Mechanics of Fiber Reinforced Polymer Composite Structures
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Lecture – 31
Transverse Deflection

In today’s lecture, we will discuss the transverse deflection of laminate.

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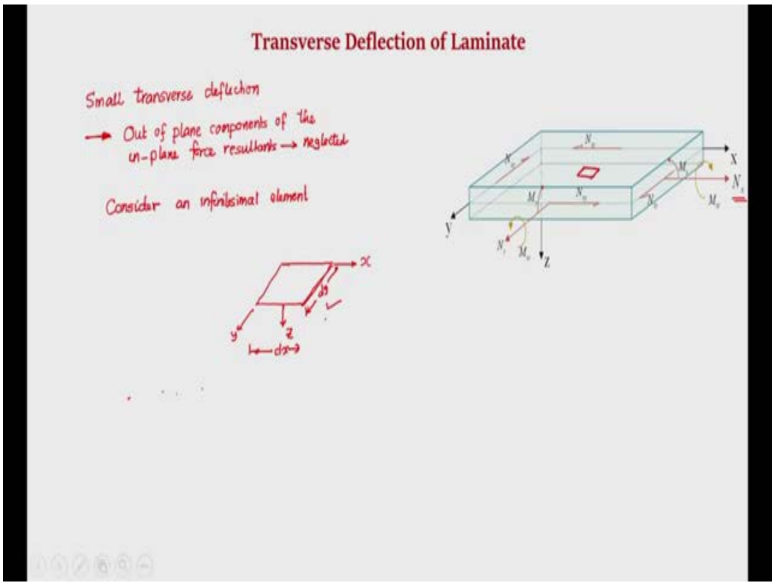

Transverse deflection of laminate is obtained using the classical lamination theory and in conjunction with the equilibrium equations. In classical lamination theory, we considered a laminate where the stresses in each layer are actually represented by the force and moment resultants acting in the mid plane of the laminate. However, the transverse shear stresses were not considered. But in order to analyze the transverse deflection, the transverse shear stress resultant also needs to be considered.

We will consider small transverse deflection and because it is small, the out of plane component of the in-plane force resultants are neglected. Considering an infinitely small element from this laminated plate as shown in the Fig. represented by its mid surface, the force resultants (N_x, N_y, N_{xy}) and moment resultants (M_x, M_y, M_{xy}) are as shown in the Fig.

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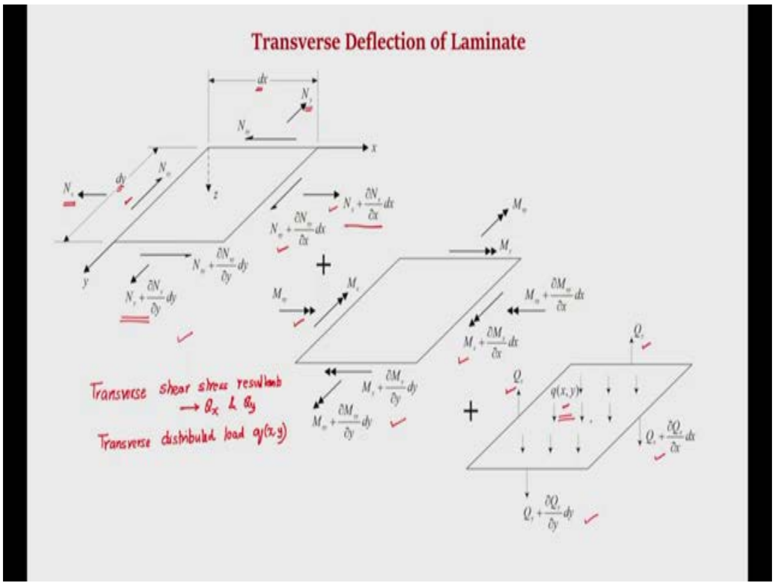

Because we will be analyzing the transverse deflection, therefore, in addition, we have also considered a transverse distributed load q_{xy}. And the transverse shear stress resultant Q_x and Q_y. So, considering a small element whose this length is *dx* and width is *dy*, we could write the forces at the two edges as N_x ($x=0$) and at a distance of *dx* from this the force is $N_x + \frac{\partial N_x}{\partial x}dx$ (Taylor's theorem). We could write N_y (at $y=0$) is the force resultant and at a distance of *dy*, the force resultant is $N_y + \frac{\partial N_y}{\partial y}dy$.

Similarly, the in-plane shear resultants are N_{xy} and this $N_{xy} + \frac{\partial N_{xy}}{\partial x}dx$ at a distance of *dx*.

In the same way the transverse shear stress resultant it is Q_x ($x=0$) and $Q_x + \frac{\partial Q_x}{\partial x}dx$ at a distance of *dx*. Similarly, this Q_y ($y=0$) and at a distance of *dy* it is $Q_y + \frac{\partial Q_y}{\partial y}dy$.

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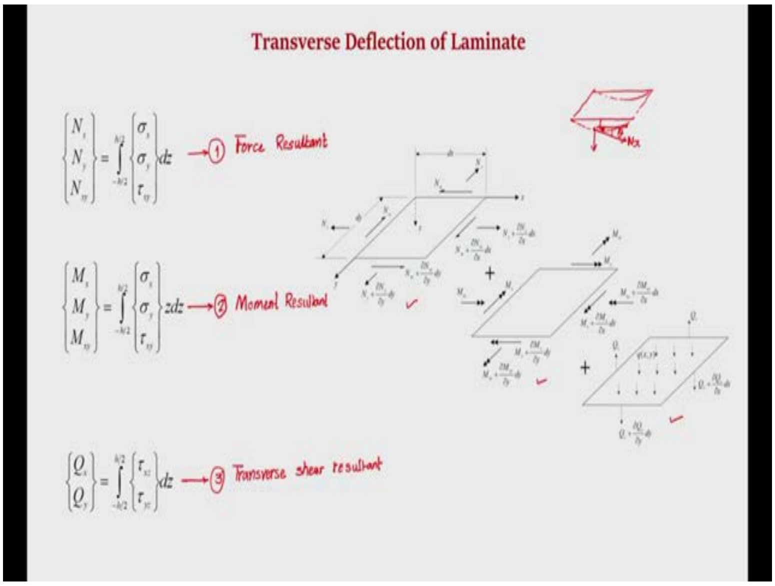

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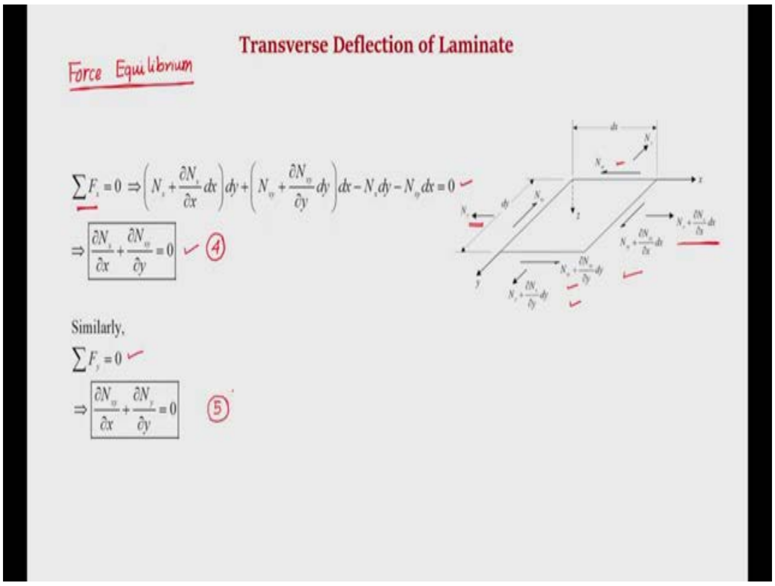

Now considering the force as well as moment equilibrium.

First, we consider the force equilibrium along x- direction, $\sum F_x = 0$ and it results in

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0.$$

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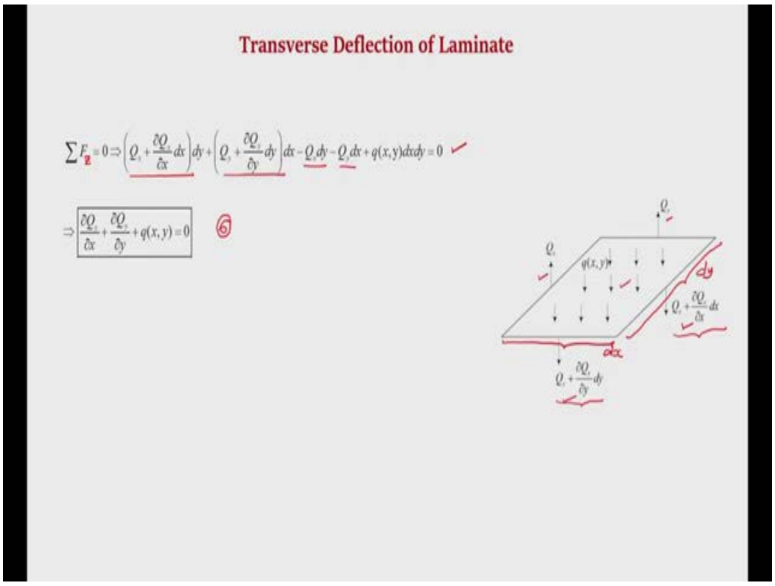

Similarly, considering the force equilibrium along y- direction, $\sum F_y = 0$ results in

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

Considering the force equilibrium along z- direction, Q_x ($x=0$) and Q_y ($y=0$) are along negative z and $Q_x + \frac{\partial Q_x}{\partial x} dx$ (at $x+dx$) and s $Q_y + \frac{\partial Q_y}{\partial y} dy$ (at $y+dy$) are along positive z. Note that, all this force and moment resultants are actually per unit length. Therefore, in force equilibrium, we must actually represent the force, multiplying it by the length. Also, *q(x,y)* is the distributed load and multiplied by the area *q(x,y)dxdy* is the force. So,

$$\sum F_z = 0 \Rightarrow \left(Q_x + \frac{\partial Q_x}{\partial x} dx \right) dy + \left(Q_y + \frac{\partial Q_y}{\partial y} dy \right) dx - Q_x dy - Q_y dx + q(x, y) dxdy = 0$$

$$\Rightarrow \boxed{\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q(x, y) = 0} \qquad (6)$$

It is important that in force equilibrium, it must be total force and not the force per unit length. So, we get three equations each from equilibrium along x, y and z, direction force equilibrium.

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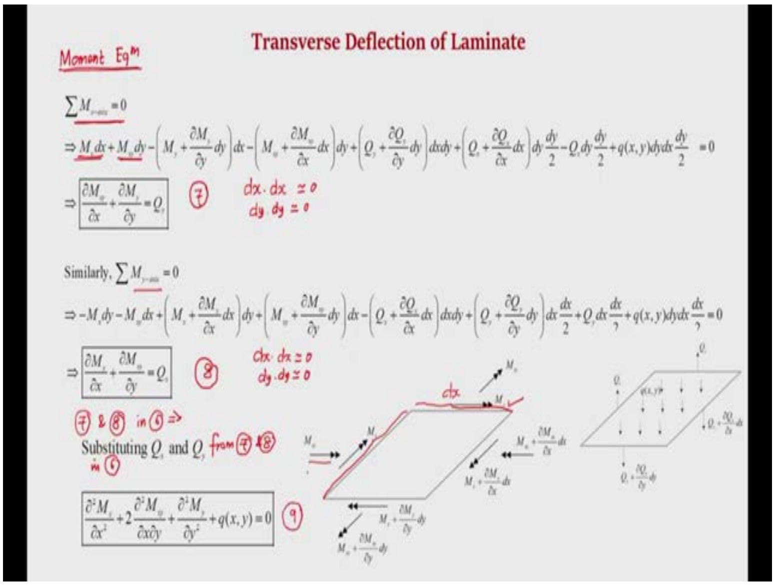

Now, considering the moment equilibrium about x axis

$$\sum M_{x-axis} = 0$$

$$\Rightarrow M_y dx + M_{xy} dy - \left(M_y + \frac{\partial M_y}{\partial y} dy \right) dx - \left(M_{xy} + \frac{\partial M_{xy}}{\partial x} dx \right) dy + \left(Q_y + \frac{\partial Q_y}{\partial y} dy \right) dx dy + \left(Q_x + \frac{\partial Q_x}{\partial x} dx \right) dy \frac{dy}{2} - Q_x dy \frac{dy}{2} + q(x,y) dy dx \frac{dy}{2} = 0$$

$$\Rightarrow \boxed{\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = Q_y} \qquad (7)$$

Now, considering the moment equilibrium about y axis

$$\sum M_{y-axis} = 0$$

$$\Rightarrow -M_x dy - M_{xy} dx + \left(M_x + \frac{\partial M_x}{\partial x} dx \right) dy + \left(M_{xy} + \frac{\partial M_{xy}}{\partial y} dy \right) dx - \left(Q_x + \frac{\partial Q_x}{\partial x} dx \right) dx dy + \left(Q_y + \frac{\partial Q_y}{\partial y} dy \right) dx \frac{dx}{2} + Q_y dx \frac{dx}{2} + q(x,y) dy dx \frac{dx}{2} = 0$$

$$\Rightarrow \boxed{\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x} \qquad (8)$$

Important to note that M_x and M_y are moments per unit length and in moment equilibrium, also, we must write the total moment and not the moment per unit length. Putting (7) and (8) in (6) leads to

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q(x,y) = 0 \qquad (9)$$

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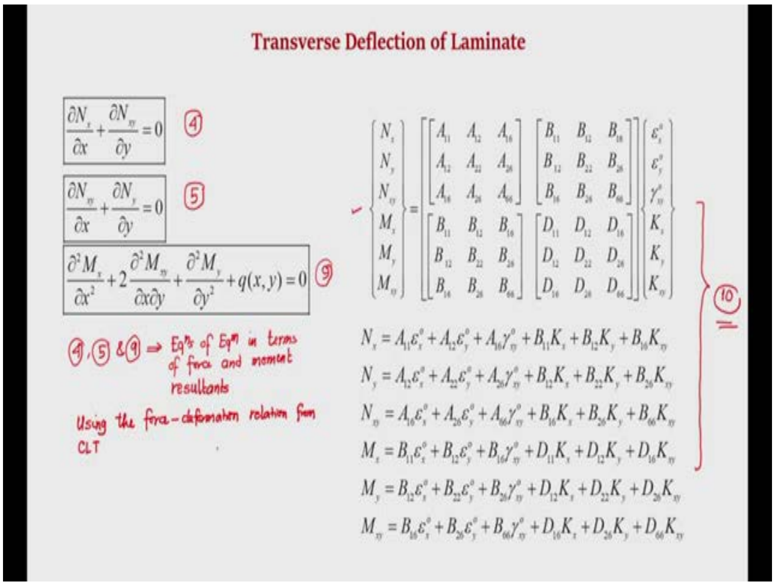

Therefore, (4), (5) and (9) are the equations of equilibrium in terms of force and moment resultants. Now, using the force deformation relation from classical lamination theory, where the force resultant and moment resultants are related to the mid-surface strains and curvatures by ABBD matrix as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{13} & A_{23} & A_{66} \end{bmatrix} & \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{13} & B_{23} & B_{66} \end{bmatrix} \\ \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{13} & B_{23} & B_{66} \end{bmatrix} & \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{13} & D_{23} & D_{66} \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix}$$

and expanding this, we get

$$\begin{aligned} N_x &= A_{11}\varepsilon_x^o + A_{12}\varepsilon_y^o + A_{16}\gamma_{xy}^o + B_{11}K_x + B_{12}K_y + B_{16}K_{xy} \\ N_y &= A_{12}\varepsilon_x^o + A_{22}\varepsilon_y^o + A_{26}\gamma_{xy}^o + B_{12}K_x + B_{22}K_y + B_{26}K_{xy} \\ N_{xy} &= A_{16}\varepsilon_x^o + A_{26}\varepsilon_y^o + A_{66}\gamma_{xy}^o + B_{16}K_x + B_{26}K_y + B_{66}K_{xy} \\ M_x &= B_{11}\varepsilon_x^o + B_{12}\varepsilon_y^o + B_{16}\gamma_{xy}^o + D_{11}K_x + D_{12}K_y + D_{16}K_{xy} \\ M_y &= B_{12}\varepsilon_x^o + B_{22}\varepsilon_y^o + B_{26}\gamma_{xy}^o + D_{12}K_x + D_{22}K_y + D_{26}K_{xy} \\ M_{xy} &= B_{16}\varepsilon_x^o + B_{26}\varepsilon_y^o + B_{66}\gamma_{xy}^o + D_{16}K_x + D_{26}K_y + D_{66}K_{xy} \end{aligned} \quad (10)$$

So, using (10), we can write these equations of equilibrium in terms of strains as

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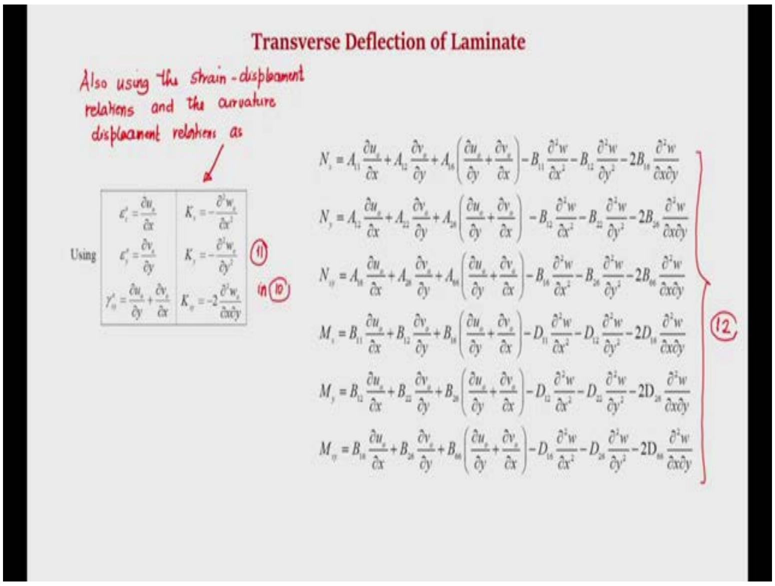

Using the strain displacement and the curvature displacement relations as

$$\boxed{\begin{array}{c:c} \varepsilon_x^o = \dfrac{\partial u_o}{\partial x} & K_x = -\dfrac{\partial^2 w_o}{\partial x^2} \\ \varepsilon_y^o = \dfrac{\partial v_o}{\partial y} & K_y = -\dfrac{\partial^2 w_o}{\partial y^2} \\ \gamma_{xy}^o = \dfrac{\partial u_o}{\partial y} + \dfrac{\partial v_o}{\partial x} & K_{xy} = -2\dfrac{\partial^2 w_o}{\partial x \partial y} \end{array}} \quad (11)$$

And putting (11) in (10), we can write this force and moment resultants in terms of and u_0, v_0 and w the mid-surface displacements as

$$\begin{aligned}
N_x &= A_{11}\frac{\partial u_o}{\partial x} + A_{12}\frac{\partial v_o}{\partial y} + A_{16}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - B_{11}\frac{\partial^2 w}{\partial x^2} - B_{12}\frac{\partial^2 w}{\partial y^2} - 2B_{16}\frac{\partial^2 w}{\partial x \partial y} \\
N_y &= A_{12}\frac{\partial u_o}{\partial x} + A_{22}\frac{\partial v_o}{\partial y} + A_{26}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - B_{12}\frac{\partial^2 w}{\partial x^2} - B_{22}\frac{\partial^2 w}{\partial y^2} - 2B_{26}\frac{\partial^2 w}{\partial x \partial y} \\
N_{xy} &= A_{16}\frac{\partial u_o}{\partial x} + A_{26}\frac{\partial v_o}{\partial y} + A_{66}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - B_{16}\frac{\partial^2 w}{\partial x^2} - B_{26}\frac{\partial^2 w}{\partial y^2} - 2B_{66}\frac{\partial^2 w}{\partial x \partial y} \\
M_x &= B_{11}\frac{\partial u_o}{\partial x} + B_{12}\frac{\partial v_o}{\partial y} + B_{16}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} - 2D_{16}\frac{\partial^2 w}{\partial x \partial y} \\
M_y &= B_{12}\frac{\partial u_o}{\partial x} + B_{22}\frac{\partial v_o}{\partial y} + B_{26}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2} - 2D_{26}\frac{\partial^2 w}{\partial x \partial y} \\
M_{xy} &= B_{16}\frac{\partial u_o}{\partial x} + B_{26}\frac{\partial v_o}{\partial y} + B_{66}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - D_{16}\frac{\partial^2 w}{\partial x^2} - D_{26}\frac{\partial^2 w}{\partial y^2} - 2D_{66}\frac{\partial^2 w}{\partial x \partial y}
\end{aligned} \quad (12)$$

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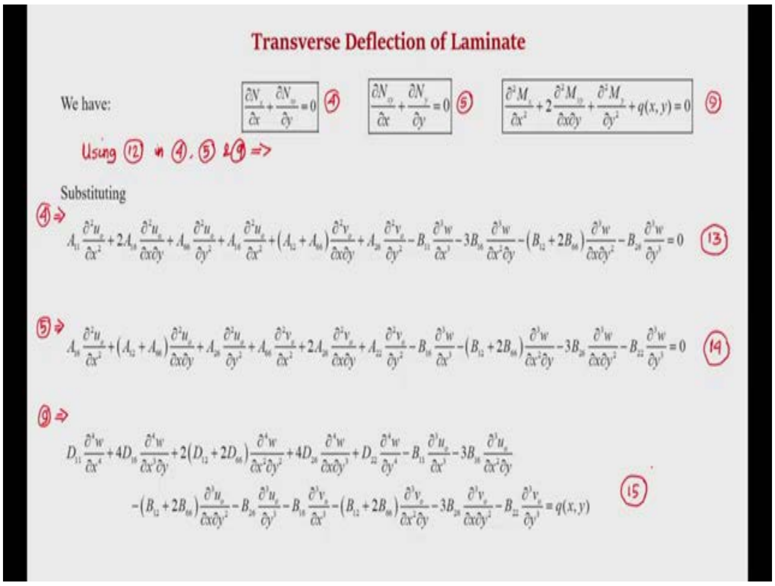

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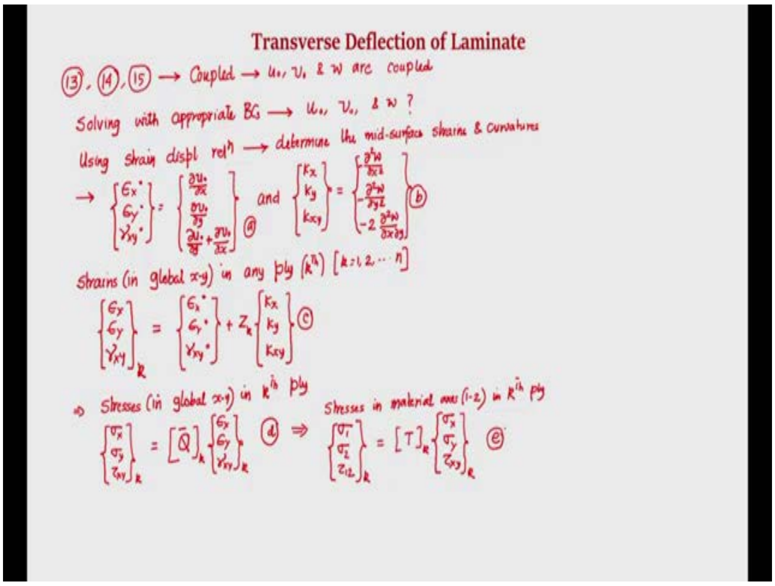

Using these relations from (12) in (4), (5) and (9), we get (13), (14) and(15) respectively as

$$A_{11}\frac{\partial^2 u_o}{\partial x^2}+2A_{16}\frac{\partial^2 u_o}{\partial x\partial y}+A_{66}\frac{\partial^2 u_o}{\partial y^2}+A_{16}\frac{\partial^2 u_o}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 v_o}{\partial x\partial y}+A_{26}\frac{\partial^2 v_o}{\partial y^2}-B_{11}\frac{\partial^3 w}{\partial x^3}-3B_{16}\frac{\partial^3 w}{\partial x^2\partial y}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w}{\partial x\partial y^2}-B_{26}\frac{\partial^3 w}{\partial y^3}=0 \quad (13)$$

$$A_{16}\frac{\partial^2 u_o}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 u_o}{\partial x\partial y}+A_{26}\frac{\partial^2 u_o}{\partial y^2}+A_{66}\frac{\partial^2 v_o}{\partial x^2}+2A_{26}\frac{\partial^2 v_o}{\partial x\partial y}+A_{22}\frac{\partial^2 v_o}{\partial y^2}-B_{16}\frac{\partial^3 w}{\partial x^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 w}{\partial x\partial y^2}-B_{22}\frac{\partial^3 w}{\partial y^3}=0 \quad (14)$$

$$\begin{aligned}&D_{11}\frac{\partial^4 w}{\partial x^4}+4D_{16}\frac{\partial^4 w}{\partial x^3\partial y}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w}{\partial x^2\partial y^2}+4D_{26}\frac{\partial^4 w}{\partial x\partial y^3}+D_{22}\frac{\partial^4 w}{\partial y^4}-B_{11}\frac{\partial^3 u_o}{\partial x^3}-3B_{16}\frac{\partial^3 u_o}{\partial x^2\partial y}\\&\quad-\left(B_{12}+2B_{66}\right)\frac{\partial^3 u_o}{\partial x\partial y^2}-B_{26}\frac{\partial^3 u_o}{\partial y^3}-B_{16}\frac{\partial^3 v_o}{\partial x^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 v_o}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 v_o}{\partial x\partial y^2}-B_{22}\frac{\partial^3 v_o}{\partial y^3}=q(x,y)\end{aligned} \quad (15)$$

So, they are coupled in the sense that u_0, v_0 and w are coupled and solving these three equations with appropriate boundary conditions, we could obtain, u_0, v_0 and w. However, it is not that easy to solve these coupled equations analytically. Many a times, numerical methods are used.

So, once we have u_0, v_0 and w, using strain-displacement relationship and the curvature displacement relationship, we can determine the mid-surface strains and curvatures as

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} \text{ and } \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ -2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \quad (16)$$

Therefore, the strains (in global x-y) in any ply or lamina (k^{th} ply, k=1,2,3, …,n) could be determined as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + Z_k \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} \quad (17)$$

Now, knowing the global strain in each ply, we could determine the global stresses in the k^{th} ply (k=1,2,3, …,n) using

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \left[\bar{Q}\right]_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k \quad (18)$$

So, we could obtain the global stresses in each ply and from these global stresses (x-y) we could obtain the stresses in material axis (1-2) in the k^{th} ply using the stress transformation as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}_k = \left[T\right]_k \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k \quad (19).$$

Knowing the stresses in each ply in the material axis, we can apply appropriate failure theory to assess the failure or safety of each lamina. Here in addition to u_0 and v_0 which we could determine in the case of classical lamination theory, we could also determine w the transverse deflection. The whole procedure discussed here is for small transverse deformation.

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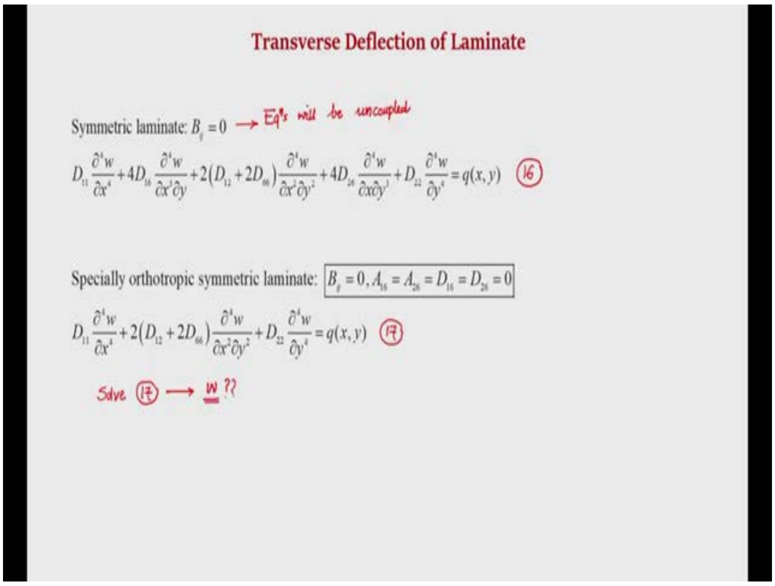

However, considering a more simplified situation of a symmetric laminate ($B_{ij} = 0$), these three equations will be uncoupled. In addition if we consider a specially orthotropic laminate which has only 0^0 and 90^0 lamina, there is no lamina level shear extension coupling. and $A_{16} = A_{26} = D_{16} = D_{26} = 0$. Therefore, (15) reduces to the equation (20) which is the equation representing the transverse displacement as the unknown as

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2\left(D_{12} + 2D_{66}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} = q(x,y) \quad (20)$$

and solving (20) with appropriate boundary conditions, we could determine the transverse displacement *w*.

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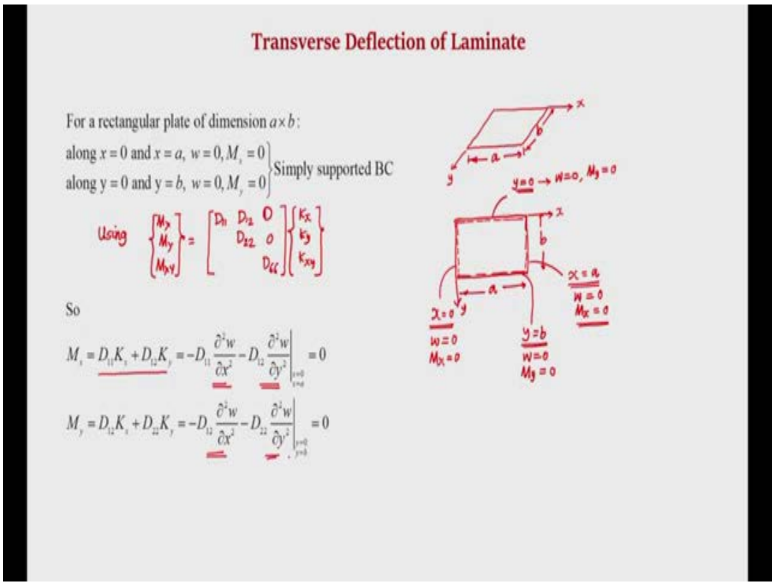

Considering a rectangular plate of dimension ($a \times b$), represented by the middle surface only (as shown in Fig.) and considering all the edges are simply supported, the boundary conditions are

For a rectangular plate of dimension $a \times b$:

$$\left.\begin{array}{l} \text{along } x = 0 \text{ and } x = a,\ w = 0, M_x = 0 \\ \text{along y} = 0 \text{ and y} = b,\ w = 0, M_y = 0 \end{array}\right\} \text{Simply supported BC}$$

In the case of symmetric laminate we could use this decoupled moment curvature relationship

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ & D_{22} & 0 \\ & & D_{66} \end{bmatrix} \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$$

And write the boundary conditions in terms of displacements as

$$\begin{aligned} M_x &= D_{11}K_x + D_{12}K_y = -D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2}\bigg|_{\substack{x=0\\x=a}} = 0 \\ M_y &= D_{12}K_x + D_{22}K_y = -D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2}\bigg|_{\substack{y=0\\y=b}} = 0 \end{aligned} \qquad (21)$$

Using these boundary conditions, we could solve this equation to find out what is the transverse deflection w.

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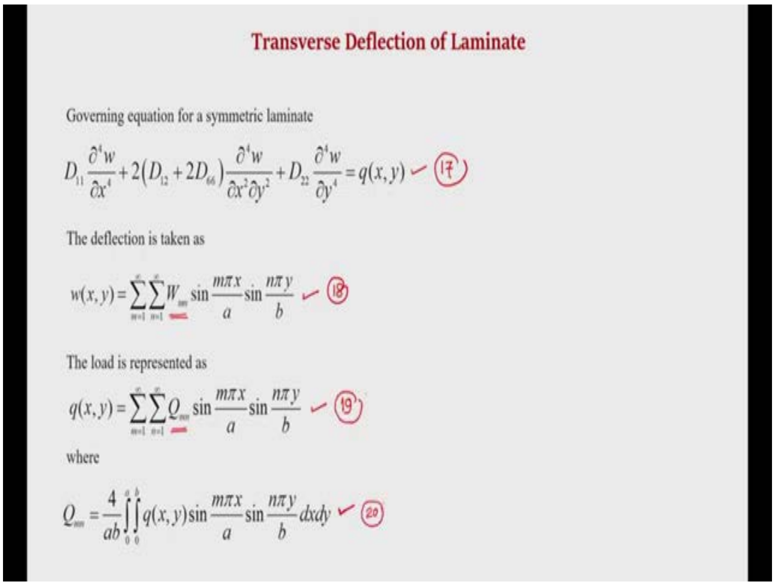

There are different methods for solution of this equation. However, double Fourier sine series is one of the simple methods of solving this, where the transverse deflection is represented as a double Fourier sine series as

$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (22)$$

which automatically satisfies the displacement boundary conditions (check that $x = 0$ and $x = a$ and $y = 0$ and $y = b$, $w = 0$).

Similarly, the distributed load is also represented by a double Fourier sine series as

$$q(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Q_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (23)$$

where this Q_{mn} could be shown to be evaluated by using this integral

$$Q_{mn}=\frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy \quad (24)$$

We will not go into the derivation of this. This is done in the theory of plates. But this is how we can actually solve this. We can write this displacement as a double Fourier sine series, as well as the distributed load as double Fourier sine series where W_{mn} and Q_{mn} are the coefficients of the displacement series and the load series.

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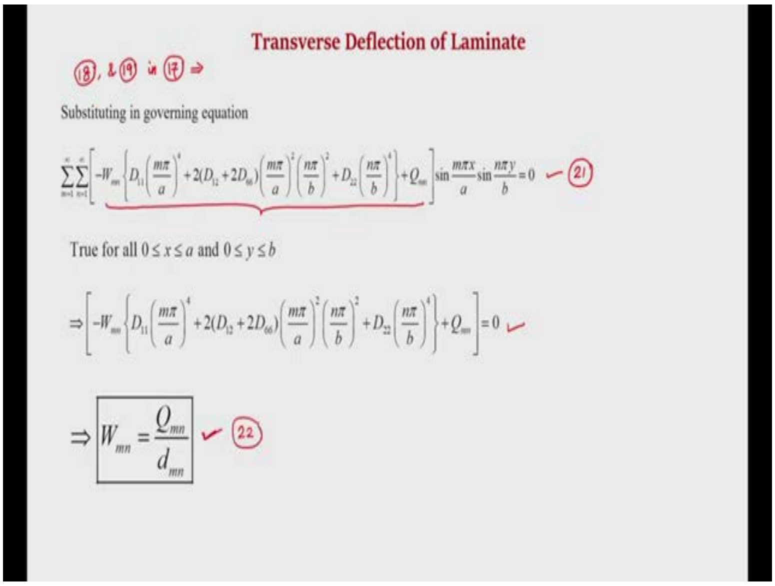

Now, putting (22), (23) and (24) in (20), we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[-W_{mn}\left\{D_{11}\left(\frac{m\pi}{a}\right)^4+2\left(D_{12}+2D_{66}\right)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2+D_{22}\left(\frac{n\pi}{b}\right)^4\right\}+Q_{mn}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}=0 \quad (25)$$

Now, because (25) is true for all values of $0\le x\le a$ and $0\le y\le b$

$$\Rightarrow\left[-W_{mn}\left\{D_{11}\left(\frac{m\pi}{a}\right)^4+2(D_{12}+2D_{66})\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2+D_{22}\left(\frac{n\pi}{b}\right)^4\right\}+Q_{mn}\right]=0$$

$$\Rightarrow \boxed{W_{mn} = \frac{Q_{mn}}{d_{mn}}} \quad (26)$$

where

$$d_{mn} = D_{11}\left(\frac{m\pi}{a}\right)^4 + 2(D_{12}+2D_{66})\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^4$$

$$d_{mn} = \frac{\pi^4}{a^4}\left[D_{11}m^4 + 2(D_{12}+2D_{66})m^2n^2R^2 + D_{22}n^4R^4\right] \qquad (27)$$

where

$R = \dfrac{a}{b} \rightarrow$ plate aspect ratio

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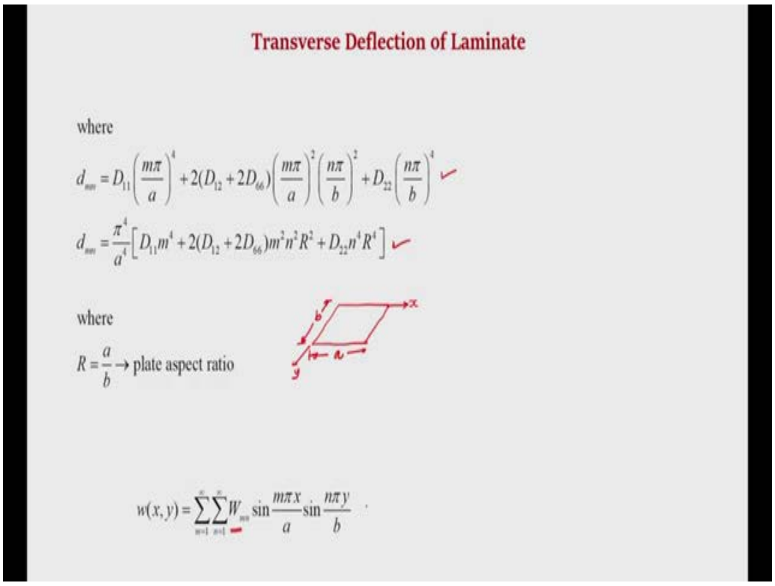

So, knowing d_{mn} (from (27)) and Q_{mn} (from integral shown in (24)), we can determine the transverse deflection of a laminate. But this is for small transverse deflection.