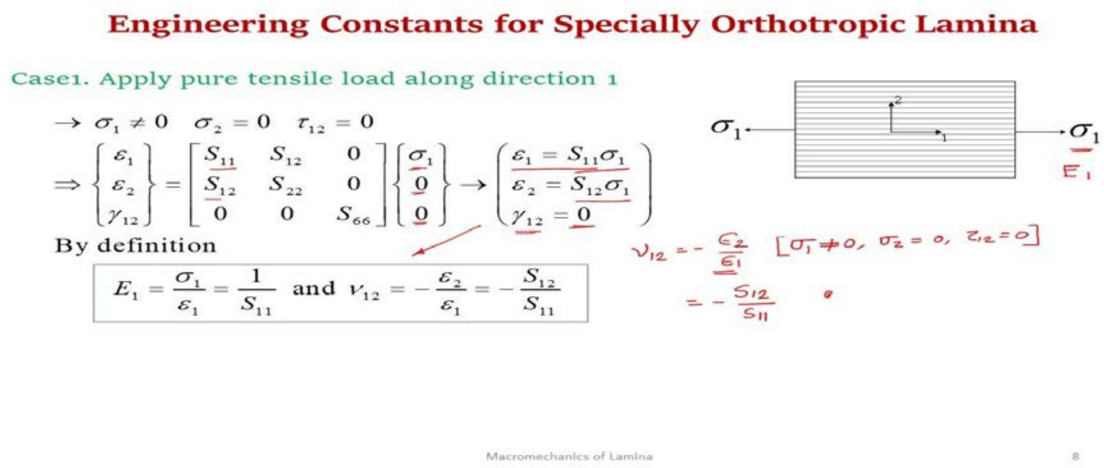

Suppose we apply a pure tensile load along direction 1, along the longitudinal direction; we apply only σ_1 and other 2 stresses viz. σ_2 and τ_{12} are 0. Therefore,

$$\rightarrow \sigma_1 \neq 0 \quad \sigma_2 = 0 \quad \tau_{12} = 0$$

$$\Rightarrow \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ 0 \\ 0 \end{Bmatrix} \rightarrow \begin{pmatrix} \varepsilon_1 = S_{11}\sigma_1 \\ \varepsilon_2 = S_{12}\sigma_1 \\ \gamma_{12} = 0 \end{pmatrix}$$

Now, this lamina is orthotropic. We have discussed that in an orthotropic material, there is no shear-extension coupling, ie. if we apply normal stress, there will not be any shear strain unlike a fully anisotropic material where all the components of stresses are actually coupled with each other.

So, naturally, here if we apply only σ_1, it did not result in any shear strain.

Now, we have applied σ_1 as the stress along 1 and the corresponding strain is ε_1. By definition, Young's modulus along 1, E_1 is defined as the ratio of σ_1/ε_1. Similarly, by definition of Poisson's ratio, $\nu_{12} = -\varepsilon_2/\varepsilon_1$ when only σ 1 is applied, meaning $\sigma_2 = 0$, $\tau_{12} = 0$.

By definition $\quad E_1 = \frac{\sigma_1}{\varepsilon_1} = \frac{1}{S_{11}}$ and $\nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1} = -\frac{S_{12}}{S_{11}}$

So, we obtain the relationship between the engineering constants and the coefficients of the compliance matrix.

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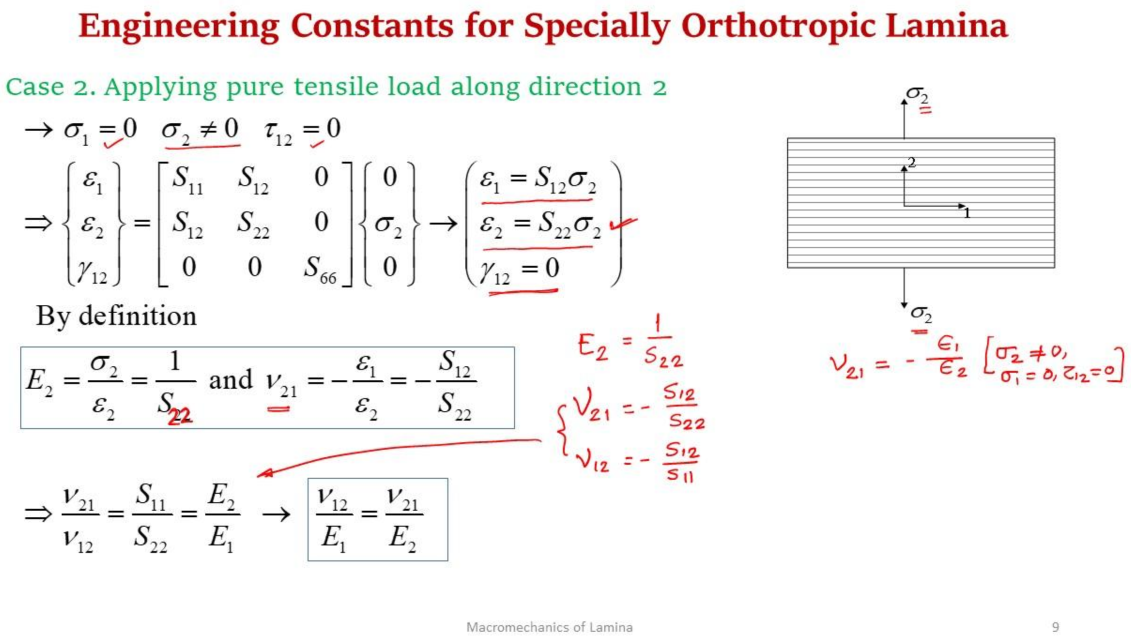

Next, what we can do is, suppose we apply only σ_2 along the transverse direction, so,

$$\rightarrow \sigma_1 = 0 \quad \sigma_2 \neq 0 \quad \tau_{12} = 0$$

$$\Rightarrow \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} 0 \\ \sigma_2 \\ 0 \end{Bmatrix} \rightarrow \begin{pmatrix} \varepsilon_1 = S_{12}\sigma_2 \\ \varepsilon_2 = S_{22}\sigma_2 \\ \gamma_{12} = 0 \end{pmatrix}$$

Therefore, again using the stress-strain relationship in terms of compliance matrix, we can write that σ_2 will lead to a strain along 1, which is $\varepsilon_1 = S_{12}\sigma_2$, and of course, a direct strain along 2, which is $\varepsilon_2 = S_{22}\sigma_2$, and there will be no shear strain because it is orthotropic.

By definition

$$E_2 = \frac{\sigma_2}{\varepsilon_2} = \frac{1}{S_{11}} \text{ and } \nu_{21} = -\frac{\varepsilon_1}{\varepsilon_2} = -\frac{S_{12}}{S_{22}}$$

$$\Rightarrow \frac{\nu_{21}}{\nu_{12}} = \frac{S_{11}}{S_{22}} = \frac{E_2}{E_1} \rightarrow \boxed{\frac{\nu_{12}}{E_1} = \frac{\nu_{12}}{E_2}}$$

If you remember, we have also obtained this relation between major and minor Poisson's ratio in the case of orthotropic material in general. So, even though we have major and minor Poisson's ratios, but they are not independent and are related by the ratio of E_1 and E_2.

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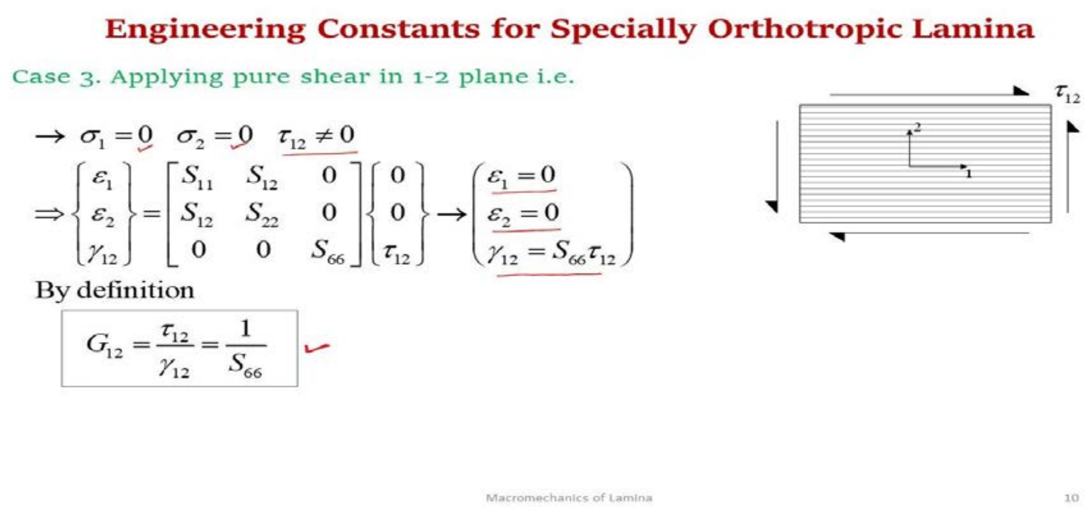

Next, suppose we apply a pure shear in an orthotropic lamina, that means, only $\tau_{12} \neq 0$, there is no normal stress; again, using the stress-strain relationship in terms of the compliance matrix,

$$\rightarrow \sigma_1 = 0 \quad \sigma_2 = 0 \quad \tau_{12} \neq 0$$

$$\Rightarrow \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \tau_{12} \end{Bmatrix} \rightarrow \begin{pmatrix} \varepsilon_1 = 0 \\ \varepsilon_2 = 0 \\ \gamma_{12} = S_{66}\tau_{12} \end{pmatrix}$$

Only shear stress is applied, therefore, there will not be any normal strain as it is orthotropic material, but there will be corresponding shear strain and the corresponding shear strain is $S_{66}\tau_{12}$. Now, going by the definition of shear modulus as the ratio of the shear stress to shear strain. By definition

$$G_{12} = \frac{\tau_{12}}{\gamma_{12}} = \frac{1}{S_{66}}$$

Thus we have established a relationship between the elements of the compliance matrix and the engineering constants for an orthotropic lamina.

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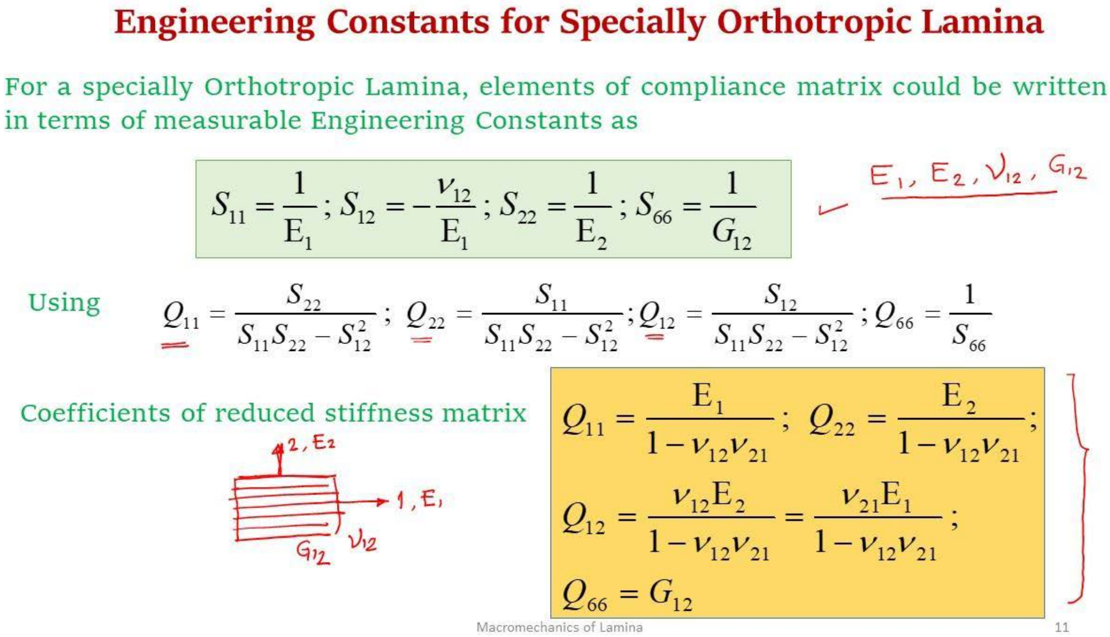

Therefore, for a specially orthotropic lamina (analysis axes 1-2-3 is also the direction of orthotropy) the elements of compliance matrix could be written in terms of engineering constants as

$$S_{11}=\frac{1}{E_1}; S_{12}=-\frac{\nu_{12}}{E_1}; S_{22}=\frac{1}{E_2}; S_{66}=\frac{1}{G_{12}}$$

where E_1 and E_2 are the Young's modulus along the longitudinal and transverse direction respectively, ν_{12} is the Poisson's ratio with reference to 1-2 plane and G_{12} is the shear modulus in the plane 1-2. We have not shown ν_{21} here, because, this is dependent on ν_{12}. Therefore, we actually need only 4 independent engineering constants, E_1, E_2, ν_{12}, G_{12}, which are related to the elements of the compliance matrix. These could also be written in terms of elements of reduced stiffness matrix, because reduced stiffness matrix and compliance matrix terms are related as $[Q]=[S]^{-1}$.

$$Q_{11}=\frac{E_1}{1-\nu_{12}\nu_{21}};\ Q_{22}=\frac{E_2}{1-\nu_{12}\nu_{21}};$$
$$Q_{12}=\frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}=\frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}};$$
$$Q_{66}=G_{12}$$

So, these are the elements of reduced stiffness matrix in terms of the engineering constants.

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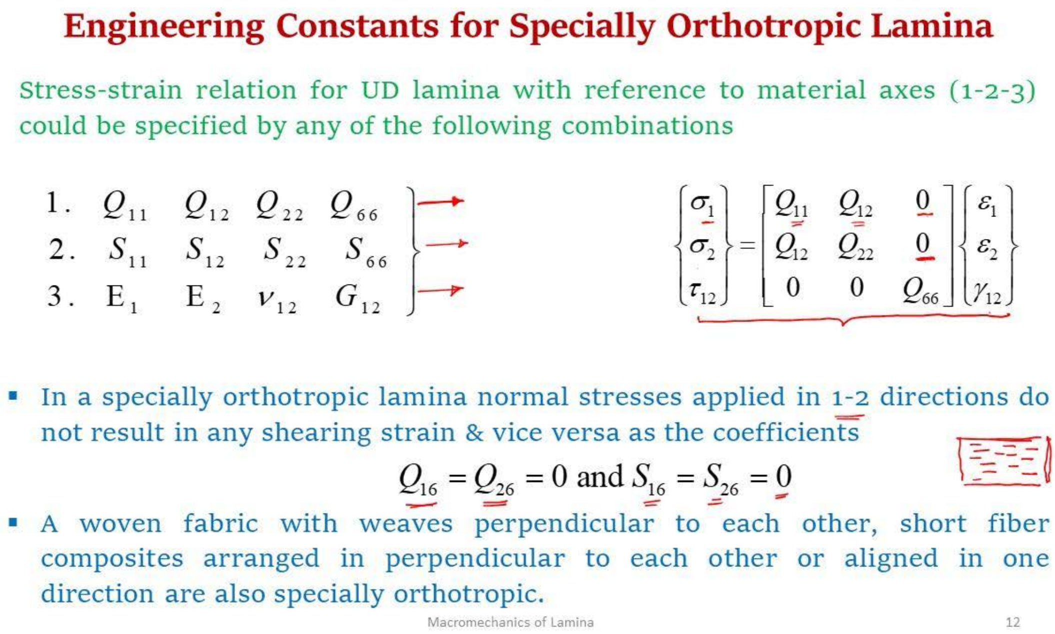

So, the stress-strain relationship for a unidirectional specially orthotropic lamina in the material axes 1-2 could be specified by any one of the following three combinations:

$$\left.\begin{array}{lllll} 1. & Q_{11} & Q_{12} & Q_{22} & Q_{66} \\ 2. & S_{11} & S_{12} & S_{22} & S_{66} \\ 3. & E_1 & E_2 & \nu_{12} & G_{12} \end{array}\right\}$$

We can write either in terms of these 4 reduced stiffness matrix elements or the compliance matrix or these 4 engineering constants.

Now, what is important here is that, again, in a specially orthotropic lamina, the normal stress is applied in 1-2 directions do not result in any shearing stain and vice versa. The reason is, Q_{16}, Q_{26} and S_{16}, S_{26} are 0. Had those not been 0, there would have been a relation between σ_1 and γ_{12} and vice versa. Thus there is no shear-extension coupling in a specially orthotropic lamina. Even though we have done this only for a unidirectional lamina and a continuous fiber lamina, even in a short fiber composites arranged perpendicular to each other or aligned in one direction are also specially orthotropic. If they are perfectly aligned, then they also show orthotropic properties.

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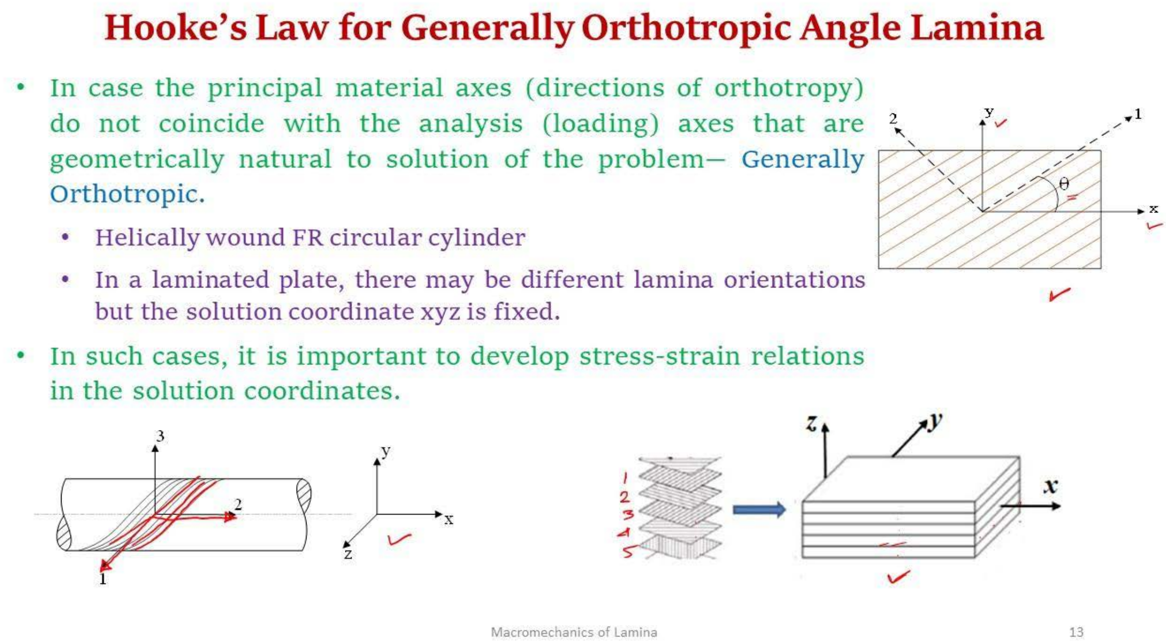
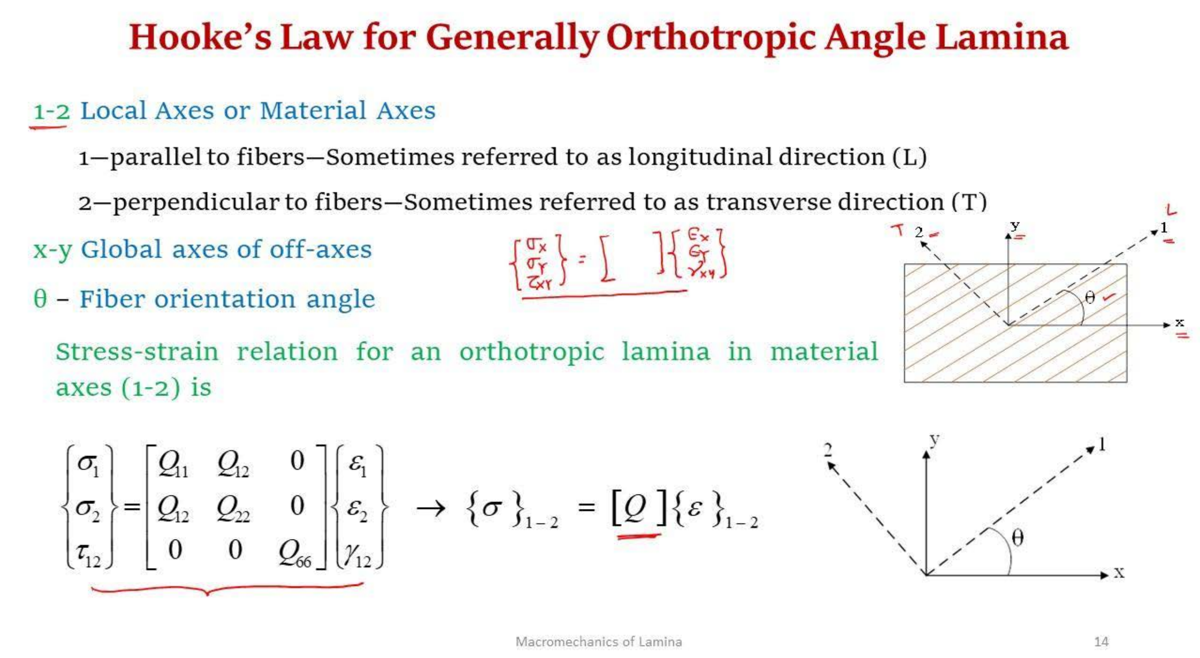


We have developed the relationship for a lamina with reference to its direction of orthotropy ie for a specially orthotropic lamina. Now, there are cases when the principal material axes (1-2_3) do not coincide with the analysis axes (x-y-z) that are geometrically natural to the solution of certain problems. For example, suppose helically wound fiber reinforced circular cylinder (as in Fig.) and we would like to analyze this with reference to x-y-z plane, not with reference to 1-2 plane. So, a lamina where the material axes actually do not coincide with the analysis axes is generally orthotropic.

Another example is, suppose, as you know that a laminate actually consists of large number of laminas stacked together and each of these laminae would have different fiber orientation. Now, suppose there is a laminate (as in Fig.), with laminae 1, 2, 3, 4, 5. Now, these 5 laminae have different orientations. Maybe one of these is having 0°, whose 1-2-3 coincides with this x-y-z, but suppose the second one is 45°, then, for this x-y do not coincides with 1-2.

For a particular lamina, we would always like to have the solutions with reference to x-y, which is natural to this problem and do not coincide with the 1-2. So, it is a specially orthotropic lamina. We have obtained the stress-strain relationship in a lamina with reference to 1-2 plane. So, in such cases, it is important that we develop the stress-strain relationship with respect to x-y which is not coincident with 1-2.

So, this is what we are going to do now for a generally orthotropic lamina, or an angle lamina where the material axes (1-2-3) make certain angle with this analysis axes (x-y-z). In this case, the θ = fiber angle. **(Refer Slide Time: 36:13)**

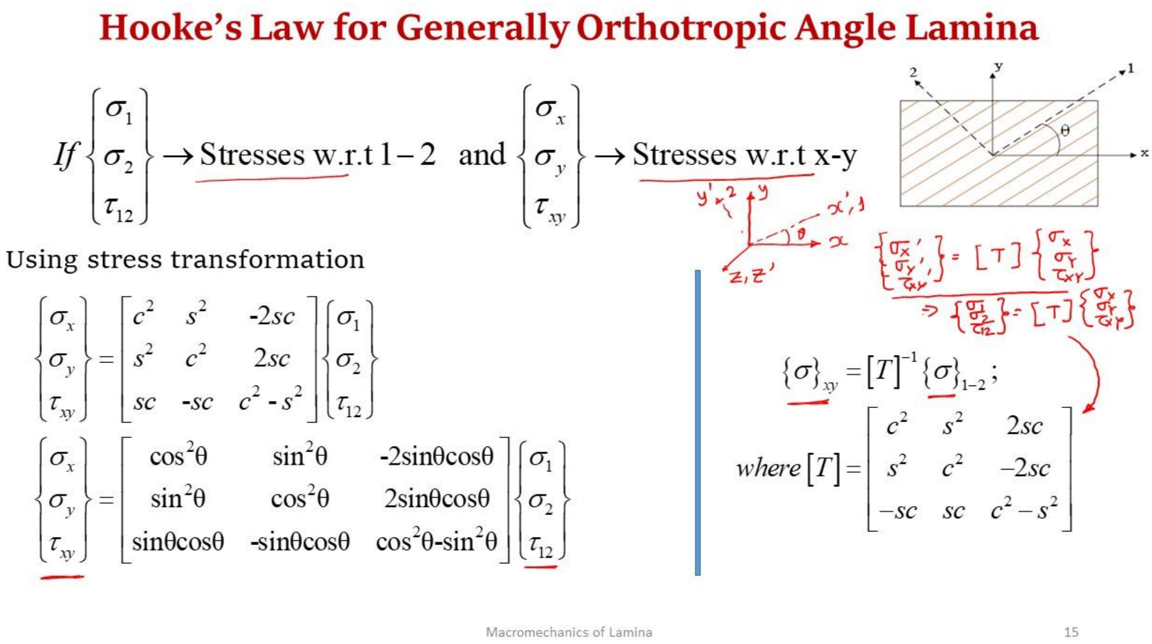


Referring to the Fig., 1-2 is the local axes or material axes for a lamina, where 1 is parallel to fiber (also referred to as longitudinal direction) and 2 is perpendicular to the fiber; sometimes referred to as transverse direction). This 1-2 axes is called the local axes or material axes and x-y which is the solution axes or the analysis axes is called global axes or off-axes. θ is called the fiber orientation angle. If θ = 0, then x coincides with 1 and it is specially orthotropic lamina.

Now, our objective here is to develop a relationship between the stresses and the corresponding strains with respect to x-y. We have already done with reference to 1-2, but now we want to do with reference to x-y.

Now, we know that we can transform stresses and strain ie. if we know the state of stress with respect to a Cartesian coordinates x-y-z, we can actually write down the stresses with respect to any rotated axis x2-y2-z2, provided we know the direction cosines. We will be using this to establish a relationship between the stresses and strains with respect to x-y plane.

Now, already we have developed the relations between the stresses and the corresponding strains that with reference to 1-2 axes in terms of the reduced stiffness matrix [Q]. **(Refer Slide Time: 38:41)**



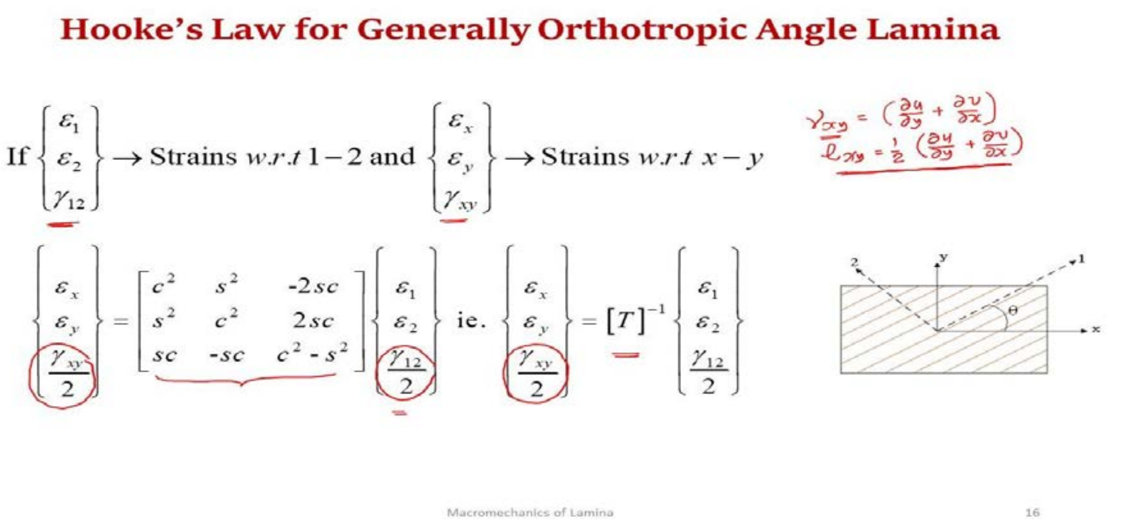
Now, suppose σ_1, σ_2, τ_{12} are the stresses with respect to 1-2 axes and σ_x, σ_y, τ_{xy} are the stresses with respect to x-y axes. Here 1-2 is obtained by rotation (by θ) about z-axis. Say, basically this is x, this is y and say this is z; we rotate this x2, y2 by θ, so that it is a rotation about z-axis. Now, in this case, this x2 is 1 and y2 is 2. So, we know that how the stresses transform.

So, σ_1, σ_2, τ_{12}, are actually related to σ_x, σ_y, τ_{xy} by means of this transformation matrix as

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -2sc\\ s^2 & c^2 & 2sc\\ sc & -sc & c^2-s^2\end{bmatrix}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix}$$ or in short

$$\{\sigma\}_{xy} = [T]^{-1}\{\sigma\}_{1-2};$$

Where $$[T] = \begin{bmatrix} c^2 & s^2 & 2sc\\ s^2 & c^2 & -2sc\\ -sc & sc & c^2-s^2\end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta\\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta\\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta-\sin^2\theta\end{bmatrix}$$ is the stress transformation matrix. Therefore, we can write the stresses with reference to x-y by taking $[T]^{-1}$ in terms of the stresses with reference to 1-2. **(Refer Slide Time: 41:13)**



So, having done that, similarly,we can also perform the stain transform analogous to this as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{\gamma_{xy}}{2} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{\gamma_{12}}{2} \end{Bmatrix} \text{ ie. } \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{\gamma_{xy}}{2} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{\gamma_{12}}{2} \end{Bmatrix} \text{ where}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \to \text{Strains } w.r.t\ 1-2 \text{ and } \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \to \text{Strains } w.r.t\ x-y$$

It is important that we remember it is a tensorial transformation, but this γ is the engineering shear strain and we know that this γ_{xy} is actually ™u/™y + ™v/™x, but actually the tensorial strain is nothing but (™u/™y + ™v/™x)/2. Therefore, to use the same transformation matrix, we have to divide the shear strain by 2.

So, we can use the same transformation but if we write only γ_{xy}, it will be wrong.

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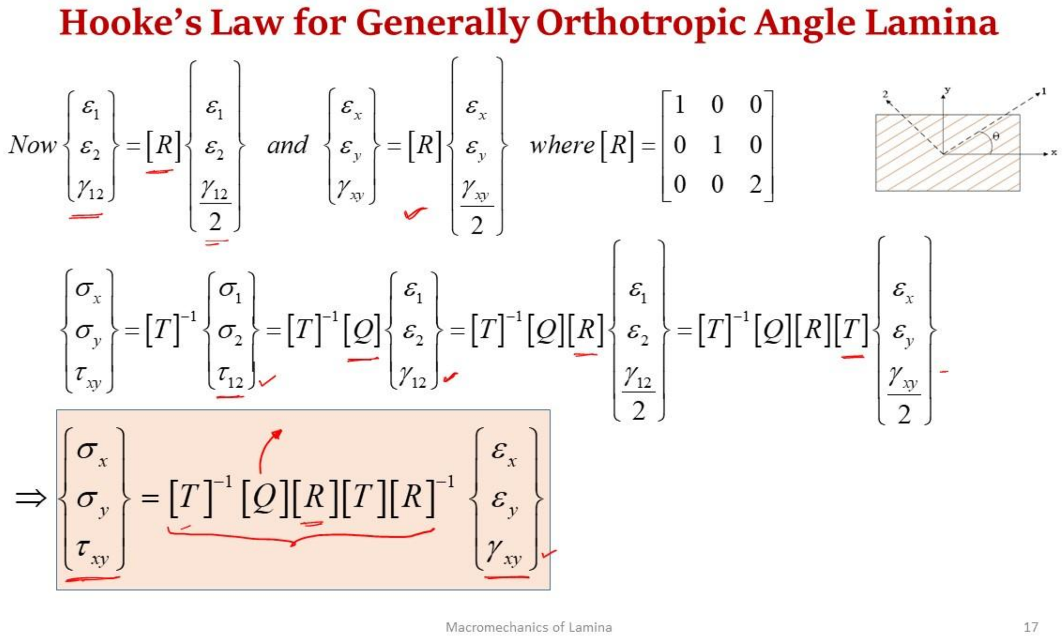

Now, our objective is to relate the stresses in the x-y to the corresponding strain and vice versa and we could do that as follows.

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [R] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{\gamma_{12}}{2} \end{Bmatrix} \quad and \quad \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [R] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{\gamma_{xy}}{2} \end{Bmatrix} \quad where\,[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [T]^{-1}[Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [T]^{-1}[Q][R] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{\gamma_{12}}{2} \end{Bmatrix} = [T]^{-1}[Q][R][T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{\gamma_{xy}}{2} \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1}[Q][R][T][R]^{-1} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

This is how we obtain the stresses in the x-y plane are actually related to the corresponding strains in the x-y plane by means of this multiplication of 5 matrices.

Note that in this, the [Q] is the reduced stiffness matrix, which is the property of that lamina, and could be expressed in terms of the engineering constants. So, if we multiply this, matrices $[T]^{-1}[Q][R][T][R]^{-1}$ we can write the stresses in terms of strains, with respect to x-y as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \rightarrow \{\sigma\}_{xy} = \left[\overline{Q}\right]\{\varepsilon\}_{xy}$$

$$where \begin{Bmatrix} \overline{Q}_{11} = Q_{11}c^4 + 2\left(Q_{12} + 2Q_{66}\right)s^2c^2 + Q_{22}s^4 \\ \overline{Q}_{12} = \left(Q_{11} + Q_{22} - 4Q_{66}\right)c^2s^2 + Q_{12}\left(s^4 + c^4\right) \\ \overline{Q}_{22} = Q_{11}s^4 + 2\left(Q_{12} + 2Q_{66}\right)c^2s^2 + Q_{22}c^4 \\ \overline{Q}_{16} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)s\ c^3 - \left(Q_{22} - Q_{12} - 2Q_{66}\right)s^3c \\ \overline{Q}_{26} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)s^3c\ - \left(Q_{22} - Q_{12} - 2Q_{66}\right)s\ c^3 \\ \overline{Q}_{66} = \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)c^2s^2 + Q_{66}\left(s^4 + c^4\right) \end{Bmatrix}$$

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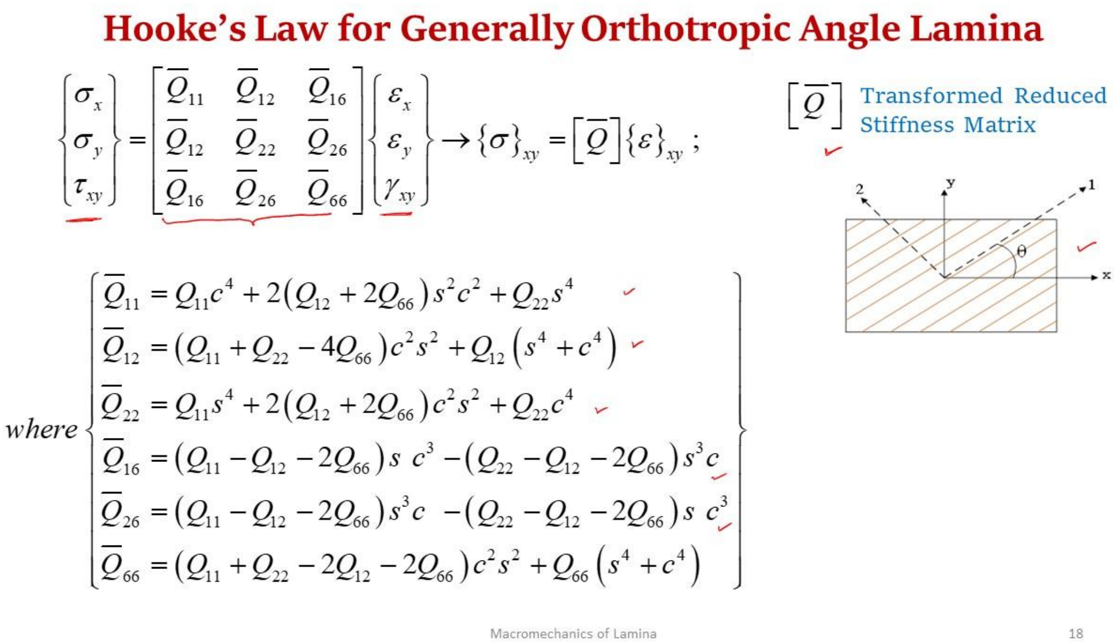

They are related to the strains with respect to x-y by this matrix, $\overline{[Q]}$. This $\overline{Q}$ is called the transformed reduced stiffness matrix. If you remember, Q was the reduced stiffness matrix, because itwas reduced from 3-dimension to 2-dimension. Now, because it is transformed from 1-2 to x-y, using stress and strain transformation, therefore, it is transformed reduced stiffness matrix. So, could be seen that all the terms of $\overline{[Q]}$ are nothing but a functions of Q_{11}, Q_{12}, Q_{22}, Q_{66} and sinθ, cosθ. This is because, [R] consists of numbers and [T] is actually function of sinθ and cosθ. Therefore, you can see that all these elements of the reduced transformed stiffness matrix are actually expressed in terms of the reduced stiffness matrix and θ, the fiber orientation angle or the angle which the 1-2 axes make with the x-y axes.

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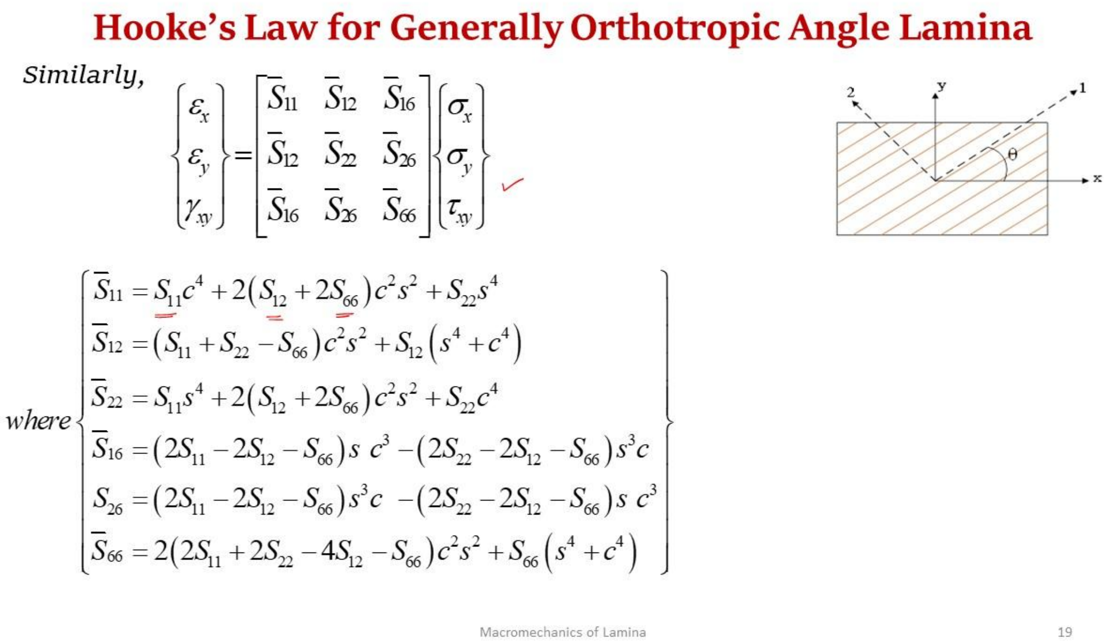

Similarly, we can also write the transformed compliance matrix as

$$
\begin{aligned}
\bar{S}_{11} &= S_{11}c^4 + 2(S_{12} + 2S_{66})c^2s^2 + S_{22}s^4 \\
\bar{S}_{12} &= (S_{11} + S_{22} - S_{66})c^2s^2 + S_{12}(s^4 + c^4) \\
\bar{S}_{22} &= S_{11}s^4 + 2(S_{12} + 2S_{66})c^2s^2 + S_{22}c^4 \\
\bar{S}_{16} &= (2S_{11} - 2S_{12} - S_{66})s\,c^3 - (2S_{22} - 2S_{12} - S_{66})s^3c \\
S_{26} &= (2S_{11} - 2S_{12} - S_{66})s^3c - (2S_{22} - 2S_{12} - S_{66})s\,c^3 \\
\bar{S}_{66} &= 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})c^2s^2 + S_{66}(s^4 + c^4)
\end{aligned}
$$

Again, the terms of the transformed compliance matrix are also functions of S_{11}, S_{12}, S_{22}, S_{66} and $\cos\theta$, $\sin\theta$.

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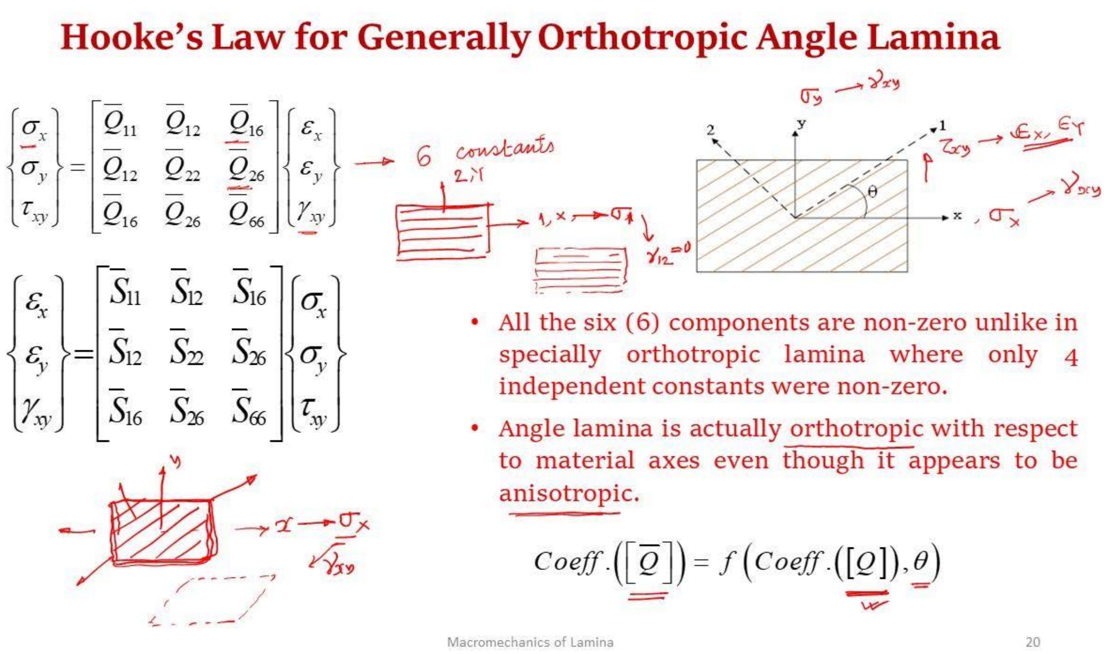

Therefore, Hooke's law for a generally orthotropic lamina could be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \text{ and } \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

It is important to notre that unlike specially orthotropic lamina, wher Q_{16}, $Q_{26} = 0$, here all the 6 components are non-zero that means six constants are required to relate the stresses and strains. For specially orthotropic lamina (stress-strain relationship with reference to 1-2), only four constants are required. Because here two more non zero terms Q_{16} and Q_{26} are also there. The significance of these terms are that they actually relate the normal stresses to shear strains and vice versa. That means, now if we apply a normal stress along x, σ_x, that will also lead to a shear strain γ_{xy} in addition to normal strains. Similarly, if we apply a normal stress along y, σ_y, that will also lead to a shear strain γ_{xy} in addition to normal strains. If we apply a pure shear stress τ_{xy}, that will also lead to normal strain ε_x, ε_y in addition to shear strain. So, there is shear extension coupling now with reference to x-y.

But we saw that in orthotropic lamina actually, there is no shear-extension coupling. Thus apparently it looks like that the angle lamina is anisotropic. But it behaves anisotropically because we are actually trying to analyze with reference to analysis axes which are not the orthotropic / material axis. Therefore, an angle lamina, actually it is not anisotropic but orthotropic with respect to the material axes even though it appears to be anisotropic. If we actually load this along its direction of orthotropy, it will still behave as an orthotropic material and there will not be any shear-extension coupling. So, it is important to notice here that, even though $\overline{[Q]}$ has 6 elements, these 6 elements are function of the 4 independent elements Q_{11}, Q_{12}, Q_{22}, Q_{66} and θ. Therefore essentially, we need actually 4 independent elastic constants to specify an orthotropic lamina. But then, to characterize it with reference to x-y axes which do not coincide with 1-2, we need these 6 constants which are actually functions of these 4 independent elastic constants.

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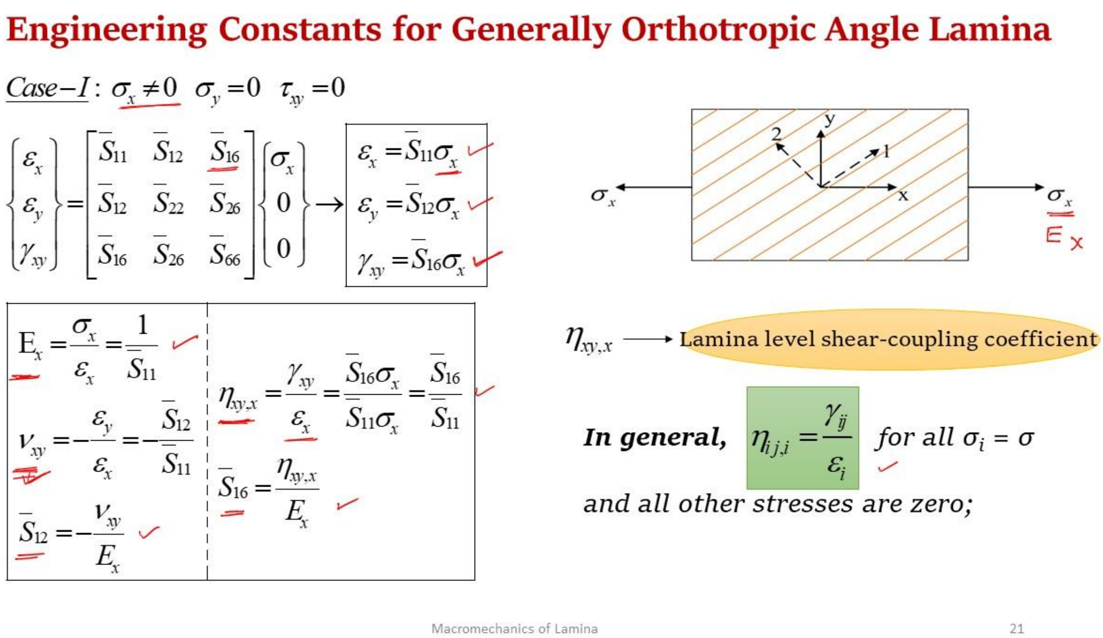

Now, having understood this stress-strain relationship using reduced transformed stiffness and transformed compliance matrix, let us try to understand the the engineering constants with reference to x-y axes for a generally orthotropic angle lamina.

First, in case 1, we apply only σ_x and if we apply this stress-strain relationship, this leads to a normal strain along x, this ε_x, a normal strain along y, ε_y.

$\underline{Case-I}: \ \sigma_x \neq 0 \quad \sigma_y = 0 \quad \tau_{xy} = 0$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{16} \\ \overline{S}_{12} & \overline{S}_{22} & \overline{S}_{26} \\ \overline{S}_{16} & \overline{S}_{26} & \overline{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ 0 \\ 0 \end{Bmatrix} \rightarrow \boxed{\begin{array}{l} \varepsilon_x = \overline{S}_{11}\sigma_x \\ \varepsilon_y = \overline{S}_{12}\sigma_x \\ \gamma_{xy} = \overline{S}_{16}\sigma_x \end{array}}$$

$\mathrm{E}_x = \frac{\sigma_x}{\varepsilon_x} = \frac{1}{\overline{S}_{11}}$ $\nu_{xy} = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\overline{S}_{12}}{\overline{S}_{11}}$ $\overline{S}_{12} = -\frac{\nu_{xy}}{E_x}$	$\eta_{xy,x} = \frac{\gamma_{xy}}{\varepsilon_x} = \frac{\overline{S}_{16}\sigma_x}{\overline{S}_{11}\sigma_x} = \frac{\overline{S}_{16}}{\overline{S}_{11}}$ $\overline{S}_{16} = \frac{\eta_{xy,x}}{E_x}$

As could be seen that in addition to the normal strains, it also leads to a shear strain γ_{xy}, (because of non-zero S_{16} and S_{26} terms). Therefore, going by definition,

$$\mathrm{E}_x = \frac{\sigma_x}{\varepsilon_x} = \frac{1}{\overline{S}_{11}}$$

$$\nu_{xy} = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\overline{S}_{12}}{\overline{S}_{11}}$$

$$\overline{S}_{12} = -\frac{\nu_{xy}}{E_x}$$

Now, because a normal stress causes a shear strain, that must also be characterized. This is called shear coupling coefficient, denoted as $\eta_{xy,x}$ which decides that if we apply a normal stress along x, what will be the shear strain along x-y. This is as follows:

$$\eta_{xy,x} = \frac{\gamma_{xy}}{\varepsilon_x} = \frac{\overline{S}_{16}\sigma_x}{\overline{S}_{11}\sigma_x} = \frac{\overline{S}_{16}}{\overline{S}_{11}}$$

$$\overline{S}_{16} = \frac{\eta_{xy,x}}{E_x}$$

So, in general, the shear coupling coefficient is $\eta_{ij,i} = \frac{\gamma_{ij}}{\varepsilon_i}$. Where, i,j could be 1, 2 and 6. Thus we have established the Young's modulus along x, Poisson's ratio ν_{xy} and in addition, we have also obtained the shear coupling coefficient in the x-y plane, $\eta_{xy,x}$.

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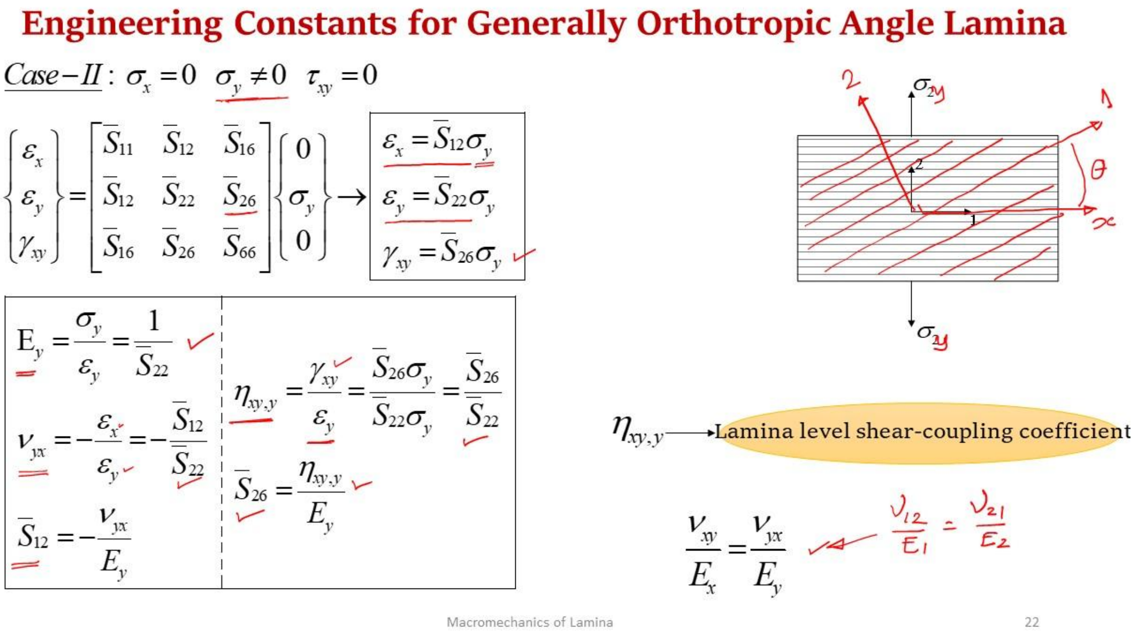

Next, in case 2, we apply only σ_y

$$\underline{Case-II}:\ \sigma_x=0 \quad \sigma_y\neq 0 \quad \tau_{xy}=0$$

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\begin{bmatrix}\bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16}\\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26}\\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66}\end{bmatrix}\begin{Bmatrix}0\\ \sigma_y\\ 0\end{Bmatrix}\rightarrow \boxed{\begin{aligned}\varepsilon_x&=\bar{S}_{12}\sigma_y\\ \varepsilon_y&=\bar{S}_{22}\sigma_y\\ \gamma_{xy}&=\bar{S}_{26}\sigma_y\end{aligned}}$$

So, going by definition, again Young's modulus along y

$$\boxed{\begin{aligned}E_y&=\frac{\sigma_y}{\varepsilon_y}=\frac{1}{\bar{S}_{22}}\\ \nu_{yx}&=-\frac{\varepsilon_x}{\varepsilon_y}=-\frac{\bar{S}_{12}}{\bar{S}_{22}}\\ \bar{S}_{12}&=-\frac{\nu_{yx}}{E_y}\end{aligned}}$$

Again, we have the shear coupling, now it is $\eta_{xy,y}$ defined as

$$\eta_{xy,y}=\frac{\gamma_{xy}}{\varepsilon_y}=\frac{\bar{S}_{26}\sigma_y}{\bar{S}_{22}\sigma_y}=\frac{\bar{S}_{26}}{\bar{S}_{22}}$$

$$\bar{S}_{26}=\frac{\eta_{xy,y}}{E_y}$$

We could also see that $\frac{\nu_{xy}}{E_x}=\frac{\nu_{yx}}{E_y}$ is also true for generally orthotropic lamina with reference to x-y.

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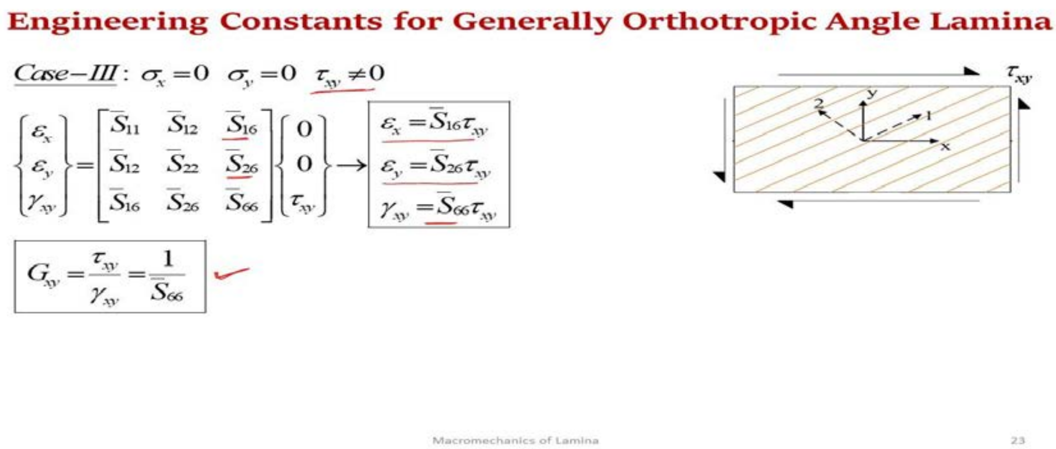

And in case 3, we apply a pure shear τ xy and naturally because the shear coupling coefficient is there, therefore, it leads to a normal strain along x, normal strain along y in addition to the direct shear strain, because of the shear stress. Going by the definition of shear modulus we get

$\underline{Case-III}: \ \sigma_x = 0 \quad \sigma_y = 0 \quad \tau_{xy} \neq 0$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \tau_{xy} \end{Bmatrix} \rightarrow \boxed{\begin{matrix} \varepsilon_x = \bar{S}_{16}\tau_{xy} \\ \varepsilon_y = \bar{S}_{26}\tau_{xy} \\ \gamma_{xy} = \bar{S}_{66}\tau_{xy} \end{matrix}}$$

$$\boxed{G_{xy} = \frac{\tau_{xy}}{\gamma_{xy}} = \frac{1}{\bar{S}_{66}}}$$

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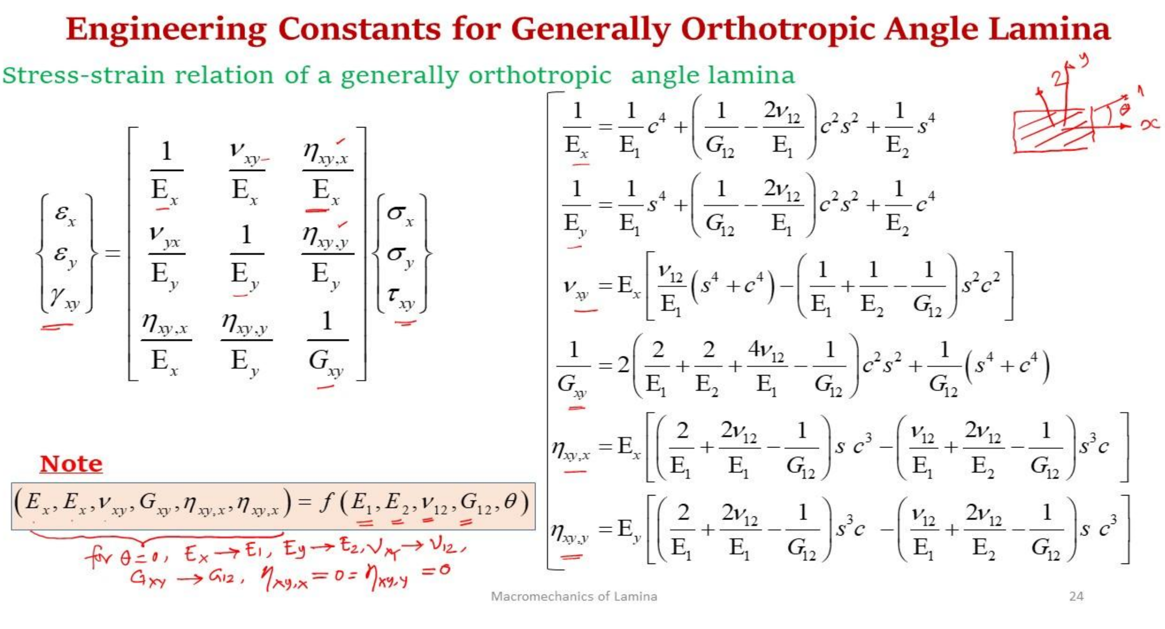

So, in general, the stress-strain relationship in terms of engineering constants for a generally orthotropic angle lamina (where 1-2 do not coincide with x-y are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{\mathrm{E}_x} & \dfrac{\nu_{xy}}{\mathrm{E}_x} & \dfrac{\eta_{xy,x}}{\mathrm{E}_x} \\ \dfrac{\nu_{yx}}{\mathrm{E}_y} & \dfrac{1}{\mathrm{E}_y} & \dfrac{\eta_{xy,y}}{\mathrm{E}_y} \\ \dfrac{\eta_{xy,x}}{\mathrm{E}_x} & \dfrac{\eta_{xy,y}}{\mathrm{E}_y} & \dfrac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

Again E_x, E_y, ν_{xy}, G_{xy} and $\eta_{xy,x}$ and $\eta_{xy,y}$ could actually be related to E_x, E_y, ν_{xy}, G_{xy} and (as

$$\frac{1}{\mathrm{E}_x} = \frac{1}{\mathrm{E}_1}c^4 + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{\mathrm{E}_1}\right)c^2s^2 + \frac{1}{\mathrm{E}_2}s^4$$

$$\frac{1}{\mathrm{E}_y} = \frac{1}{\mathrm{E}_1}s^4 + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{\mathrm{E}_1}\right)c^2s^2 + \frac{1}{\mathrm{E}_2}c^4$$

$$\nu_{xy} = \mathrm{E}_x\left[\frac{\nu_{12}}{\mathrm{E}_1}\left(s^4 + c^4\right) - \left(\frac{1}{\mathrm{E}_1} + \frac{1}{\mathrm{E}_2} - \frac{1}{G_{12}}\right)s^2c^2\right]$$

$$\frac{1}{G_{xy}} = 2\left(\frac{2}{\mathrm{E}_1} + \frac{2}{\mathrm{E}_2} + \frac{4\nu_{12}}{\mathrm{E}_1} - \frac{1}{G_{12}}\right)c^2s^2 + \frac{1}{G_{12}}\left(s^4 + c^4\right)$$

$$\eta_{xy,x} = \mathrm{E}_x\left[\left(\frac{2}{\mathrm{E}_1} + \frac{2\nu_{12}}{\mathrm{E}_1} - \frac{1}{G_{12}}\right)s\,c^3 - \left(\frac{\nu_{12}}{\mathrm{E}_1} + \frac{2\nu_{12}}{\mathrm{E}_2} - \frac{1}{G_{12}}\right)s^3c\right]$$

$$\eta_{xy,y} = \mathrm{E}_y\left[\left(\frac{2}{\mathrm{E}_1} + \frac{2\nu_{12}}{\mathrm{E}_1} - \frac{1}{G_{12}}\right)s^3c - \left(\frac{\nu_{12}}{\mathrm{E}_1} + \frac{2\nu_{12}}{\mathrm{E}_2} - \frac{1}{G_{12}}\right)s\,c^3\right]$$

So, even though there are 6 engineering constants, but actually these 6 engineering constants are nothing but functions of the 4 independent engineering constants. Therefore, even though there is a shear coefficient, it is still orthotropic, but the axes of orthotropy do not coincide with the direction of loading. If we apply load along the direction of orthotropy, it will still show no shear-extension coupling.

So, what we have learnt in this lecture is that, we have obtained the stress-strain relationship for lamina, both with reference to the material axes 1-2 as well as with reference to the analysis axes x-y which may not coincide with the material axes.

We understood that, 4 independent engineering elastic constants are required to characterise a lamina and those could be related to the measurable engineering constants like E_1, E_2, ν_{12}, G_{12}. While the stress-strain relationship for a generally orthotropic lamina, (where the analysis axes or loading axes do not coincide with the material axes), there are 6 constants in the compliance and reduced transformed stiffness matrix. However, those 6 constants are actually functions of the 4 independent elastic constants for a specially orthotropic lamina and θ.

Now, if you put θ=0, we will get back the same; like, if we put theta is equal to 0, this E_x, E_y, ν_{xy}, G_{xy} and $\eta_{xy,x}$ and $\eta_{xy,y}$ will be nothing but E_1, E_2, ν_{12}, G_{12} and 0. So, basically, for an orthotropic lamina, we need 4 independent elastic constants and that there are 4 engineering constants.