Mechanics of Fiber Reinforced Polymer Composite Structures
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Module - 2
Review of Elasticity
Lecture - 04
Orthotropic Materials

Welcome to the second lecture of the second module and we have been discussing the review of elasticity.

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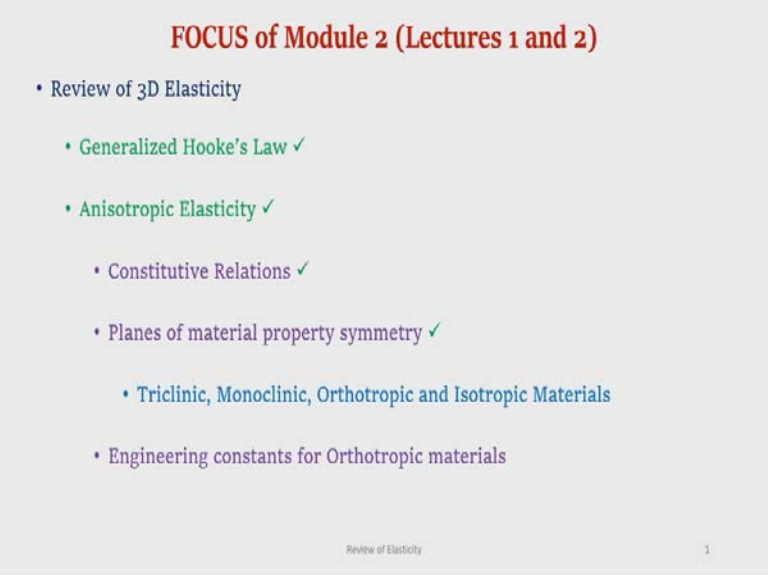
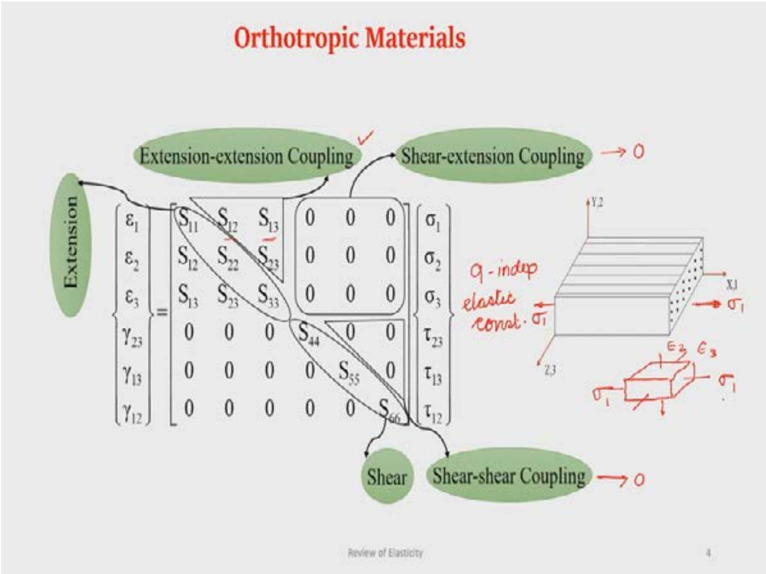


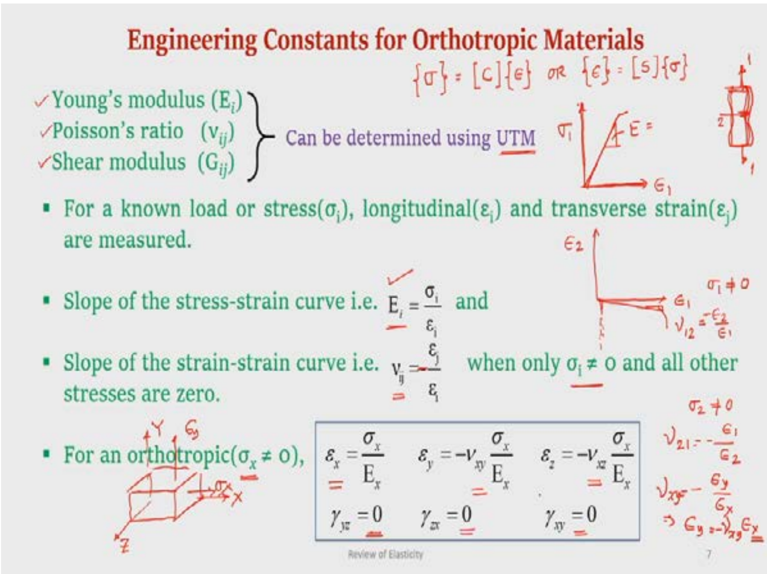
In our last lecture, we understood the generalized Hooke's law; that means, the relation between the stresses and strains in terms of stiffness and compliance like we understood that at a point there are nine stress components and corresponding nine strain components and therefore they are related by stiffness or compliance matrix which has eighty one elastic constants.

But then, we also understood that for symmetric stress and strain tensors, there are six stresses and six strains and therefore the number of elastic constants is thirty six. Again, from the energy consideration, we have showed that the stiffness or compliance matrix is also symmetric and therefore, the number of independent elastic constant because of the symmetry is actually twenty one. So, in the last class we understood this and then we have actually introduced an important concept of planes of material property symmetry and we saw that as a consequence of existence of planes of material property symmetry, the number of independent elastic constants is actually reduced.

We have seen that based on the existence of number of planes of material property symmetry, the material could be triclinic or anisotropic. When there is one plane of material property symmetry, it is monoclinic. Then we have three mutually perpendicular planes of material property symmetry; it is orthotropic. And then we have isotropic material feature, we all know. So, today we will discuss the engineering constants for orthotropic materials. **(Refer Slide Time: 09:34)**



Engineering Constants for Orthotropic Materials (Refer Slide Time: 16:19)



The simple way to write the relation between stresses and strains are in terms of stiffness matrix or compliance matrix where the six components of stress are related to corresponding six components of strain. So, in order to characterize the material we need to know the elements of the stiffness or the compliance matrix. In other word, subjected to stress, if we want to know what the strains are, we must know each element (elastic constants) of this stiffness or compliance matrix. However, these elastic constants are actually not measurable quantities and the measurable quantities from which these elastic constants could be obtained are termed as **engineering constants** which could be determined experimentally in laboratories by conducting experiments. There are relationship between the **engineering constants** and these **elastic constants**. The elastic constants are actually expressed in terms of engineering constants. For example, for a simple axially loaded bar, if we want to know what is the longitudinal strain, we must know what is the Young's modulus (E) for the material of the bar and we could determine the lateral strain by knowing the Poisson's ratio (ν) of the material. We know that both E and ν could be measured in laboratory using universal testing machine (UTM). As shown in the Fig., by applying stress along the longitudinal direction and measuring the corresponding longitudinal strain, the slope of the stress strain curve gives us E. Similarly by simultaneously measuring lateral strain and plotting the lateral strain versus longitudinal strain, the slope of the curve gives us ν. Similarly using torsion test we could determine G from the slope of the shear stress and shear strain curve. For an isotropic material G could be expressed in terms of E and ν. We could now write the stress strain relationship for 2D where S11, S12, … are expressed in terms of the engineering constants E, ν and G.

For isotropic materials, there are two independent elastic constants and hence two independent engineering constants viz. Young's modulus, E and Poisson's ratio, ν. By conducting just a single experiment in an UTM, the Young's modulus and Poisson's ratio can be determined and the shear modulus could be expressed in terms of Young's modulus and Poisson's ratio. However, for orthotropic materials, there are nine independent elastic constants and hence nine engineering constants and we will see how for an orthotropic material these elastic constants could be expressed in terms of engineering constants.

As already discussed, engineering constants are those which could be actually measured in the laboratory. Say for example, in the case of isotropic materials conducting a simple tensile test in UTM, we could obtain E and ν and derive G from those. We have also seen how those are related to the elements of the compliance matrix. Same principle applies to orthotropic materials also where tests are conducted by loading in three mutually perpendicular planes (directions of orthotropy) and three direction-dependent Young's moduli and Poisson's ratios could be determined. Note that in the case of orthotropic materials, the relationship between the E, ν and G (for isotropic materials) does not hold good and three direction-dependent Gs are to be determined by separate tests.

Referring to the Fig., suppose in an orthotropic material, the load is applied along 1 (or X), and the corresponding stress is σ_1 and the measured strain along 1 (or X) is, then the slope of the stress-strain curve (= $\frac{\sigma_1}{\varepsilon_1}$) gives us Young's modulus along 1. Due to Poisson's effect there will be strain along 2 (or Y) and 3 (or Z). Suppose the strain along 2 due to only load along 1 is ε_2, then the Poisson's ratio in the plane 12 is $\nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1}$ or $\varepsilon_2 = -\nu_{12}\varepsilon_1 = -\nu_{12}\frac{\sigma_1}{E_1}$ Suppose, we apply in this direction, say this is 1, and we also keep a strain gauge along direction 2 and we keep track of what is the strain along direction 2. This is how the Poisson's ratio is defined and in general $\nu_{ij} = -\frac{\varepsilon_j}{\varepsilon_i}$ (when only $\sigma_i \neq 0$ is acting), Suppose, the strain along direction 2 is this. I will do a separate diagram maybe; similarly, we can draw the longitudinal strain versus the lateral strain. Therefore, in the case of an orthotropic material, the slope of the stress-strain curve gives us Young's modulus in a particular direction $E_i = \frac{\sigma_i}{\varepsilon_i}$ (σ_i is the applied stress and ε_i is the corresponding strain in that direction) and the slope of the longitudinal strain versus lateral strain curve gives us the Poisson's ration as, that is $\nu_{ij} = -\frac{\varepsilon_j}{\varepsilon_i}$ (when only σ_i is applied).

The shear moduli G_{ij} need to be determined from separate tests. It is to noted that $\nu_{ij} \neq \nu_{ji}$ (for isotropic material of course they are same).

Considering an orthotropic material with x plane, y plane and z plane are the planes of orthotropy. Suppose, we apply a stress σ_x along X and all other stresses are zero, that will lead to a strain along X which is given by $\frac{\sigma_x}{E_x}$. Now, because of the Poisson's effect, application of σ_x along x will lead to strains along Y which is $\varepsilon_y = -\nu_{xy}\varepsilon_x = -\nu_{xy}\frac{\sigma_x}{E_x}$. Similarly, because of σ_x, there will be a strain along Z which is given by again in terms of Poisson's ratio $\varepsilon_z = -\nu_{xz}\varepsilon_x = -\nu_{xz}\frac{\sigma_x}{E_x}$.

Note that even though the procedure followed is same as that in isotropic material, but here the Poisson's ratios in XY and XZ planes are different. Similarly, if we apply stress σ_y along Y and all other stresses are zero, that will lead to a strain along Y which is given by $\frac{\sigma_y}{E_y}$. Now, because of the Poisson's effect, application of σ_y along Y will lead to strains along X which is $\varepsilon_x = -\nu_{yx}\varepsilon_y = -\nu_{yx}\frac{\sigma_y}{E_y}$. Similarly, because of σ_y, there will be a strain along Z which is given by again in terms of Poisson's ratio $\varepsilon_z = -\nu_{yz}\varepsilon_x = -\nu_{yz}\frac{\sigma_y}{E_y}$. Extending the same if we apply only stress σ_z along Z and all other stresses are zero, that will lead to a strain along Z which is given by $\frac{\sigma_z}{E_z}$. Now, because of the Poisson's effect, application of σ_z along Z will lead to strains along X which is $\varepsilon_x = -\nu_{zx}\varepsilon_z = -\nu_{zx}\frac{\sigma_z}{E_z}$ and there will be a strain along X which is given by again in terms of Poisson's ratio $\varepsilon_x = -\nu_{xz}\varepsilon_z = -\nu_{xz}\frac{\sigma_z}{E_z}$.

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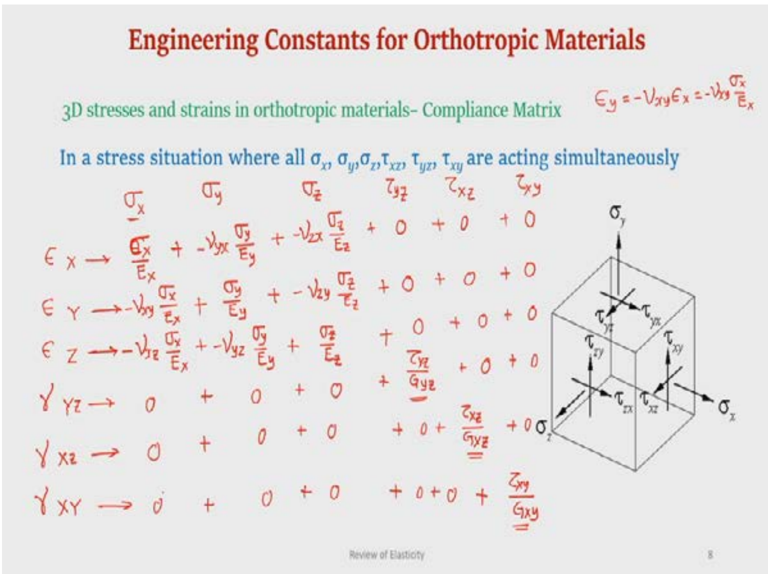
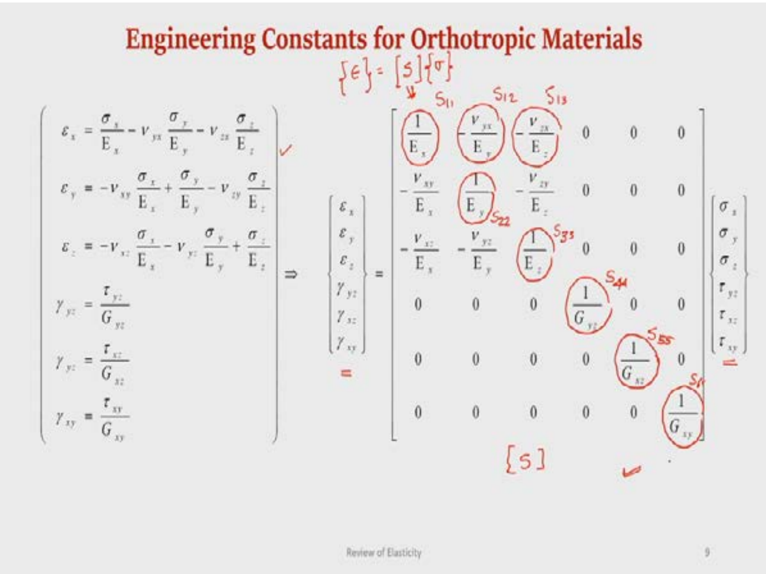


Now, suppose it is also subjected to shear stress along plane YZ, τ_{yz}, that will lead to a direct shear strain $\gamma_{yz} = \frac{\tau_{yz}}{G_{yz}}$. Suppose it is also subjected to shear along plane XZ τ_{xz}, therefore, in the XY plane it will lead to shear strain $\gamma_{xz} = \frac{\tau_{xz}}{G_{xz}} \frac{\tau_{xz}}{G_{xz}}$. Similarly, subjected to shear stress τ_{xy}, leads to $\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}}$. It is to be noted that for an orthotropic material there is no shear extension coupling and hence application of normal stresses does not result in shear strain and applications of shear stresses does not result in normal strains. In addition there is no shear-shear coupling ie application of shear stress in one plane only results in shear strain in that plane and not in other planes.

Extending this, if all the three normal stresses and three shear stresses are applied simultaneously, then we could obtain the total strains by using the method of superposition and therefore we can add this as follows. From this we could write the six components of strains in terms of six components of stresses resulting in what is called the compliance matrix. Her it could be noted that the elements of this compliance matrix are now expressed in terms of nine engineering constants.



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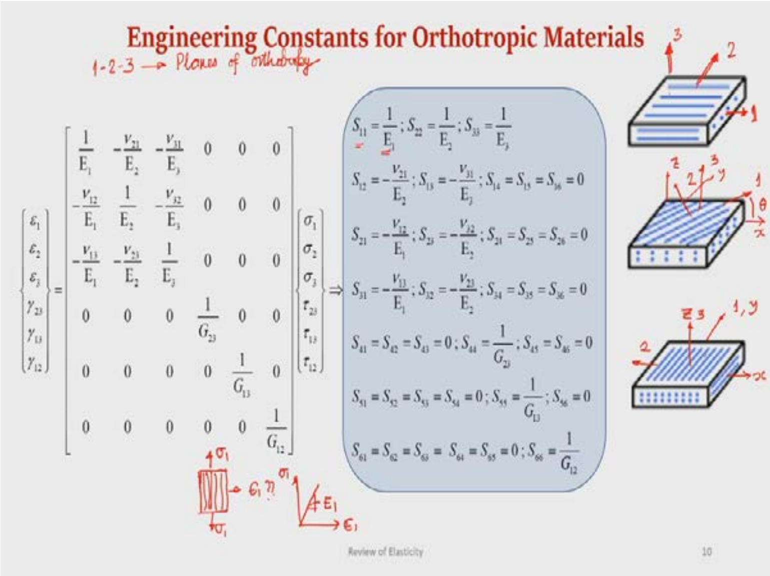
Therefore the elements of compliance matrix S11, S12, S13, … S66 could now be expressed in terms of the engineering constants as follows. As discussed earlier it could be seen that in this the shear-extension coupling as well as the shear-shear coupling terms are zero though extension-extension coupling (due to Poisson's effect) terms do exist.



So, we could actually express elements of the compliance matrix in terms of engineering constants. What are these engineering constants? Three Young's moduli, E_x, E_y, E_z; three Poisson's ratios, ν_{xz}, ν_{yz}, ν_{xy}; and three shear moduli, G_{yz}, G_{xz} and G_{xy}.

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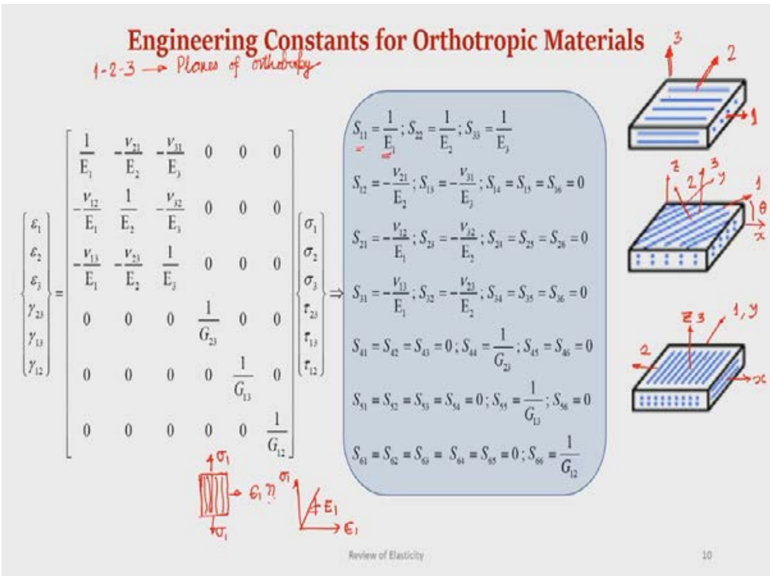

Note that X, Y and Z planes are the three mutually perpendicular planes of material property symmetry in an orthotropic material. Generally 1, 2, 3 are designated as the planes of orthotropy ie. 1 means the corresponding plane where 1 is the surface normal to the plane 2-3, 2 is the surface normal to the plane 1-3 and 3 is the surface normal to the plane 1-2. With reference to the Fig., generally in an orthotropic lamina, X-Y-Z are conventionally used for analysis axis or loading axis. As shown in Fig., sometimes 1, 2, 3 may coincide with X, Y, Z, but not necessarily always. Say for example, if we have a lamina where the fibers are actually oriented in a defined direction, 1, 2, 3 have a different orientation with respect to X, Y, Z. 1, 2, 3 are the directions of orthotropy, sometimes also called **principal material direction**. So, now we can write that $S_{11} = 1/E_1$, $S_{22}=1/E_2$ and , $S_{33}=1/E_3$. E_1, E_2, E_3 are the Young's moduli along directions 1,2,3 respectively. Similarly, we could express other terms S12, S13 … S66 in terms of Young’s moduli, Poisson's ratios and the shear moduli. So, these are the nine engineering constants which could be actually measured in the laboratory. For example if we need to determine what is E_1 for a lamina, we make specimen from the this lamina, load it along direction 1in UTM (following certain standards) and from the strain gauge reading, we get ε_1 lamina and from this stress-strain curve we get E_1 and from lateral strain longitudinal strain curve we get ν_{12} Similarly, by loading along directions 2 and 3, we get E_2 and E_3 and other two Poisson’s ratios and by conducting shear tests, we could get three shear moduli. From these engineering constants we could obtain the compliance and stiffness matrix.

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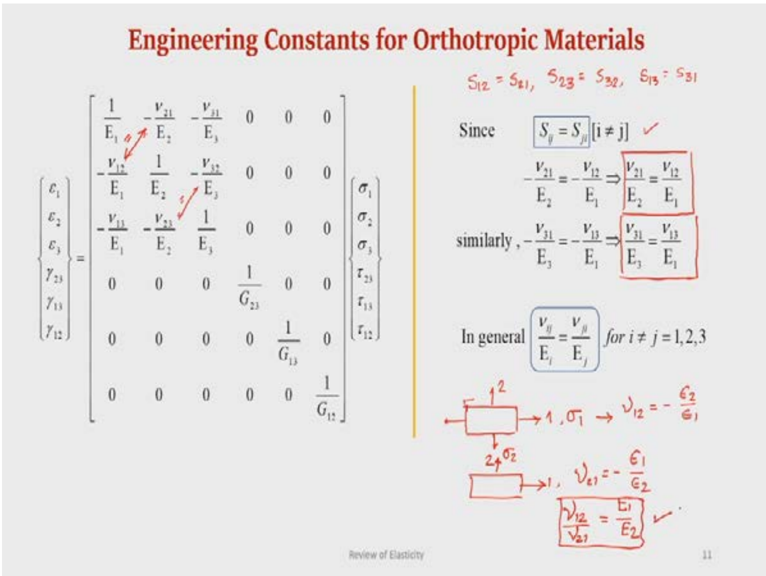

Now, as already discussed, compliance and stiffness matrix are symmetric. Therefore, $S_{12} = S_{21}$ and in general, $S_{ij} = S_{ji}$ [i ≠ j], meaning $S_{12} = S_{21}$, $S_{23} = S_{32}$, $S_{13} = S_{31}$ and this leads to a relationship which is known as reciprocal relations. That implies that $\dfrac{\nu_{ij}}{E_i} = \dfrac{\nu_{ji}}{E_j}$. As we have discussed earlier that ν_{12} and ν_{21} are not same but they are not independent and are actually related by this relation $\dfrac{\nu_{21}}{E_2} = \dfrac{\nu_{12}}{E_1}$. Similarly, ν_{13} and ν_{31} are also bound by this relation. So, in general, $\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}$. Recall the definition of Poisson's ratio and see what are ν_{12} and ν_{21}.

Suppose this is our direction 1, this is our direction 2; suppose if we apply then $\nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1}$ when only σ_1,is applied and all other stresses are zero and $\nu_{21} = -\frac{\varepsilon_1}{\varepsilon_2}$ when only σ_2 is applied and all other stresses are zero.

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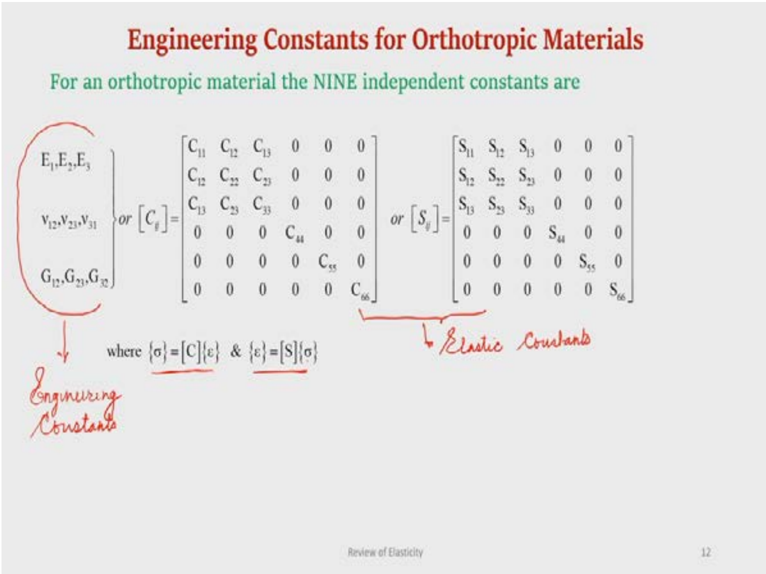

So, in general, we can write the stress strain relations as $\{\sigma\} = [C]\{\varepsilon\}$ and $\{\varepsilon\} = [S]\{\sigma\}$. Having understood the relationship between the engineering constants and the elastic constants we could obtain the elements of the compliance matrix in terms of engineering constants. The fact that the stiffness and compliance matrices are mutually invertible ie.$[C] = [S]^{-1}$, we can write the elements of the stiffness matrix in terms of the compliance matrix or vice versa.

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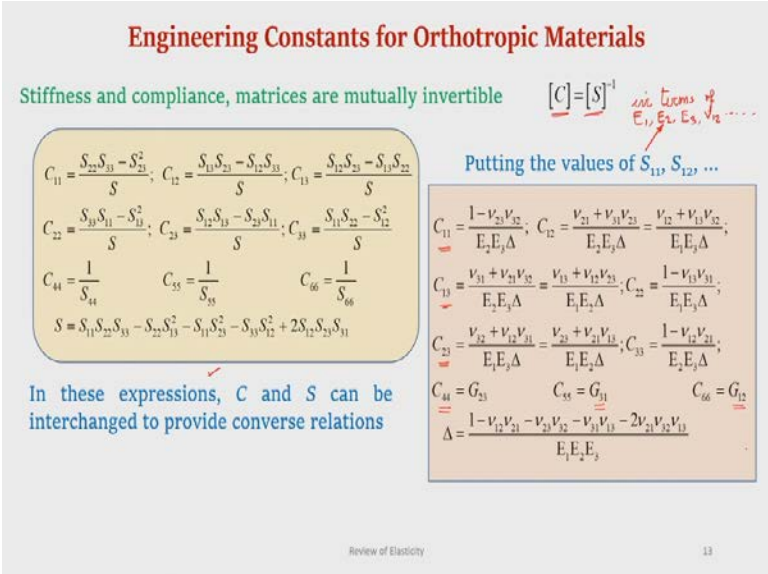

Now that we know these elements of compliance matrix in terms of E_1, E_2, E_3, ν_{12}, ν_{23}, ν_{31}, G_{12}, G_{23}, G_{31}; if we put those, we get the elements of the stiffness matrix in terms of the engineering constants. We could write the elements of the stiffness matrix in terms of engineering constants by these expressions. So, therefore, we obtain the relationship between the engineering constants and the elements of the stiffness and the compliance matrices.

Restrictions on Engineering Constants

Now, we have nine independent engineering constants or elastic constants for orthotropic materials, there are some restrictions on the possible values of those. Before discussing those restrictions, let us just revisit the restrictions on elastic constants for isotropic material with which we are more conversant. We know that G and E have to be always positive. If we apply a tensile stress that will lead to a tensile strain and therefore, E must be positive. Similarly, G must also be positive.

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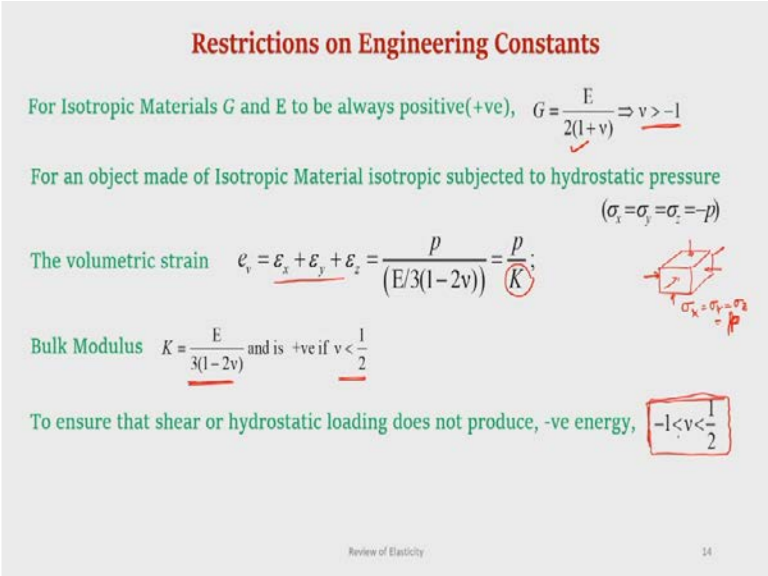

Now for an isotropic material, the relationship between G, E and ν given by this $G=\frac{E}{2(1+\nu)}$.

For G to be positive and E to be positive, $\nu > -1$ it can never be less than −1. Now, suppose an isotropic material which is actually subjected to hydrostatic stress (equal stress in all directions). For example if we put an object under water, it is subjected to hydrostatic stress and is equal to pressure p. If it is subjected to hydrostatic pressure like this, then the volumetric strain is given by $\frac{p}{K}$ where this K is the bulk modulus. Now K must be positive because if it is subjected to compression, its volume must decrease and if it is subjected to hydrostatic tension (say in the case of balloon), volume must increase. For K to be positive in this expression, $K=\frac{E}{3(1-2\nu)}$, $\nu<\frac{1}{2}$ i.e. ν can never be more than 0.5. So, the restrictions on Poisson's ratio in an isotropic material is $-1<\nu<\frac{1}{2}$. So, if we determine the values of Young's modulus and Poisson's ratio for a particular isotropic material by conducting experiments in UTM and the Poisson’s ratio must be within this bound. If someone tells that the Poisson’s ratio for an isotropic material like steel is 0.8, we can immediately infer that there is some issue in the calculation or in the data acquisition because we know that for a material like steel, the Poisson's ratio cannot be more than 0.5. Therefore, this restriction gives us an additional check on the measured properties.

Similarly, there are restrictions on engineering constants of orthotropic materials. Again, the philosophy is same that means, all the diagonal elements of [C] and [S] must be positive. The argument is that if we apply a positive normal stress along a particular direction, that must lead to corresponding positive normal strain. If we apply tensile stress, it cannot lead to compression strain in that direction; therefore, they must be positive.

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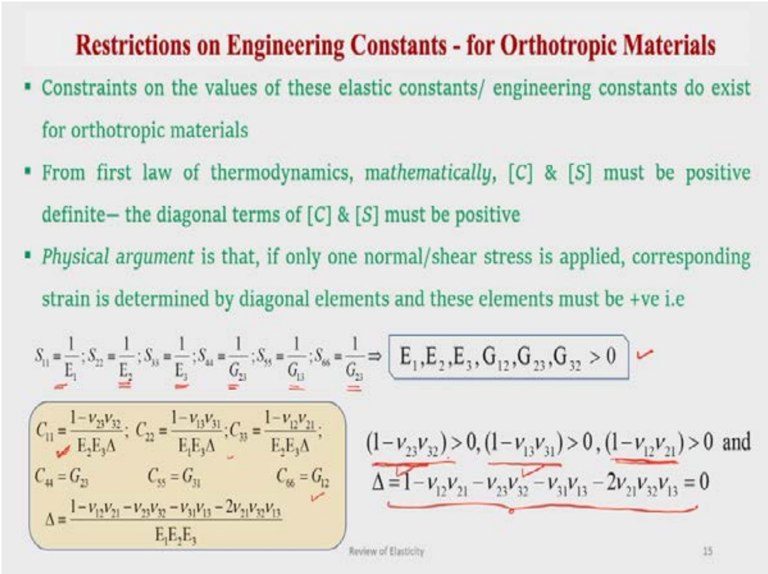

In order to have the positive diagonal elements the conditions are is that $E_1, E_2, E_3, G_{12}, G_{23}, G_{32} > 0$. Similarly, if we write the stiffness matrix in terms of the engineering constants and for each diagonal element to be positive, gives rise to these conditions

$(1-\nu_{23}\nu_{32}) > 0$, $(1-\nu_{13}\nu_{31}) > 0$, $(1-\nu_{12}\nu_{21}) > 0$ and .

$\Delta = 1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{21}\nu_{32}\nu_{13} = 0$

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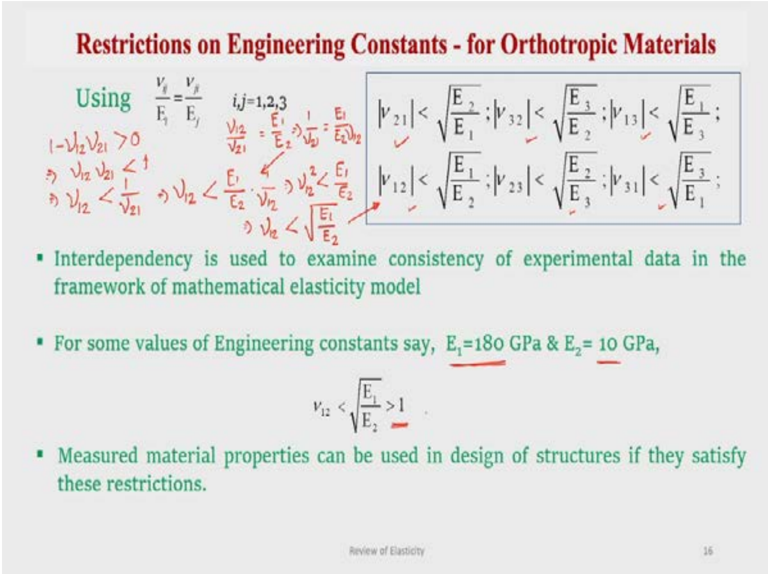

Considering $(1-\nu_{12}\nu_{21})>0$, means $\nu_{12}\nu_{21}<1 \Rightarrow \nu_{12}<\dfrac{1}{\nu_{21}}$; and we know that

$$\frac{\nu_{12}}{\nu_{21}}=\frac{E_1}{E_2} \Rightarrow \frac{1}{\nu_{21}}=\frac{E_1}{E_2}\left(\frac{1}{\nu_{12}}\right) \Rightarrow \nu_{12}<\frac{E_1}{E_2}\left(\frac{1}{\nu_{12}}\right) \Rightarrow \nu_{12}^2 \leq \frac{E_1}{E_2} \Rightarrow \nu_{12}<\sqrt{\frac{E_1}{E_2}}\,.$$

Therefore, $(1-\nu_{12}\nu_{21})>0$ leads to relationship; $\nu_{12}<\sqrt{\dfrac{E_1}{E_2}}$. Similarly, by taking other conditions, we get the following relationships. **(Refer Slide Time: 46:55)**

$$|\nu_{21}|<\sqrt{\frac{E_2}{E_1}};|\nu_{32}|<\sqrt{\frac{E_3}{E_2}};|\nu_{13}|<\sqrt{\frac{E_1}{E_3}};$$
$$|\nu_{12}|<\sqrt{\frac{E_1}{E_2}};|\nu_{23}|<\sqrt{\frac{E_2}{E_3}};|\nu_{31}|<\sqrt{\frac{E_3}{E_1}};$$

That means, there are relationships between the nine engineering constants, which serve as restrictions on the values the engineering constants could take.

These interdependencies are used to examine the consistencies of the experimentally determined engineering constants. Suppose we take an orthotropic material to the laboratory and try to find out those nine engineering constants. Once we get this for a large number of data, we must see that those values actually satisfy this interdependency to be consistent with the mathematical theory of elasticity. If they do not satisfy, then there is something wrong. Again, one important thing is that; suppose, say for example, for an orthotropic material, suppose Young's modulus E1 is 180 GPa and Young's modulus E2 is 10 GPa (a typical example for a graphite epoxy lamina), then the Poisson's ratio determined must satisfy the following relations, $\nu_{12} < \sqrt{\frac{E_1}{E_2}} > 1$.

Notice here that in isotropic elasticity, we know that the Poisson's ratio cannot be 1, it cannot be greater than 0.5, but in the example above, it is possible in the case of orthotropic material. However, they have to be checked for satisfying the other conditions as well. So, knowing these constraints in the engineering constants for orthotropic materials, once they satisfy these data, the material properties obtained from the laboratory could be used for design and analysis structures with confidence.

So, what we have learnt today is that, what are the engineering constants for orthotropic materials; there are 9 engineering constants corresponding to 9 elastic constants and we have also established the relationship between the engineering constants and the elastic constants; that is the elements of the compliance and the stiffness matrix. And finally, we also understood the existence of restrictions on the engineering constants for orthotropic materials and those restrictions are useful to check the consistency and accuracy of the experimentally observed data from the laboratory.