Mechanics of Fiber Reinforced Polymer Composite Structures

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Lecture – 32
Buckling and Free Vibration

Hello and welcome to the second lecture of the 12th module.

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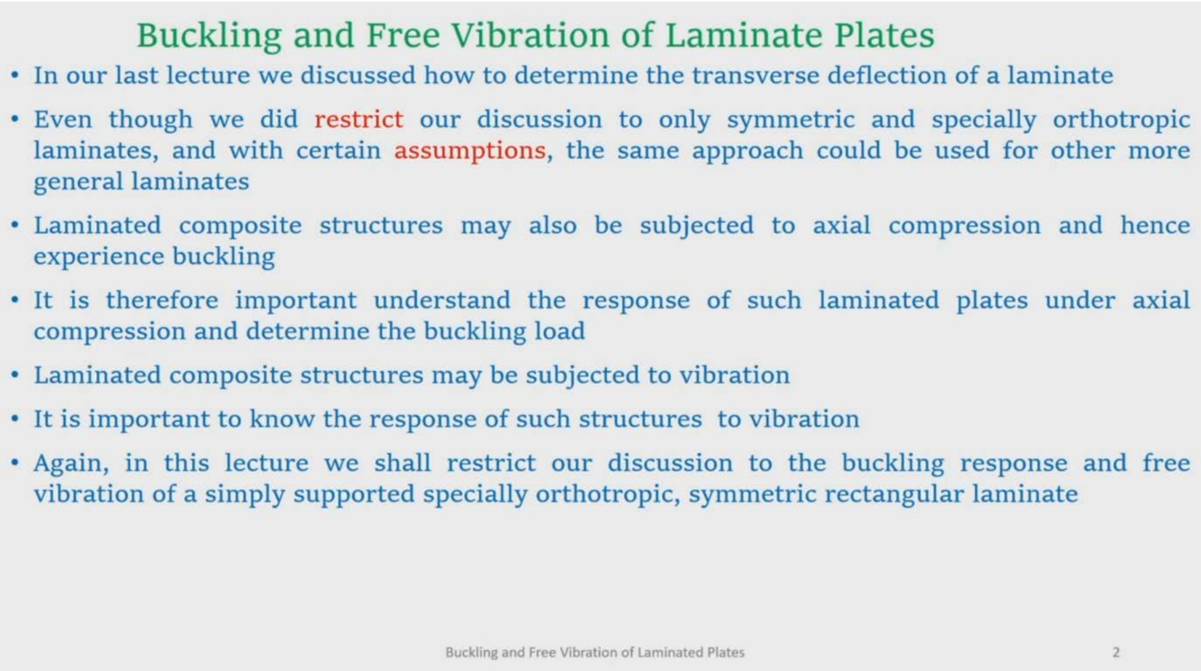

So, in our last lecture we have discussed how to determine the transverse deflection of a laminate using classical lamination theory. In order to determine the transverse deflection, the transverse shear resultants were introduced in the classical lamination theory. We did however restrict our discussion to only symmetric and specially orthotropic laminate, and with the inherent assumptions in classical lamination theory. However, even though we have restricted our discussion to symmetric special orthotropic and rectangular laminated plate, the same approach could actually be used for other more general laminates.

Now, laminated composite structures may also be subjected to axial compression and whenever there is an axial compression the slender structures actually experiences buckling. Therefore, it is important to understand the response of such laminated plates under axial compression and more importantly to determine the critical buckling load. In undergraduate strength of materials, column buckling has already been discussed.

When the axial compression load actually exceeds certain critical value which is called critical buckling load, the column goes to instability. Similarly, for a laminated composite plate, we need to actually determine what is the buckling load?

In addition, laminated composite structures may also be subjected to vibration and therefore it is important to know the response of such structures to vibration. Because it is important to understand that when a laminated plate or for that matter, any component is subjected to vibrations. We always try to see that the forcing frequency is far away from the natural frequency otherwise resonance occurs. Therefore, we will restrict our discussions to I mean only to determination of natural frequency of laminated composite plates.

In this lecture address these two issues viz. the buckling and free vibration of FRP laminates. Again, we shall restrict our discussions only to the buckling response and free vibration response of a simply supported specially orthotropic symmetric rectangular laminate since the objective here has been to understand basically different parameters and properties in a laminated composite plate influence the free vibration and buckling response of the laminate.

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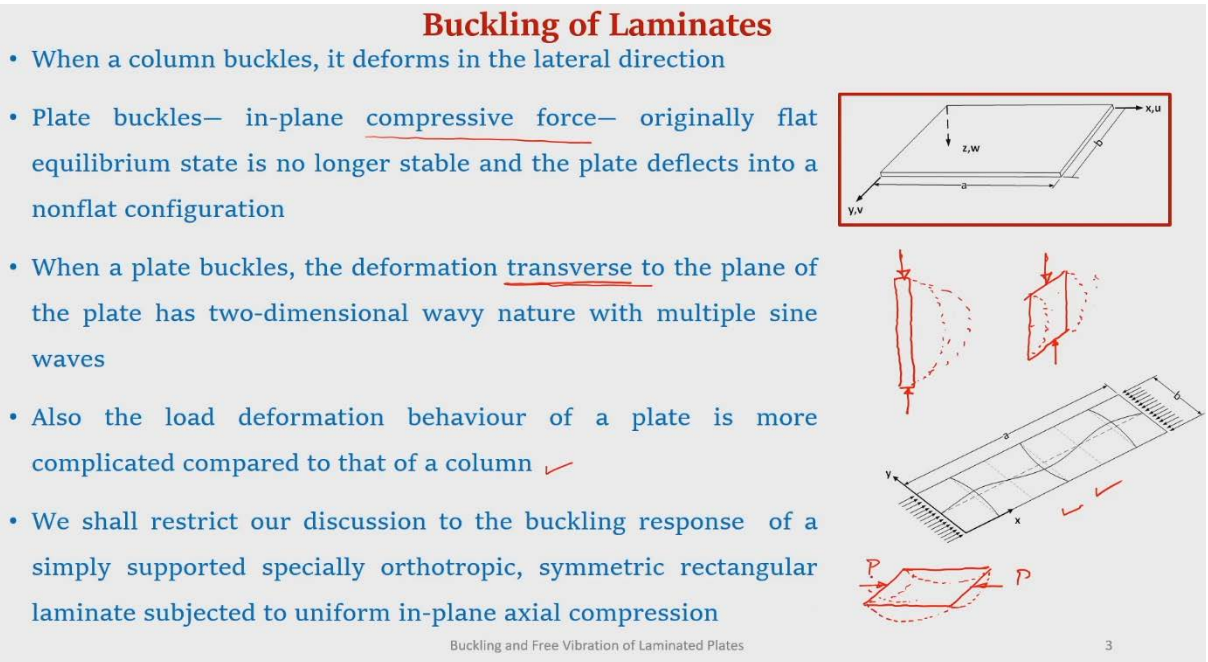

Therefore, to start with, let us first see the buckling of laminated plates in brief. We know when a slender column buckles it actually deforms in the lateral direction as it experiences bending. Therefore, a plate also, similarly buckles when the in-plane compressive force is so large that the originally flat equilibrium state is no longer stable and the plate deflects into a non-flat configuration. Similarly considering a rectangular plate experiencing an axial compression the plate is initially flat. If the load is so large that after some time the transverse deformation becomes unstable. That means even if the load is constant it keeps on the transverse deflection keeps on increasing that is what is called instability.

The difference between this the column, buckling and plate buckling is when a plate buckles the deformation transfers to the plane of the plate has two dimensional wavy nature with multiple sine waves (as shown in Fig.), the edges may have different boundary conditions like simply supported, clamped etc. Now, this could be actually represented by two dimensional multiple sin waves, as it is shown in the figure.

That, of course, depends upon what is the actual load. Also, the load deformation behaviour of a plate is more complicated compared to that of a column. Again, we will restrict our discussions to the buckling response of a simply supported, specially orthotropic, symmetric rectangular laminate, subjected to uniform in-plane axial compression.

A plate could be also a circular plate and the response will be different but we are restricting our discussions to only a rectangular plate. Also, we are considering a laminate which is spatially orthotropic. That means it is only 0/90 layers, meaning that there is no shear extension coupling and symmetric. That means there is no bending extension coupling. However, the approach that will be followed could also be applied for a general laminate.

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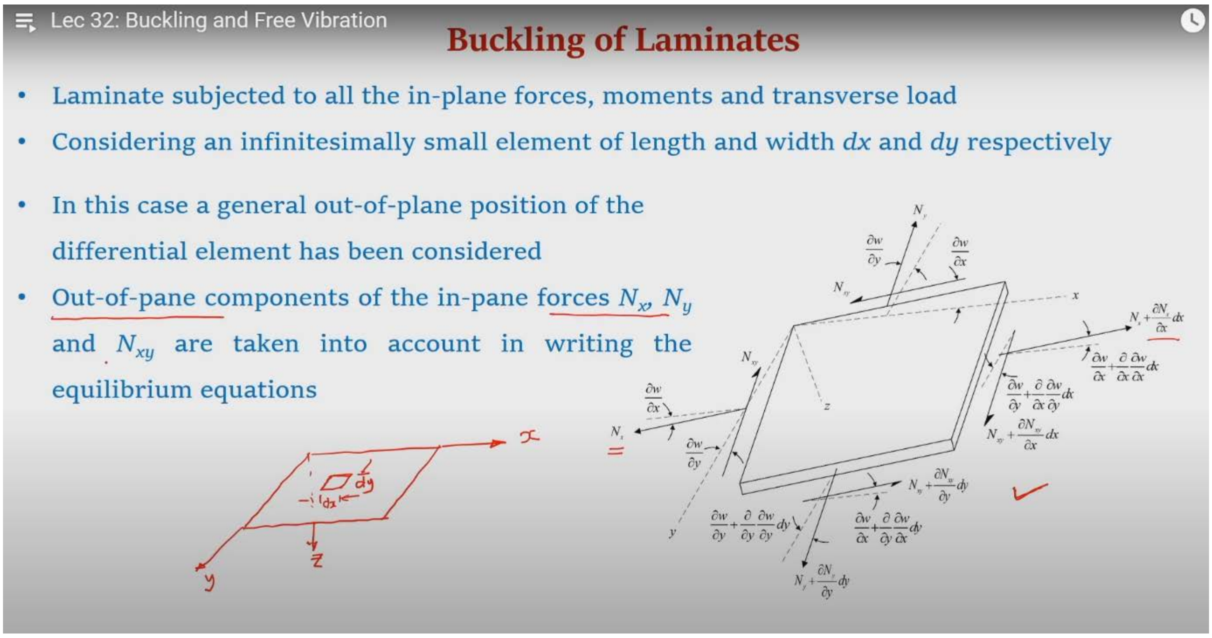

Now, let us consider a laminate subjected to all the in-plane forces, N_x, N_y, N_{xy} , moments M_x, M_y, M_{xy} and transverse load. Now, considering a general laminate and we represent this by means of its mid plane, as done in classical dimension theory. Coordinates x, y and z are fixed at the mid surface it is subjected to all kinds of in-plane force and moment resultants and uniformly distributed transverse load.

Now, considering a small element from this laminate with length, dx width dy, the forces are as shown in the Figure.

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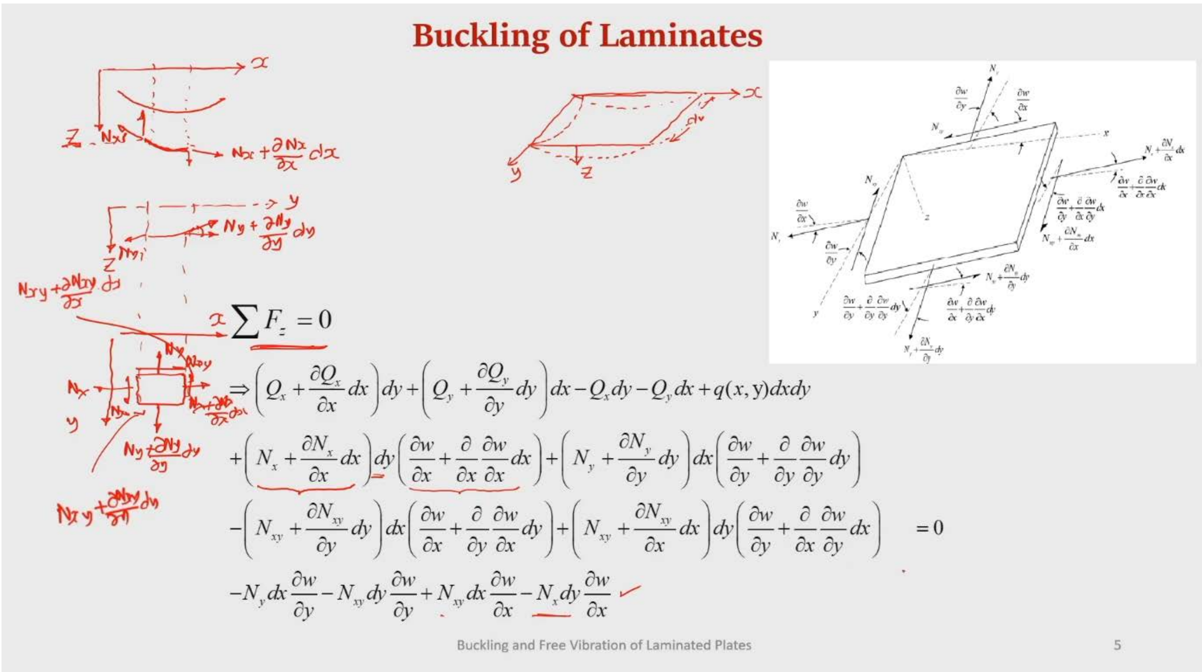

Now, under this load, the deformed configuration of the element is shown in Fig. Now, considering a section in the xz plane, at x it is actually experiencing N_x and at a distance of dx, this is $N_x + (\partial N_x/\partial x)dx$. (Using Taylor's series, only first term, continuous function)

Similarly, considering yz plane, this is N_y and at a distance of dy this is $N_y + (\partial N_y/\partial y)dy$. Considering the xy plane suppose we look from top xy, for same the elemental length, dx, dy, this is N_{xy} incremented as $N_{xy} + (\partial N_{xy}/\partial x)dx$ along x and N_{xy} incremented as $N_{xy} + (\partial N_{xy}/\partial y)dy$ along dy.

So, what is important here is that out of plane component of in-plane forces as considered meaning now N_x is not along x. It is making certain angle with x axis. Similarly, N_y is not no more along y. It is making certain angle. What is that angle that we have shown that rotation is $(\partial w/\partial x)$, $(\partial w/\partial y)$.

Therefore, the appropriate components of forces along x, y and z will have to be taken in equilibrium equations. We have used, for small θ, sin(θ) is θ and Cos(θ)=1 and the product terms like $dx \cdot dx$ and $dy \cdot dy$=0.

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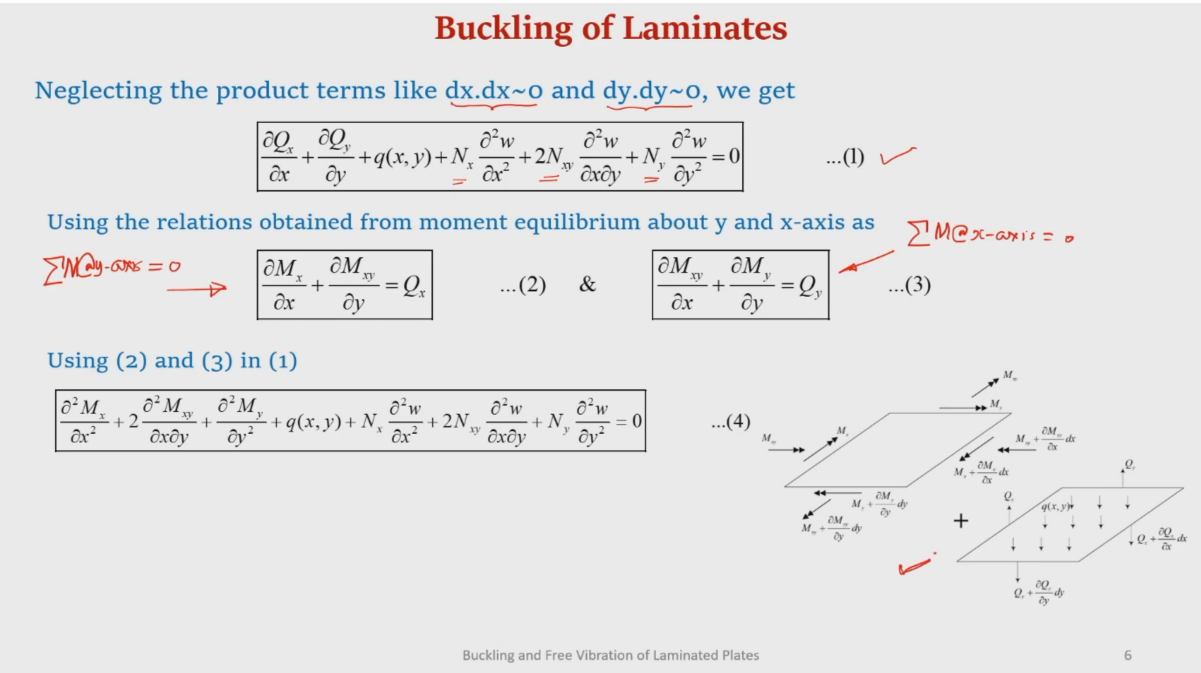

Note that component of N_x, N_y and N_{xy} along the z (which were neglected in transverse deflection) are considered here. Considering Fz=0 and putting $dx \cdot dx$ and $dy \cdot dy$=0 we obtain

Eqn (1) and moment equilibrium about y- and x- axes gives Eqn(2) and Eqn(3). Using (2) and (3) in (1), we obtain (4).

Now, using classical lamination theory we have the relationship between force and moment resultants with the mid surface strains and curvatures by so, called ABBD matrix of the laminate (5) expressing N_x, N_y, N_{xy}, M_x, M_y, M_{xy} in terms of the mid surface strains and curvatures. Again, using the strain displacement and curvature deflection relations (6) we obtain (7) as the relations between force/moment resultants and the displacements.

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Buckling of Laminates

and using strain displacement and curvature displacement relations (6) in (5)

from CLT

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{13} & A_{23} & A_{66} \end{bmatrix} & \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{13} & B_{23} & B_{66} \end{bmatrix} \\ \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{13} & B_{23} & B_{66} \end{bmatrix} & \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{13} & D_{23} & D_{66} \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix} \quad (5)$$

$$\left.\begin{aligned} \varepsilon_x^o &= \frac{\partial u_o}{\partial x}, \quad K_x = -\frac{\partial^2 w_o}{\partial x^2} \\ \varepsilon_y^o &= \frac{\partial v_o}{\partial y}, \quad K_y = -\frac{\partial^2 w_o}{\partial y^2} \\ \gamma_{xy}^o &= \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}, \quad K_{xy} = -2\frac{\partial^2 w_o}{\partial x \partial y} \end{aligned}\right\} \quad (6)$$

We obtain

$$\left.\begin{aligned} N_x &= A_{11}\frac{\partial u_o}{\partial x} + A_{12}\frac{\partial v_o}{\partial y} + A_{16}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - B_{11}\frac{\partial^2 w}{\partial x^2} - B_{12}\frac{\partial^2 w}{\partial y^2} - 2B_{16}\frac{\partial^2 w}{\partial x \partial y} \\ N_y &= A_{12}\frac{\partial u_o}{\partial x} + A_{22}\frac{\partial v_o}{\partial y} + A_{26}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - B_{12}\frac{\partial^2 w}{\partial x^2} - B_{22}\frac{\partial^2 w}{\partial y^2} - 2B_{26}\frac{\partial^2 w}{\partial x \partial y} \\ N_{xy} &= A_{16}\frac{\partial u_o}{\partial x} + A_{26}\frac{\partial v_o}{\partial y} + A_{66}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - B_{16}\frac{\partial^2 w}{\partial x^2} - B_{26}\frac{\partial^2 w}{\partial y^2} - 2B_{66}\frac{\partial^2 w}{\partial x \partial y} \\ M_x &= B_{11}\frac{\partial u_o}{\partial x} + B_{12}\frac{\partial v_o}{\partial y} + B_{16}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} - 2D_{16}\frac{\partial^2 w}{\partial x \partial y} \\ M_y &= B_{12}\frac{\partial u_o}{\partial x} + B_{22}\frac{\partial v_o}{\partial y} + B_{26}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2} - 2D_{26}\frac{\partial^2 w}{\partial x \partial y} \\ M_{xy} &= B_{16}\frac{\partial u_o}{\partial x} + B_{26}\frac{\partial v_o}{\partial y} + B_{66}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - D_{16}\frac{\partial^2 w}{\partial x^2} - D_{26}\frac{\partial^2 w}{\partial y^2} - 2D_{66}\frac{\partial^2 w}{\partial x \partial y} \end{aligned}\right\} \quad (7)$$

$w_0 = w$

$w(x, y, z) = w_0$

Buckling and Free Vibration of Laminated Plates 7

In equation (7), we could express N_x, N_y, N_{xy}, M_x, M_y, M_{xy} in terms of the displacement components. Note here that we have been using w_o and w, same because in classical lamination theory, we have one of the assumptions was that w does not depend upon the the thickness, the z component of displacement, is independent of z, meaning that the z component of mid surface displacement and z component of displacement at any other, point along the thickness is same. So, putting (7) in (4) ie. putting M_x, M_y, M_{xy} in terms of the displacement in (4) we get equation (8).

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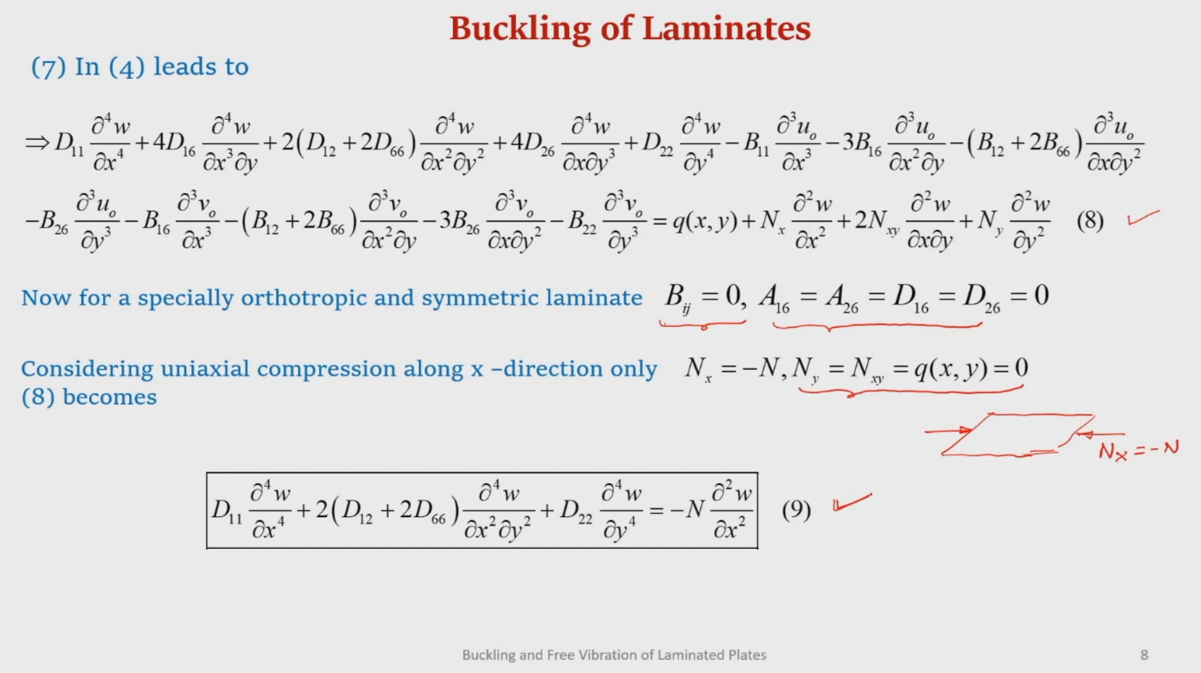

Note that equation (8) is actually **obtained** as a consequence of force equilibrium in the z directions but this is a coupled equation where u_o, v_o are also there.

Now, for a specially orthotropic and symmetric laminate, B_{ij}=0 i.e. all the elements of bending extension coupling matrix [B] are 0 and since it is specially orthotropic, A_{16}, A_{26}, D_{16}, D_{26}, the shear extension and twisting bending coupling are also 0. Therefore, this equation (8) actually gets simplified to this equation (9).

Now because we are actually dealing with the buckling of a plate, N_x is non-zero and other forces are all zero, that means the plate experiences only axial compression.

Therefore, this is $N_x = -N$, $N_y = N_{xy} = q(x,y) = 0$ ($q(x,y)$ is **uniformly** distributed transverse load). Therefore we get this equation (9) from (8).

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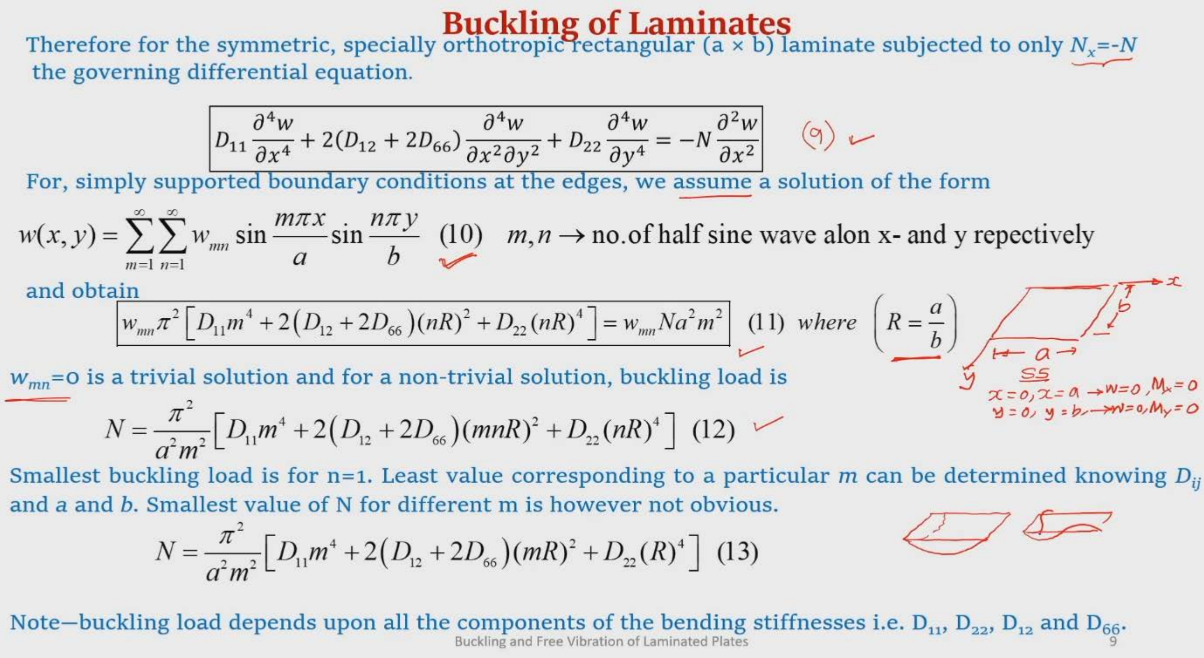

Eqn (9) is actually for a symmetric special orthopaedic rectangular laminate, subjected to $N_x = -N$. That means axial compression load. The governing differential equation is this equation number (9). Now, for a simply supported boundary conditions at the edges, we can solve this using double sine series already discussed in the lecture of transverse deflection. Therefore considering a simply supported boundary conditions at the edges, the solution is assumed of the form as (10) which satisfies this boundary conditions.

Putting (10) in (9) that is using the double sine series we obtain this equation (11), where R is equal to ratio of length to the width of the plate, a/b is known as aspect ratio of the plate sometimes.

Now, in this equation, (11), one of the solution is, of course, $w_{mn} = 0$ but this is trivial. That means there is no deflection. Therefore, for non-trivial solution we obtain (12) whcih is the buckling load and note that it is actually a function of m and n the number of half sine wave along the x and y direction respectively.

Therefore, if $n = 1$, this is the smallest buckling load and the least value corresponding to a particular a m can be determined knowing, D_{11}, D_{12} and D_{22} and D_{66}. The values of D_{11}, D_{12}, D_{22}, D_{66} could be determined knowing the stacking sequence of the laminate and the individual the reduced transform stiffness matrix of the constituent lamina. So for a laminate, knowing

D_{11}, D_{12}, D_{22}, D_{66} and knowing R, the smallest buckling load corresponding to n = 1 and for different m could be obtained. Note that he smallest value of N for different m is, however, not very obvious.

What is important here to note that the buckling load actually depends upon all the components of bending stiffness D_{11}, D_{22}, D_{12}, D_{66}. Therefore, depending upon the stacking sequence of the laminate, it will be different as the elements of [D] matrix actually depend not only on the in the properties of the constituent laminar but also their stacking sequence.

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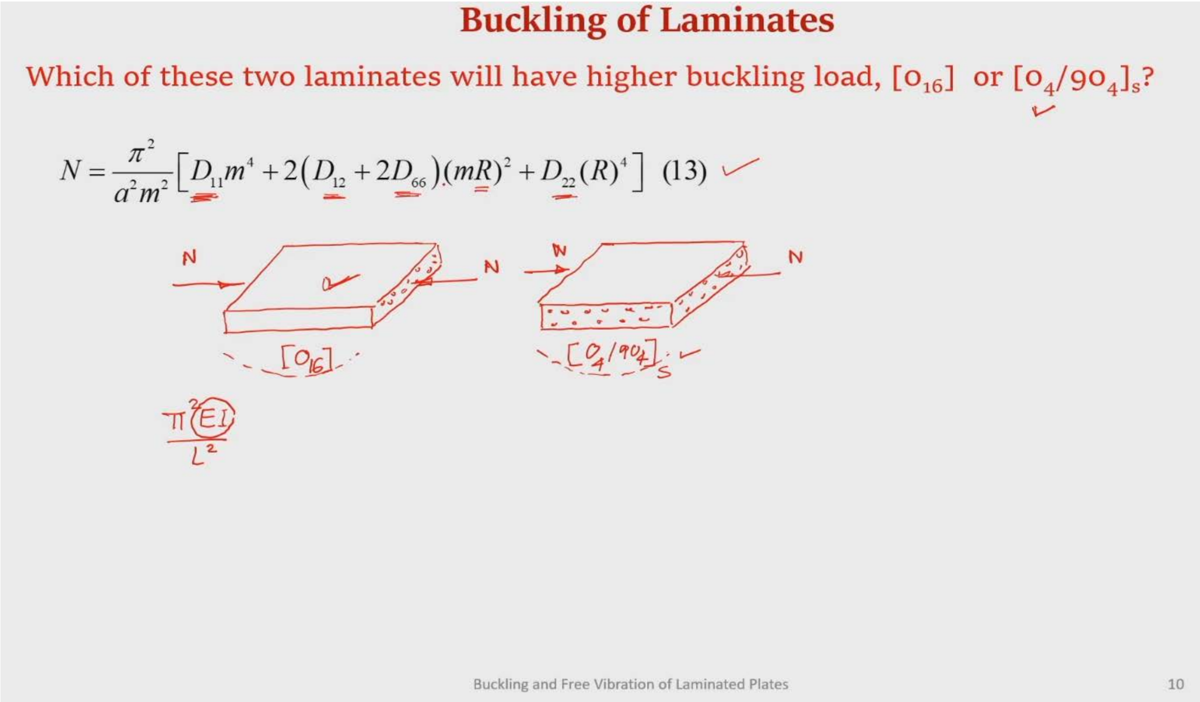

Now, the smallest buckling load is a function of the elements of [D] matrix, and of course, the plate dimension R, the aspect ratio. Now, suppose we have two laminates. One is all 0˚ layer we have total sixteen 0˚ layer ($[0^\circ_{16}]$) and this is subjected to axial load and in other with identical dimension, identical thickness, identical material but the stacking sequence is different say, $[0^\circ_4/90^\circ_4]_S$. Now, in which case, we expect the buckling load to be higher.

That means from the design point of view, we always try to see that the buckling load is higher. Now, when a column buckles, naturally bending stiffness (EI) decides the buckling load for column buckling as the critical buckling load (for both ends pinned) is

$$\frac{\pi^2 E\, I}{L^2}$$

Now, for a given L, more is EI, that is the bending stiffness, more is the buckling load.

Therefore, we may be tempted to think that the [0°_{16}] will have higher buckling load because all are 0˚. But it is not so, because in the case of a laminated plate it is not only D_{11}, it is also decided by D_{22}, D_{12}, D_{66}. Therefore, we may try using this formula, in which case actually the buckling load will be higher.

It is important to understand that for a plate of course because it is subjected to boundary conditions at the edge the buckling load is actually a function of all these stiffnesses D_{11}, D_{12}, D_{22}, D_{66}. Therefore, it is not very straightforward to tell that wherever D_{11} is high, the buckling load will be higher.

We have restricted to rectangular, symmetric special orthotropic laminate. But the same principle could actually be applied to other types of laminates. But the equations will be more involved, especially when they are coupled. It is not very easy to solve those equations analytically but numerical methods could be used but the principle remains same.

So, now next, we will consider the free vibration, basically transverse vibration of laminated plate.

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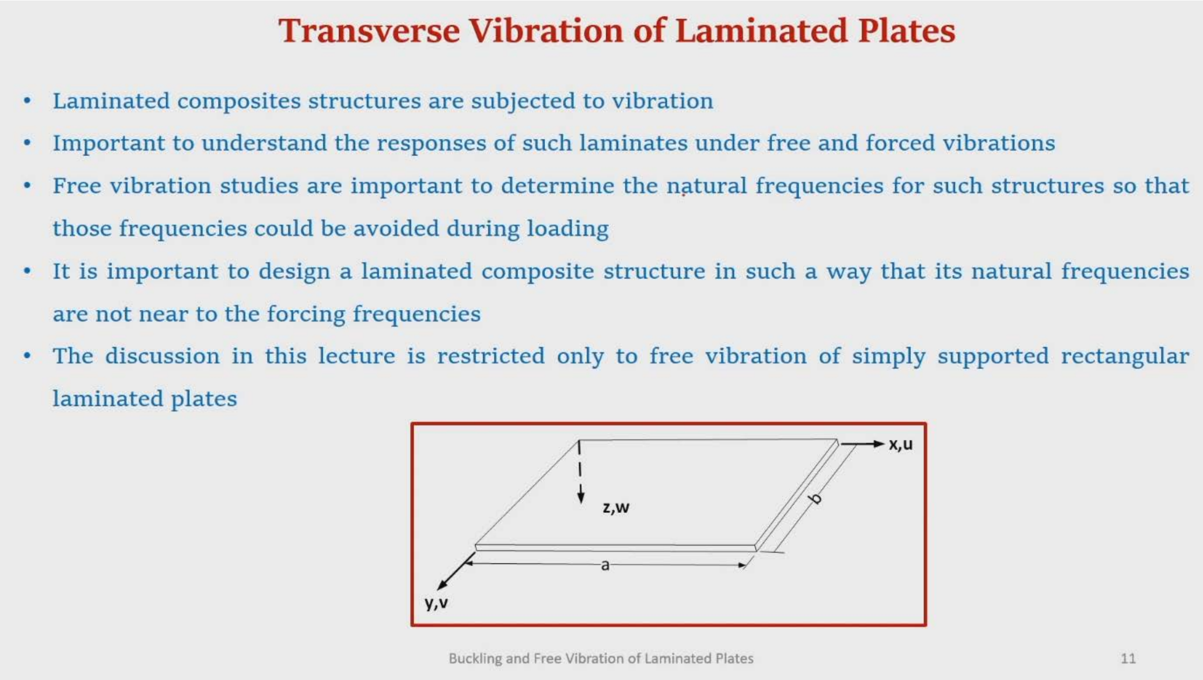

Now, laminated composite structures are also subjected to vibrations. It is important to understand the response of such laminates under free and forced vibrations. Now, free vibration studies are important to determine the natural frequencies for such structures. So that these frequencies could be avoided during loading. That means whenever there is a force vibration, we ensure that the forcing frequency is far away from the natural frequency to avoid resonance. Therefore, it is important to design laminated composite structures in such a way that its natural frequencies are not near to the forcing frequencies. The discussion in this lecture is again restricted to free vibration of simply supported, rectangular laminated plate. Of course, symmetric and specially orthotropic.

The objective here has been that what are the factors which actually influence the natural frequency and therefore in designing a laminate we must know how to design a laminate for a required range.

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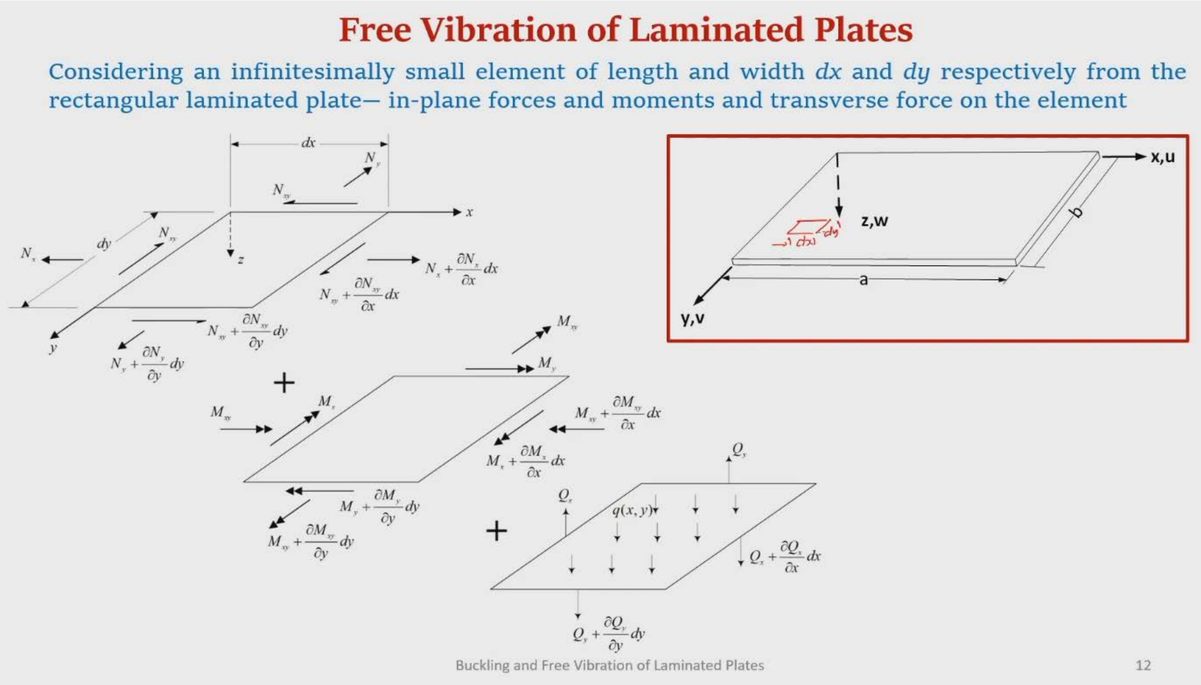

Again, we consider a rectangular eliminated plate with length a and width b and subjected to all generalized loading in-plane, force and moment resultants as well as uniformly distributed transverse load which we have also considered for transverse deflection. The laminate is represented by the mid plane as shown and considering a very small element from this laminate, with length dx and width dy the in-plane forces moments and transverse shear stress is resultants are shown in Fig.

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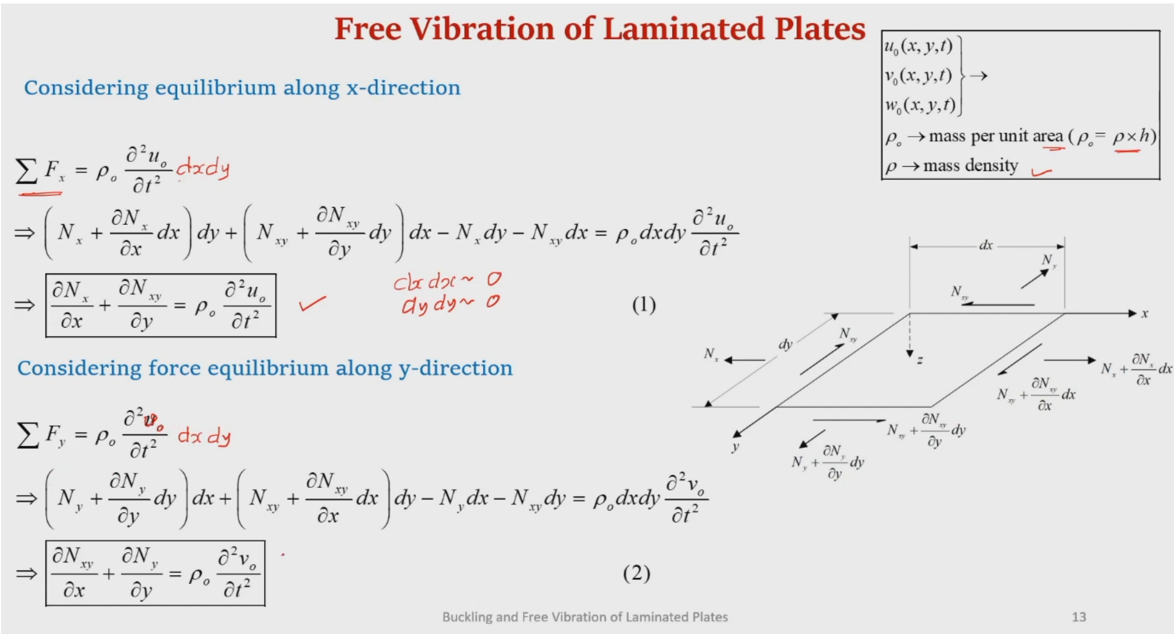

Considering the intertia force and equilibrium along x-and y- directions we obtain equations (1) and (2). Again, the smaller terms, $dx \cdot dx$, $dy \cdot dy$ (product terms) are set to 0. Note that ρ is the density and ρ_o is the mass per unit area. Also remember that this force, is actually force per unit length. Therefore, it is $\rho_o dx\, dy$ is the mass per unit length. Acceleration in the in the in the x and y directions are $\frac{\partial^2 u_o}{\partial t^2}$ *and* $\frac{\partial^2 v_o}{\partial t^2}$.

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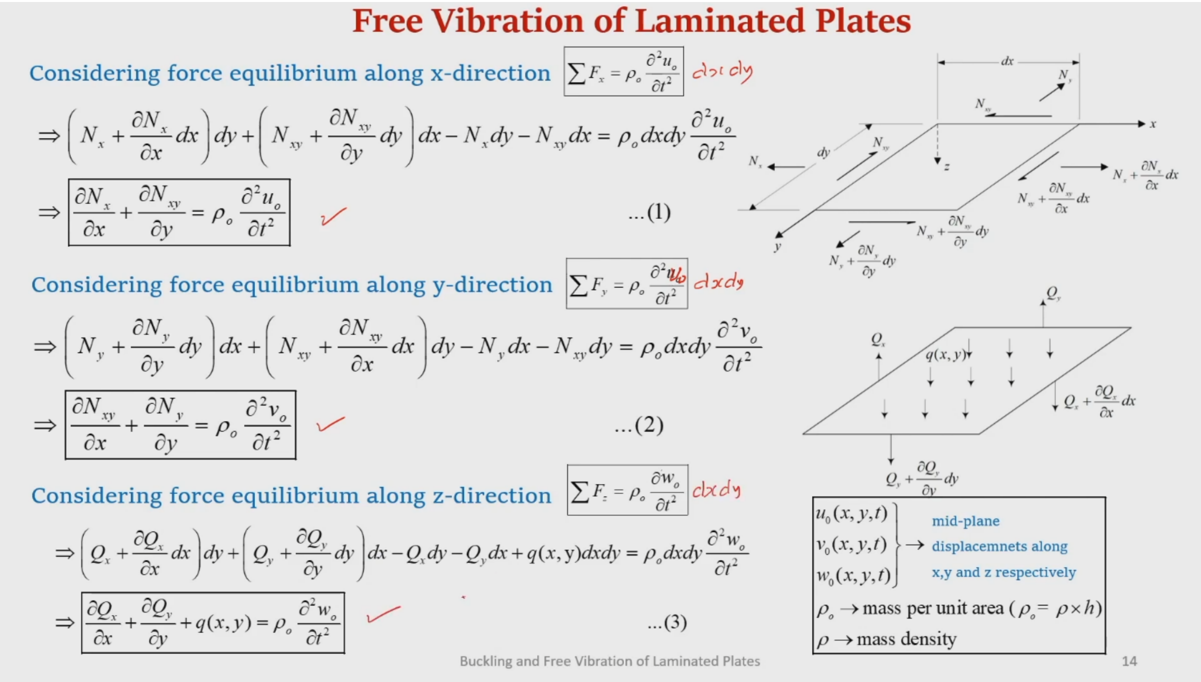

Similarly when we consider the force equilibrium along z- we get equation (3). So, we get equations (1), (2) and (3) from the equilibrium along x, y and z.

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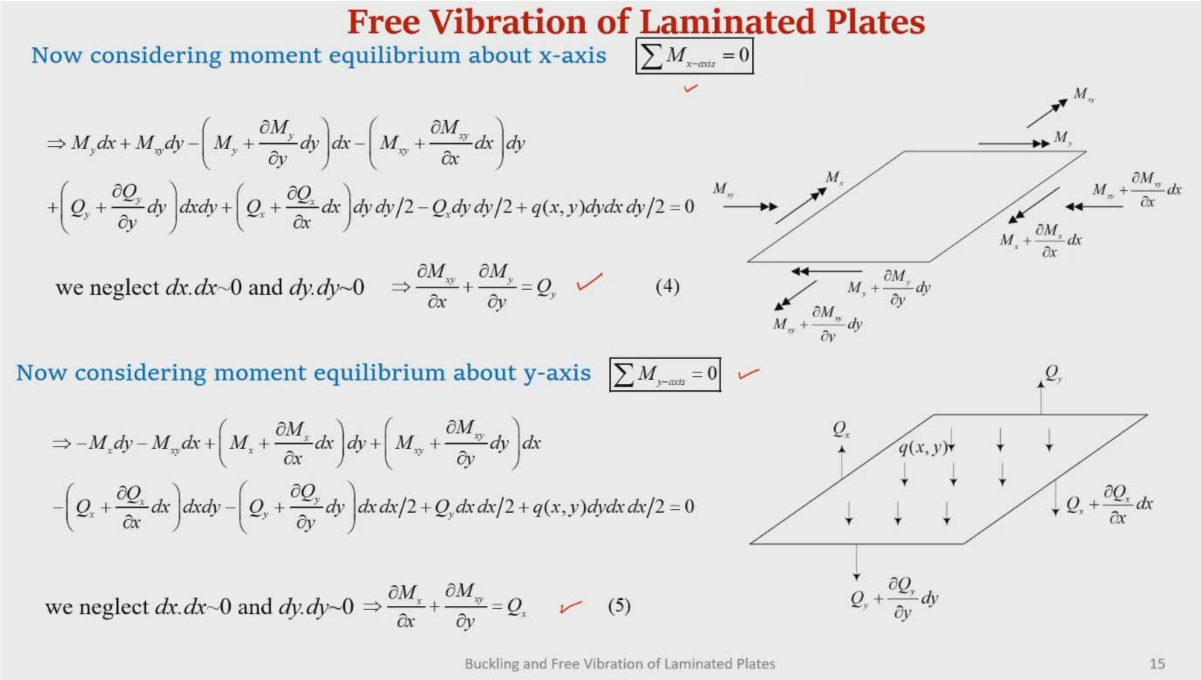

Considering moment equilibrium along x axis, and moment equilibrium about y axis, we get equations (4) and (5) respectively.

So, using this (4) and (5) in equation number (3) we get this equation (6). **(Refer slide 16)**. Now, from classical lamination theory, we have the force and moment result and expressed in terms of mid surface strains and curvatures as in equation (7) and (8).

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Free Vibration of Laminated Plates

Using (4) and (5) in (3), we obtain

$$\boxed{\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q(x,y) = \rho_o \frac{\partial^2 w_o}{\partial t^2}} \quad ...(6)$$

Now, we have

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^o \\ K \end{Bmatrix} \longleftarrow \text{CLT}$$

$$\Rightarrow \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^{\,0} \\ \varepsilon_y^{\,0} \\ \gamma_{xy}^{\,0} \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix} \quad (7)$$

$$\left.\begin{aligned} N_x &= A_{11}\varepsilon_x^{\,0} + A_{12}\varepsilon_y^{\,0} + A_{16}\gamma_{xy}^{\,0} + B_{11}K_x + B_{12}K_y + B_{16}K_{xy} \\ N_y &= A_{12}\varepsilon_x^{\,0} + A_{22}\varepsilon_y^{\,0} + A_{26}\gamma_{xy}^{\,0} + B_{12}K_x + B_{22}K_y + B_{26}K_{xy} \\ \Rightarrow N_{xy} &= A_{16}\varepsilon_x^{\,0} + A_{26}\varepsilon_y^{\,0} + A_{66}\gamma_{xy}^{\,0} + B_{16}K_x + B_{26}K_y + B_{66}K_{xy} \\ M_x &= B_{11}\varepsilon_x^{\,0} + B_{12}\varepsilon_y^{\,0} + B_{16}\gamma_{xy}^{\,0} + D_{11}K_x + D_{12}K_y + D_{16}K_{xy} \\ M_y &= B_{12}\varepsilon_x^{\,0} + B_{22}\varepsilon_y^{\,0} + B_{26}\gamma_{xy}^{\,0} + D_{12}K_x + D_{22}K_y + D_{26}K_{xy} \\ M_{xy} &= B_{16}\varepsilon_x^{\,0} + B_{26}\varepsilon_y^{\,0} + B_{66}\gamma_{xy}^{\,0} + D_{16}K_x + D_{26}K_y + D_{66}K_{xy} \end{aligned}\right\} (8)$$

Strain-Displacement and curvature displacement Relations

$$\left.\begin{aligned} &\varepsilon_x^{\,0} = \frac{\partial u_o}{\partial x},\ \varepsilon_y^{\,0} = \frac{\partial v_o}{\partial y},\ \gamma_{xy}^{\,0} = \left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) \\ &K_x = -\frac{\partial^2 w}{\partial x^2},\quad K_y = -\frac{\partial^2 w}{\partial y^2},\quad K_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} \end{aligned}\right\} (9)$$

Buckling and Free Vibration of Laminated Plates 16

Then using this (8) along with the strain displacement and curvature displacement relationship (9), we obtain (10).

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Free Vibration of Laminated Plates

Using (9) in (8), we obtain

$$\left.\begin{aligned} N_x &= A_{11}\frac{\partial u_o}{\partial x} + A_{12}\frac{\partial v_o}{\partial y} + A_{16}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - B_{11}\frac{\partial^2 w}{\partial x^2} - B_{12}\frac{\partial^2 w}{\partial y^2} - 2B_{16}\frac{\partial^2 w}{\partial x \partial y} \\ N_y &= A_{12}\frac{\partial u_o}{\partial x} + A_{22}\frac{\partial v_o}{\partial y} + A_{26}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - B_{12}\frac{\partial^2 w}{\partial x^2} - B_{22}\frac{\partial^2 w}{\partial y^2} - 2B_{26}\frac{\partial^2 w}{\partial x \partial y} \\ N_{xy} &= A_{16}\frac{\partial u_o}{\partial x} + A_{26}\frac{\partial v_o}{\partial y} + A_{66}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - B_{16}\frac{\partial^2 w}{\partial x^2} - B_{26}\frac{\partial^2 w}{\partial y^2} - 2B_{66}\frac{\partial^2 w}{\partial x \partial y} \\ M_x &= B_{11}\frac{\partial u_o}{\partial x} + B_{12}\frac{\partial v_o}{\partial y} + B_{16}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} - 2D_{16}\frac{\partial^2 w}{\partial x \partial y} \\ M_y &= B_{12}\frac{\partial u_o}{\partial x} + B_{22}\frac{\partial v_o}{\partial y} + B_{26}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2} - 2D_{26}\frac{\partial^2 w}{\partial x \partial y} \\ M_{xy} &= B_{16}\frac{\partial u_o}{\partial x} + B_{26}\frac{\partial v_o}{\partial y} + B_{66}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - D_{16}\frac{\partial^2 w}{\partial x^2} - D_{26}\frac{\partial^2 w}{\partial y^2} - 2D_{66}\frac{\partial^2 w}{\partial x \partial y} \end{aligned}\right\} (10)$$

$w = w_o$

Using (10) in (1), we obtain

$$\begin{aligned} &A_{11}\frac{\partial^2 u_o}{\partial x^2} + 2A_{16}\frac{\partial^2 u_o}{\partial x \partial y} + A_{66}\frac{\partial^2 u_o}{\partial y^2} + A_{16}\frac{\partial^2 v_o}{\partial x^2} + \left(A_{12} + A_{66}\right)\frac{\partial^2 v_o}{\partial x \partial y} + A_{26}\frac{\partial^2 v_o}{\partial y^2} \\ &- B_{11}\frac{\partial^3 w}{\partial x^3} - 3B_{16}\frac{\partial^3 w}{\partial x^2 \partial y} - \left(B_{12} + 2B_{66}\right)\frac{\partial^3 w}{\partial x \partial y^2} - B_{26}\frac{\partial^3 w}{\partial y^3} = \rho_o \frac{\partial^2 u_o}{\partial t^2} \end{aligned} \quad (11)$$

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Therefore, in (10), we can get the force and moment resultants in terms of the displacements, mid surface displacement u_o, v_o and w_o . Again note here that $w = w_o$, because w does not depend upon the z direction. The mid surface displacement is same as the displacement in any other point along the thickness. And then putting (10) ie. N_x, N_y, N_{xy}, M_x, M_y, M_{xy} in equation number (1) we get (11). (**Refer slide 14**).

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Free Vibration of Laminated Plates

Using (10) in (2), we obtain

$$A_{16}\frac{\partial^2 u_o}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 u_o}{\partial x\partial y}+A_{26}\frac{\partial^2 u_o}{\partial y^2}+A_{66}\frac{\partial^2 v_o}{\partial x^2}+2A_{26}\frac{\partial^2 v_o}{\partial x\partial y}+A_{22}\frac{\partial^2 v_o}{\partial y^2}$$
$$-B_{16}\frac{\partial^3 w}{\partial x^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 w}{\partial x\partial y^2}-B_{22}\frac{\partial^3 w}{\partial y^3}=\rho_o\frac{\partial^2 v_o}{\partial t^2} \qquad \ldots(12)$$

Using (10) in (6), we obtain

$$D_{11}\frac{\partial^4 w}{\partial x^4}+4D_{16}\frac{\partial^4 w}{\partial x^3\partial y}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w}{\partial x^2\partial y^2}+4D_{26}\frac{\partial^4 w}{\partial x\partial y^3}+D_{22}\frac{\partial^4 w}{\partial y^4}$$
$$-B_{11}\frac{\partial^3 u_o}{\partial x^3}-3B_{16}\frac{\partial^3 u_o}{\partial x^2\partial y}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 u_o}{\partial x\partial y^2}-B_{26}\frac{\partial^3 u_o}{\partial y^3}-B_{16}\frac{\partial^3 v_o}{\partial x^3}$$
$$-\left(B_{12}+2B_{66}\right)\frac{\partial^3 v_o}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 v_o}{\partial x\partial y^2}-B_{22}\frac{\partial^3 v_o}{\partial y^3}+\rho_o\frac{\partial^2 w_o}{\partial t^2}=q(x,y) \qquad \ldots(13)$$

Equations (11), (12) and (13) represent the coupled differential equations with u, v and w as unknown

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Similarly, putting (10) in (2), we get equation number (12). Then (10) in (6) get equation (13).

So, you could see that equation number (11), (12) and (13) actually represent coupled differential equations with u v w as unknown and they are coupled.

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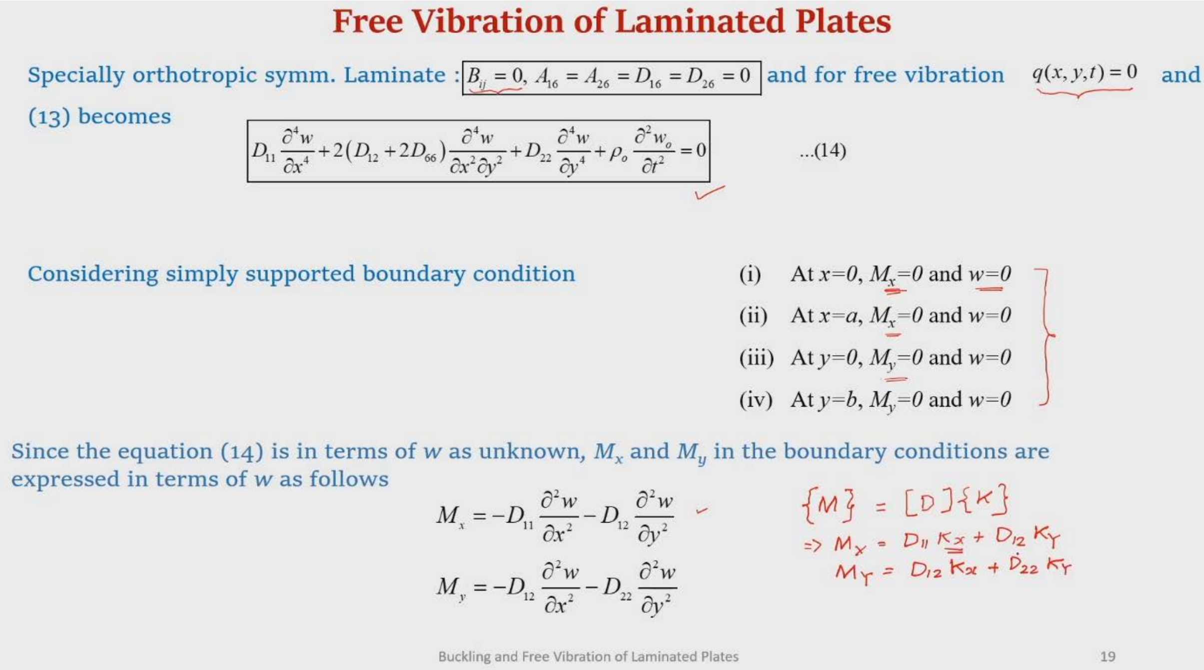

Now, for a specially orthotropic, symmetric laminate, this we can actually uncouple them because all these B_{ij} terms are 0. Also, for a special orthotropic laminate this A_{16}, A_{26}, D_{12}, D_{16}, D_{26} are 0. For free vibration there is no load, so, $q(x, y, t)$ is also 0 and putting these in (13) for transverse vibration, we obtain this equation number (14). Now, considering simply supported boundary conditions at all the edges, w = 0, as well as moment = 0. Like along x-axis w = 0 and M_x = 0, along y, w = 0 and M_y = 0. Now because this equation is actually in terms of w as unknown therefore, these force boundary conditions like M_x = 0, are also expressed in terms of displacement boundary conditions using $\{M\}=[D]\{K\}$ from the ABBD matrix (see the lower right portion in slide 19). ~~We can uncouple them~~. Therefore, we can write:

$$M_x = D_{11}K_x + D_{12}K_y \ ; \ M_y = D_{12}K_x + D_{22}K_y.$$

D_{16} is 0 because we have considered specially orthotropic and

and putting $K_x = -\frac{\partial^2 w}{\partial x^2}$, $K_y = -\frac{\partial^2 w}{\partial y^2}$ and we get

$$M_x = -D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2}$$

$$M_y = -D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2}$$

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Free Vibration of Laminated Plates

(i) At $x = 0$, $\left(-D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2}\right)_{x=0} = 0$ and $w = 0$

(ii) At $x = a$, $\left(-D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2}\right)_{x=a} = 0$ and $w = 0$

(iii) At $y = 0$, $\left(-D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2}\right)_{y=0} = 0$ and $w = 0$

(iv) At $y = b$, $\left(-D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2}\right)_{y=b} = 0$ and $w = 0$

The free vibration being harmonic in time, the solution with frequency ω is assumed to be of the form

$$w(x,y,t) = w(x,y)(A\mathrm{Cos}\,\omega t + B\mathrm{Sin}\,\omega t) \qquad (15)$$

And the problem is separated into time and space. The resulting differential equation and boundary conditions are satisfied with the following spatial distribution of transverse displacement

$$w(x,y) = W_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \qquad (16)$$

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The four boundary conditions are as in (i) – (iv). Now, for free vibration, it is being harmonic in time, the solution with frequency, ω (omega) is assumed to be of the form as in equation number (15) where this *w(x, y)* could be spatial. Distribution of *w(x, y)* could be actually taken as a double sine series as written in equation number (16) which actually satisfies this simply supported boundary conditions.

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Free Vibration of Laminated Plates

Puting (15) and (16) into (14) leads to

$$\omega^2 = \frac{\pi^4}{\rho_o a^4}\left[D_{11}m^4 + 2\left(D_{12} + 2D_{66}\right)(mnR)^2 + D_{22}(nR)^4\right] \quad (17)$$

where the various natural frequencies ω corresponds to different mode shapes (corresponding to different values of m and n in (16) and accordingly different shapes of w) and we write

$$\omega_{mn}^2 = \frac{\pi^4}{\rho_o a^4}\left[D_{11}m^4 + 2\left(D_{12} + 2D_{66}\right)(mnR)^2 + D_{22}(nR)^4\right] \quad (18)$$

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And when we put (15) and (16) in (14), we get equation (17) and (18). So, the various natural frequencies, ω, corresponding to different mode shapes basically corresponding to different values of *m* and *n*, (*m* and *n* are the number of half sin wave along x and y) and accordingly different shapes. Note that here again, natural frequency, ω is a function of D_{11}, D_{12}, D_{66} and D_{22}. So, depending upon the values of this elements of [D] matrix, they will be different.

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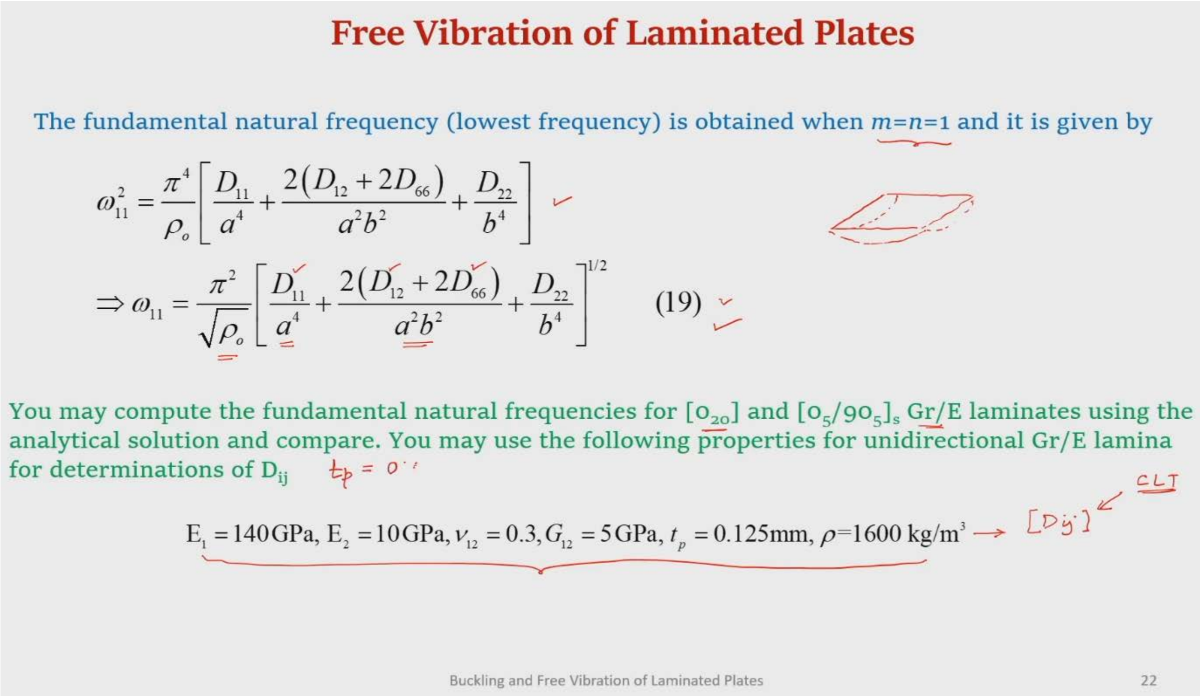

So, fundamental natural frequency of the lowest frequency is obtained when m = 1 and n = 1 and it is for m = 1 and n = 1. This means that the first natural frequency will be corresponding to one half sine wave in the x direction and one half sine wave in the y direction.

Now, this is for a rectangular plate simply supported at all edges symmetric, specially orthotropic laminate. Similar approach could be also applied to for other more general laminated plates. Maybe the equations will be more involved.

Here the objective here has been to understand the factors which actually affect the free vibration response. We could see that it is decided by the dimensions a and b and in addition, more importantly, the frequency will be decided by the stiffness and the mass. So, ρ is the density and the stiffness here for this orthotropic plate is actually all these components: D_{11}, D_{12}, D_{66} and D_{22}. So, we may actually compute the fundamental natural frequencies again two different laminates: one [0°_{20}] having 20 layers and another is [$0^\circ/90^\circ$]$_{5S}$ (symmetric laminate) to appreciate how the natural frequencies actually vary with stacking sequence. You may take the properties for graphite epoxy to find out what is [D] using classical lamination theory and ply thickness as 0.1 mm.

In today's lecture we understood what are the factors actually influence in in deciding the buckling load as well as in deciding the natural frequencies of laminated plates. Therefore, in design if we have to design a particular component with a particular requirement of buckling load and the then we can have a first-hand idea that how the stacking sequence should be.

Similarly, if we have a requirement of natural frequency for a particular, laminate. Then we understand that what are the factors which actually influence and accordingly, we may decide the stacking sequence of that particular laminate.