Mechanics of Fiber Reinforced Polymer Composite Structures
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Module - 2
Review of Elasticity
Lecture - 03
Anisotropic Elasticity

Welcome to the second module of the course "Mechanics of Fiber Reinforced Composite Structures". This module is basically on review of elasticity and there will be two lectures in this module.

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FOCUS of Module 2 (Lectures 1 and 2)

- Review of 3D Elasticity
  - Generalized Hooke's Law
  - Anisotropic Elasticity
    - Constitutive Relations
    - Planes of material property symmetry
      - Triclinic, Monoclinic, Orthotropic and Isotropic Materials
    - Engineering constants for Orthotropic materials

First, generalized Hooke's law in 3D elasticity will be discussed followed by anisotropic elasticity. Under anisotropic elasticity, starting with constitutive relations for anisotropic materials, different types of materials like triclinic, monoclinic, orthotropic and isotropic with reference to the existence of planes of material property symmetry will be discussed. This will be followed by a detailed understanding of the engineering constants for orthotropic materials. Therefore, before proceeding, to the review of 3D elasticity, a quick recapitulation of what all have been discussed in the last module may useful to maintain the flow of the lectures.

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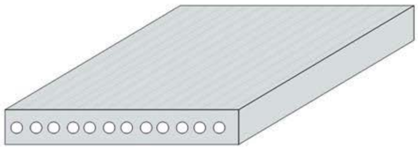
Recap of Last Module

- Discussed Composite Materials in general and FRP composites in particular
- Discussed in details the basic constituents of FRP composites
    - ie. FIBERS and MATRIX
    - LAMINA (heterogeneous and anisotropic) and LAMINATE
    - Macromechanics and Micromechanics of Lamina
    - Macromechanics of Laminate
    - Failure analysis of Laminates
- Mechanics (micro and macro) of Lamina is prerequisite to understand
    - Mechanics of Laminate
    - Mechanics of FRP composite structures
- LAMINA is Anisotropic - Anisotropic Elasticity



Recapitulation and Objective

In the last module, definition of composites with different types and classifications of composites have been discussed broadly with detailed discussions on fiber reinforced polymer composites. Basic constituents of fiber reinforced polymer composites viz. the fibers and the matrix, their types and advantages have been discussed along with the basic terminologies like lamina, laminates etc.

Basic idea of what are macromechanics and micromechanics of lamina, macromechanics of laminate, failure analysis of laminates importance and significance of those modules have been discussed in brief.

Having understood that a laminate being the basic structural component of an FRP composite structure and the fact that a laminate is actually made up of a number of laminae stacked together implies that understanding the mechanics of lamina is prerequisite to understand the mechanics of laminate and hence to understand the mechanics of fiber reinforced polymer composite structures. Again, the fact that a lamina is heterogeneous and anisotropic, to understand the mechanics of lamina, it is important to understand anisotropic elasticity. It is with this background, that anisotropic elasticity will be introduced briefly here to facilitate the understanding the mechanics of an orthotropic lamina.

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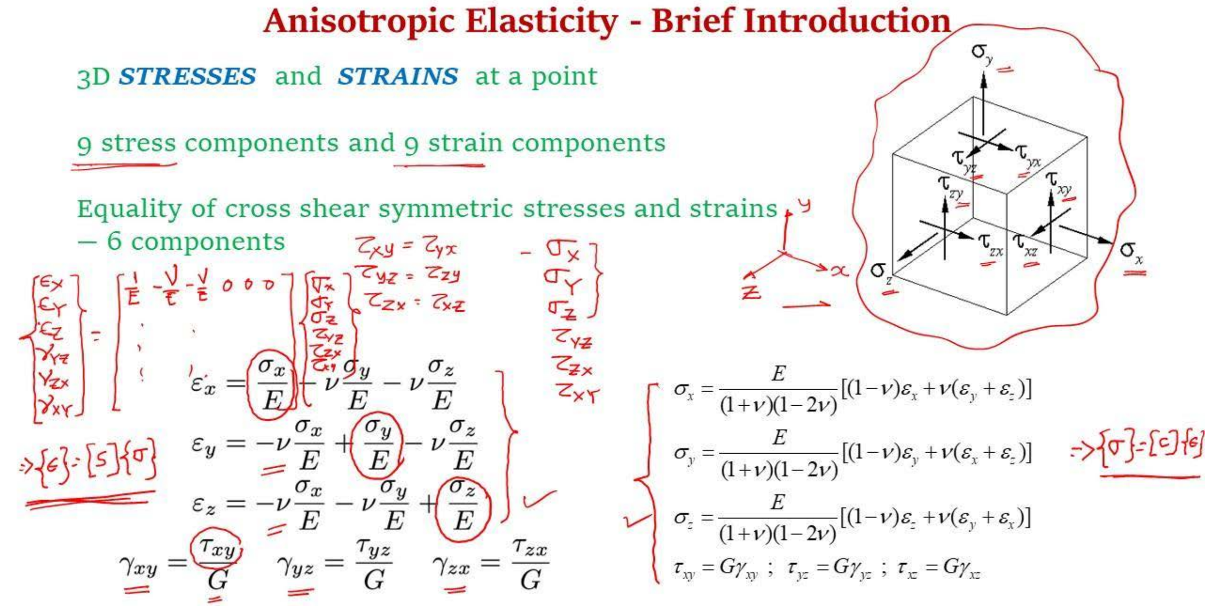
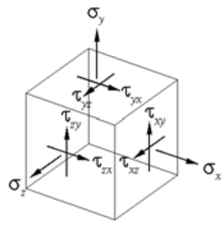


Anisotropic elasticity

Before discussing anisotropic elasticity, a brief review of stress strain relations in three dimensional elasticity of isotropic materials will be useful. With reference to the Cartesian coordinates x-y-z, the state of stress at a point in a deformable solid is defined with reference to three mutually perpendicular planes at that point. Referring to the figure, each plane has one normal stress and two shear stresses. As shown, the x-plane has normal stress σ_x and shear stresses τ_{xy} and τ_{xz}. Similarly, in the y-plane, there is normal stress σ_y and shear stresses τ_{yx} and τ_{yz} and in z-plane and in z-plane, there is normal stress σ_z and shear stresses τ_{zx} and τ_{zy}. Therefore, on three mutually perpendicular planes passing through a point, there are all together nine stress components (three normal stresses and six shear stresses) representing the state of stress at that point. Again, due to equality of cross shear (symmetric stress tensor),of $\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$ and $\tau_{zx} = \tau_{xz}$ resulting in six stresses to represent the state of state of stress at a point. Similarly, there are six strain components corresponding to the stresses. In order to understand the mechanical characterization of a material, it is important to understand the stress strain relations ie. to understand that subjected to stresses what are the stains and vice versa. Before proceeding to generalized Hooke's law, let us write the strains induced in an component made of isotropic materials subjected to all the six components of stresses. As shown in the figure, using the knowledge of basic strength of materials, the normal strains and shear strains could be written as

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$
$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$
$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$
$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)\right]$$
$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x)\right]$$
$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)\right]$$
$$\tau_{xy} = G\gamma_{xy}; \quad \tau_{yz} = G\gamma_{yz}; \quad \tau_{xz} = G\gamma_{xz}$$



Here, strains along x- are direct strain (σ_x/E) due to σ_x and due to Poisson's effect strains along x- due to σ_y ($-\nu\sigma_y/E$) and σ_z ($-\nu\sigma_z/E$). These strains are superposed to get the total normal strain ε_x. Similarly, the total normal strain along y and z are also obtained. Again, shear stresses in xy, yz and zx planes are due to the respective shear stresses in those planes only. Thus, we obtain a the six strains ε_x, ε_y, ε_z, γ_{yz}, γ_{zx}, γ_{xy} in terms of the six stresses σ_x, σ_y, σ_z, τ_{yz}, τ_{zx}, τ_{xy} and the material properties (E and ν). This could be written in matrix form as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\mathrm{E}} & -\frac{\nu}{\mathrm{E}} & -\frac{\nu}{\mathrm{E}} & 0 & 0 & 0 \\ -\frac{\nu}{\mathrm{E}} & \frac{1}{\mathrm{E}} & -\frac{\nu_{zy}}{\mathrm{E}_z} & 0 & 0 & 0 \\ -\frac{\nu}{\mathrm{E}} & -\frac{\nu}{\mathrm{E}} & \frac{1}{\mathrm{E}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

Similarly, by taking inverse, the stresses could be expressed in terms of strains. Thus the six components of stresses are related to the six components of strains. So, for an isotropic material in three-dimension, it is easy to obtain the stress strain relations from the elementary solid mechanics. This will be useful in understanding the generalized Hooke's law.

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Anisotropic Elasticity - Brief Introduction

Generalized Hooke's Law relating ***STRESSES*** to ***STRAINS*** at a point

$$\overset{9\times 1}{\sigma_{ij}} = \overset{[9\times 9]}{C_{ijkl}}\,\overset{9\times 1}{\varepsilon_{kl}} \qquad \ldots (1)$$

or
$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \qquad [\, i,j,k,l = 1,2,3 \,] \ldots (2)$$

C_{ijkl} ⟶ Stiffness components

S_{ijkl} ⟶ Compliance components

In general, it requires ***81*** elastic constants to fully characterize a material

Generalized Hooke's Law

In a Cartesian coordinate, at a point, stresses in three mutually perpendicular planes are needed to completely specify the state of stress at a point. Now, because on each plane there could be three stresses viz. one normal stress and two shear stresses, therefore, there will be nine stress components and corresponding nine strain components Hooke's law in three-dimensions could be written as

$$\begin{aligned}
\sigma_{xx} &= C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy} + C_{13}\varepsilon_{zz} + C_{14}\varepsilon_{xy} + C_{15}\varepsilon_{yz} + C_{16}\varepsilon_{zx} + C_{17}\varepsilon_{yx} + C_{18}\varepsilon_{zy} + C_{19}\varepsilon_{xz} \\
\sigma_{yy} &= C_{21}\varepsilon_{xx} + C_{22}\varepsilon_{yy} + C_{23}\varepsilon_{zz} + C_{24}\varepsilon_{xy} + C_{25}\varepsilon_{yz} + C_{26}\varepsilon_{zx} + C_{27}\varepsilon_{yx} + C_{28}\varepsilon_{zy} + C_{29}\varepsilon_{xz} \\
\sigma_{zz} &= C_{31}\varepsilon_{xx} + C_{32}\varepsilon_{yy} + C_{33}\varepsilon_{zz} + C_{34}\varepsilon_{xy} + C_{35}\varepsilon_{yz} + C_{36}\varepsilon_{zx} + C_{37}\varepsilon_{yx} + C_{38}\varepsilon_{zy} + C_{39}\varepsilon_{xz} \\
\sigma_{xy} &= C_{41}\varepsilon_{xx} + C_{42}\varepsilon_{yy} + C_{43}\varepsilon_{zz} + C_{44}\varepsilon_{xy} + C_{45}\varepsilon_{yz} + C_{46}\varepsilon_{zx} + C_{47}\varepsilon_{yx} + C_{48}\varepsilon_{zy} + C_{49}\varepsilon_{xz} \\
\sigma_{yz} &= C_{51}\varepsilon_{xx} + \ldots\ldots\ldots + C_{59}\varepsilon_{xz} \\
\sigma_{zx} &= C_{61}\varepsilon_{xx} + \ldots\ldots\ldots + C_{69}\varepsilon_{xz} \\
\sigma_{yx} &= C_{71}\varepsilon_{xx} + \ldots\ldots\ldots + C_{79}\varepsilon_{xz} \\
\sigma_{zy} &= C_{81}\varepsilon_{xx} + \ldots\ldots\ldots + C_{89}\varepsilon_{xz} \\
\sigma_{xz} &= C_{91}\varepsilon_{xx} + \ldots\ldots\ldots + C_{99}\varepsilon_{xz}
\end{aligned}$$

where these nine components of stresses are related to the corresponding nine components of strains by means of a 9 × 9 matrix. That is the components of the stresses are expressed as linear combinations of the strains and vice versa. The elements of this matrix are now understood with reference to the discussion on isotropic materials in the previous section.

$$\{\sigma\}_{9\times 1} = [C]_{9\times 9}\{\varepsilon\}_{9\times 1}$$

Here the matrix [C] is called the stiffness matrix and the matrix [S] is known as the compliance matrix. Stiffness matrix expresses the stresses in terms of the strains and the compliance matrix expresses the strains in terms of the stresses. In general, at a point there are nine stress components and corresponding nine strain components. These nine stress components are related to the corresponding nine strain components by a 9 × 9 matrix having 81 elements. These elements are known as elastic constant and 81 elastic constants are required to fully characterize a material. Therefore, in a material subjected to generalized loading, to determine the strains at a point from the stresses or vice versa these 81 elastic constants need to be known. Stresses and strains being second order tensors will have $3^2 = 9$ components in three dimension and the Generalized Hooke's law in the index notation is

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

$$\text{or} \quad \varepsilon_{ij} = S_{ijkl}\sigma_{kl} \qquad [\, i, j, k, l = 1, 2, 3 \,]$$

where the stiffness C_{ijkl} and compliance S_{ijkl} are actually a fourth order tensor and therefore, in three dimension, they will have $3^4 = 81$ elastic constants.

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Anisotropic Elasticity - Brief Introduction

$$\{\sigma\}_{9\times1} = [C]_{9\times9}\,\{\varepsilon\}_{9\times1} \qquad \dots \; 3$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{21} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} & C_{1132} & C_{1113} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} & C_{2232} & C_{2213} & C_{2221} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312} & C_{3332} & C_{3313} & C_{3321} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2331} & C_{2312} & C_{2332} & C_{2313} & C_{2321} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3131} & C_{3112} & C_{3132} & C_{3113} & C_{3121} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212} & C_{1232} & C_{1213} & C_{1221} \\ C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3231} & C_{3212} & C_{3232} & C_{3213} & C_{3221} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1331} & C_{1312} & C_{1332} & C_{1313} & C_{1321} \\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2131} & C_{2112} & C_{2132} & C_{2113} & C_{2121} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \\ \varepsilon_{32} \\ \varepsilon_{13} \\ \varepsilon_{21} \end{Bmatrix}$$

9-stress

81 Constants

9-strain

Review of Elasticity

5

Anisotropic elasticity

Anisotropy means that the properties are direction dependent. The properties of the materials are different in different directions and in such materials, in order to relate the stresses to the strains, we need these 81 independent elastic constants. If we expand the stress-strain relations written in the index notation, it will be

$$
\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{21} \end{Bmatrix} =
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} & C_{1132} & C_{1113} & C_{1121} \\
C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} & C_{2232} & C_{2213} & C_{2221} \\
C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312} & C_{3332} & C_{3313} & C_{3321} \\
C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2331} & C_{2312} & C_{2332} & C_{2313} & C_{2321} \\
C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3131} & C_{3112} & C_{3132} & C_{3113} & C_{3121} \\
C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212} & C_{1232} & C_{1213} & C_{1221} \\
C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3231} & C_{3212} & C_{3232} & C_{3213} & C_{3221} \\
C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1331} & C_{1312} & C_{1332} & C_{1313} & C_{1321} \\
C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2131} & C_{2112} & C_{2132} & C_{2113} & C_{2121}
\end{bmatrix}
\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \\ \varepsilon_{32} \\ \varepsilon_{13} \\ \varepsilon_{21} \end{Bmatrix}
$$

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Anisotropic Elasticity - Brief Introduction

Conditions of rotational equilibrium— symmetric stress and strain tensors

$$
\left.\begin{aligned} \sigma_{ij} &= \sigma_{ji} \\ \varepsilon_{ij} &= \varepsilon_{ji} \end{aligned}\right\} \quad [i \neq j] \qquad \ldots (4)
$$

Number of independent elastic constants reduces from 81 to 36

$$
\{\sigma\}_{6\times1} = [C]_{6\times6} \{\varepsilon\}_{6\times1} \qquad \ldots (5)
$$

$$
[C] = \begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\
C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\
C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\
C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\
C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\
C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212}
\end{bmatrix}
$$

Due to the equality of cross shear (obtained from the condition of moment equilibrium in absence of body moments), $\sigma_{ij} = \sigma_{ji}$ and similarly $\varepsilon_{ij} = \varepsilon_{ji}$ for $i \neq j$ and therefore, out of the nine stress/strain components, actually six are independent and this leads to symmetric stress and strain tensors. So, six components of strains are related to six components of stresses by means of these 36 independent elastic constants.

So, as the stresses and strains are reduced from 9 to 6, therefore, the number of elements of stiffness matrix also gets reduced to 36 from 81. So, for symmetric stress and strain tensors, we need 36 independent elastic constants to characterize a material.

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Anisotropic Elasticity - Brief Introduction

In contracted notation;

$$[\sigma]=\begin{pmatrix}\sigma_{11} & \tau_{12} & \tau_{13}\\ & \sigma_{22} & \tau_{23}\\ symm. & & \sigma_{33}\end{pmatrix}=\begin{pmatrix}\sigma_{1} & \sigma_{6} & \sigma_{5}\\ & \sigma_{2} & \sigma_{4}\\ symm. & & \sigma_{3}\end{pmatrix}$$

$$\text{similarly}\,[\varepsilon]=\begin{pmatrix}\varepsilon_{11} & \gamma_{12} & \gamma_{13}\\ & \varepsilon_{22} & \gamma_{23}\\ symm. & & \varepsilon_{33}\end{pmatrix}=\begin{pmatrix}\varepsilon_{1} & \varepsilon_{6} & \varepsilon_{5}\\ & \varepsilon_{2} & \varepsilon_{4}\\ symm. & & \varepsilon_{3}\end{pmatrix}$$

$$\begin{array}{l}C_{1111}=C_{11};C_{1122}=C_{12};C_{1133}=C_{13};C_{1144}=C_{41};C_{1155}=C_{15};C_{1166}=C_{16};\\ C_{2211}=C_{21};C_{2222}=C_{22};C_{2233}=C_{23};C_{2223}=C_{24};C_{2231}=C_{25};C_{2212}=C_{26};\\ \quad\vdots \\ C_{1211}=C_{61};C_{1222}=C_{62};C_{1233}=C_{63};C_{1223}=C_{64};C_{1231}=C_{65};C_{1212}=C_{66};\end{array}$$

Now using the contracted notation, C_{1111} is written C_{11}, C_{1122} is written as C_{12} and so on. Similarly, the stresses are also written in contracted notation where σ_{11}, σ_{22}, σ_{33} are written as σ_1, σ_2, σ_3 and so on. The stress strain relation in contracted notation is

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{23}\\ \tau_{13}\\ \tau_{12}\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{Bmatrix}$$

$$\underset{\text{Stiffness}}{\{\sigma\}=[C]\{\varepsilon\}} \qquad \text{and} \qquad \underset{\text{Compliance}}{\{\varepsilon\}=[S]\{\sigma\}}$$

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Anisotropic Elasticity - Brief Introduction

In contracted notation, for an anisotropic body, the ***Stress-Strain relation***

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{23}\\ \tau_{13}\\ \tau_{12}\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{Bmatrix} \quad \ldots (6)$$

$$\underset{\text{Stiffness}}{\{\sigma\}=[C]\{\varepsilon\}} \qquad \text{and} \qquad \underset{\text{Compliance}}{\{\varepsilon\}=[S]\{\sigma\}}$$

where six components of stresses are actually related to six components of strains by means of 36 independent elastic constants. This matrix [C] is the stiffness matrix and taking inverse of this yields the compliance matrix [S].

Therefore, in order to characterize an anisotropic material, we need to know these 36 independent elastic constants correlating the stresses to the corresponding strains and vice versa.

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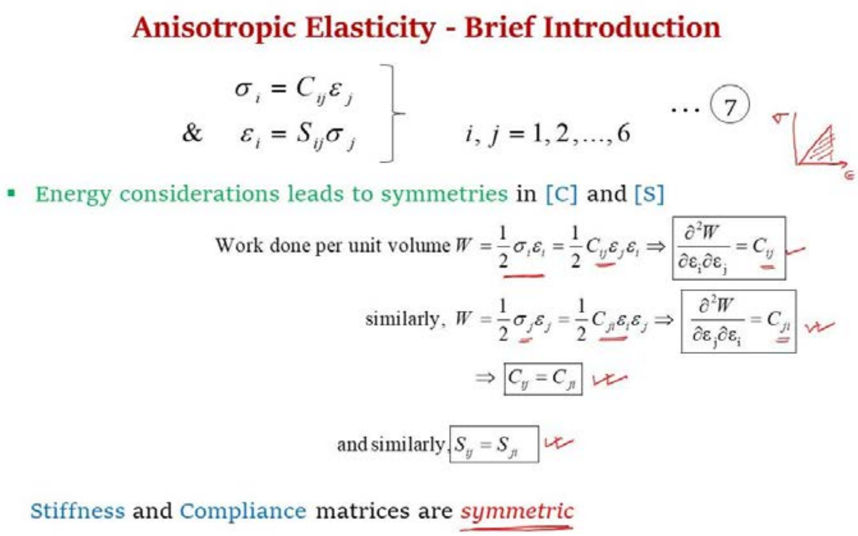

However, from the energy consideration, it could be shown that for linearly elastic material these stiffness and compliance matrices are also symmetric ie. $C_{ij} = C_{ji}$ and $S_{ij} = S_{ji}$ as follows.

$$\text{Work done per unit volume } W = \frac{1}{2}\sigma_i\varepsilon_i = \frac{1}{2}C_{ij}\varepsilon_j\varepsilon_i \Rightarrow \boxed{\frac{\partial^2 W}{\partial\varepsilon_i\partial\varepsilon_j} = C_{ij}}$$

$$\text{similarly, } W = \frac{1}{2}\sigma_j\varepsilon_j = \frac{1}{2}C_{ji}\varepsilon_i\varepsilon_j \Rightarrow \boxed{\frac{\partial^2 W}{\partial\varepsilon_j\partial\varepsilon_i} = C_{ji}}$$

$$\Rightarrow \boxed{C_{ij} = C_{ji}}$$

$$\text{and similarly, } S_{ij} = S_{ji}$$

Area under the stress strain curve is actually the strain energy stored per unit volume, So, we can write this stress in terms of strain and we get this; and if we take to succeed differentiation points with respect to ε_i and then with respect to ε_j, we get this.

Similarly, we can write this σ_j in terms of C_{ji} and ε_i; and we take second to successive differentiation with respect to ε_j and ε_i, we get C_{ji}. Now, that order of differentiation is immaterial and therefore, $C_{ji} = C_{ij}$. In the same way, we can also prove that $S_{ij} = S_{ji}$. The net result is that the stiffness matrices and the compliance matrices are symmetric.

Therefore, as a consequence of this symmetry, instead of 36, now we actually need 21 independent elastic constants to characterize an anisotropic material. That means is given the six components of stresses at a point, to determine the corresponding six components of strains, we need to know these 21 independent elastic constants.

Determination of these 21 independent elastic constant is no easy task and we have discussed with respect to isotropic material that these elements are nothing but the functions of material properties. Let us now discuss the same with respect to anisotropic materials.

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Anisotropic Elasticity - Brief Introduction

STIFFNESS and ***COMPLIANCE*** matrices— **21** independent terms or elastic constants

Linear Elasticity— General expression for ***stress-strain relation***

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ Symm. & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} \quad \ldots (8)$$

- Characteristics of ***ANISOTROPIC*** materials
- No planes of symmetry for material properties
- ***TRICLINIC*** materials

Now, what happens is, these anisotropic materials are also known as triclinic materials.

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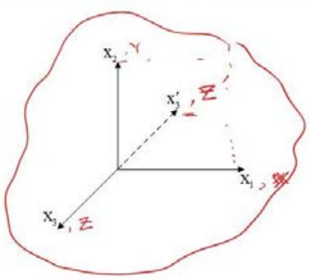
Planes of Material Property Symmetry - Monoclinic Materials

- If there is one plane of material property symmetry i.e. elastic constants to be invariant with respect to inversion of the axis perpendicular to that plane - MONOCLINIC
- If X_3 (plane 1-2) is the plane of material property symmetry, i.e. C_{ij} is invariant with respect to an inversion of the X_3 axis, then the direction cosine matrix is



$$Q_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Planes of material property symmetry

Fortunately, in many materials there exist planes of material property symmetry. Let us first understand what is actually a plane of material property symmetry? Referring to the Fig…., with reference to 3d Cartesian coordinates (X_1, X_2, X_3 or X, Y, Z), suppose for a material, with respect to a certain plane the material properties are symmetric; say for example, X_1- X_2 or the X-Y, is the plane of material property symmetry. This means that if we rotate the object with respect to this plane by 180^o, then there will be no change in material properties. Referring to the Fig…., the stiffness or compliance matrix of the material remains same when defined with respect to X_1, X_2, X_3 or X_1, X_2, $X_3^/$ ($X_3 = - X_3^/$), ie. the properties are symmetric with respect to X_3 plane or X_1-X_2 plane. X_3 is the normal to X_1-X_2 plane. Generally, plane is represented by its surface normal. So, if X_3 is the plane of material property symmetry, it means, the stiffness matrix C_{ij} or compliance matrix S_{ij} are invariant with respect to an inversion of X_3 axis. That means, instead of X_3, suppose we just invert it and we write X'_3, there will be no change in the stiffness matrix or compliance matrix ie. the material properties do not change.

Now, we know that the coordinate transformation from X_1, X_2, X_3 to X'_1, X'_2, X'_3 or say X, Y, Z to X′, Y′, Z′, could be written in terms of the direction cosine matrix is given by

$$a_{ij} = \begin{bmatrix} Cos(x_1, x'_1) & Cos(x_1, y'_1) & Cos(x_1, z'_1) \\ Cos(y_1, x'_1) & Cos(y_1, y'_1) & Cos(y_1, z'_1) \\ Cos(z_1, x'_1) & Cos(z_1, y'_1) & Cos(z_1, z'_1) \end{bmatrix}$$

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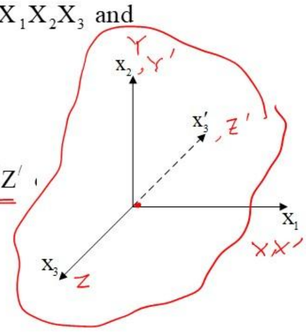
Planes of Material Property Symmetry - Monoclinic Materials

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \text{and} \begin{bmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_y & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_z \end{bmatrix}$$ stress and strains with respect to XYZ or $X_1X_2X_3$ and



$$\begin{bmatrix} \sigma_x' & \tau_{xy}' & \tau_{xz}' \\ \tau_{xy}' & \sigma_y' & \tau_{yz}' \\ \tau_{xz}' & \tau_{yz}' & \sigma_z' \end{bmatrix} \text{and} \begin{bmatrix} \varepsilon_x' & \gamma_{xy}' & \gamma_{xz}' \\ \gamma_{xy}' & \varepsilon_y' & \gamma_{yz}' \\ \gamma_{xz}' & \gamma_{yz}' & \varepsilon_z' \end{bmatrix}$$ stress and strains with respect to $X'Y'Z'$

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix}$$ is same with respect to $X'Y'Z'$ or $X_1X_2X'_3$

Material having one plane of material property symmetry – Monoclinic

Referring to the Fig…., considering a material having Z- plane as the plane of material property symmetry. For a loaded body made of this material, the stresses are strains with respect to 3D Cartesian coordinates XYZ are defined as

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \text{and} \begin{bmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_y & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_z \end{bmatrix}$$

Now, suppose the stresses and strains with respect to the transformed coordinate system X´, Y´, Z´ (where X´=X, Y´=Y, Z´= −Z) are

$$\begin{bmatrix} \sigma_x' & \tau_{xy}' & \tau_{xz}' \\ \tau_{xy}' & \sigma_y' & \tau_{yz}' \\ \tau_{xz}' & \tau_{yz}' & \sigma_z' \end{bmatrix} \text{and} \begin{bmatrix} \varepsilon_x' & \gamma_{xy}' & \gamma_{xz}' \\ \gamma_{xy}' & \varepsilon_y' & \gamma_{yz}' \\ \gamma_{xz}' & \gamma_{yz}' & \varepsilon_z' \end{bmatrix}$$

Since Z- plane is the plane of material property symmetry, the stiffness matrix remains same in both the cases we could relate the stresses and strains in both the cases by the same stiffness matrix as

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix}$$

Now, using stress and strain transformations about the rotated axis

$$\begin{bmatrix} \sigma_x^{/} & \tau_{xy}^{/} & \tau_{xz}^{/} \\ \tau_{xy}^{/} & \sigma_y^{/} & \tau_{yz}^{/} \\ \tau_{xz}^{/} & \tau_{yz}^{/} & \sigma_z^{/} \end{bmatrix} = \left[a_{ij}\right]\left[\sigma_{xyz}\right]\left[a_{ij}\right] = \begin{bmatrix} \sigma_x & \tau_{xy} & -\tau_{xz} \\ \tau_{xy} & \sigma_y & -\tau_{yz} \\ -\tau_{xz} & -\tau_{yz} & \sigma_z \end{bmatrix}$$

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Planes of Material Property Symmetry - Monoclinic Materials

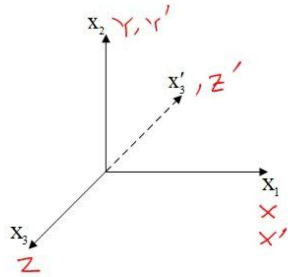
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ Sym & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix}$$

XYZ

and

$$\begin{Bmatrix} \sigma'_x \\ \sigma'_y \\ \sigma'_z \\ \tau'_{yz} \\ \tau'_{zx} \\ \tau'_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ Sym & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon'_x \\ \varepsilon'_y \\ \varepsilon'_z \\ \gamma'_{yz} \\ \gamma'_{zx} \\ \gamma'_{xy} \end{Bmatrix}$$

X'Y'Z'



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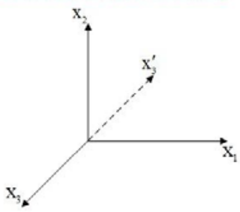
Planes of Material Property Symmetry - Monoclinic Materials

$$\begin{bmatrix} \sigma_x^{/} & \tau_{xy}^{/} & \tau_{xz}^{/} \\ \tau_{xy}^{/} & \sigma_y^{/} & \tau_{yz}^{/} \\ \tau_{xz}^{/} & \tau_{yz}^{/} & \sigma_z^{/} \end{bmatrix} = [Q]\left[\sigma_{xyz}\right][Q]^T = \begin{bmatrix} \sigma_x & \tau_{xy} & -\tau_{xz} \\ \tau_{xy} & \sigma_y & -\tau_{yz} \\ -\tau_{xz} & -\tau_{yz} & \sigma_z \end{bmatrix}$$ and



$$[Q] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

similarly

$$\begin{bmatrix} \varepsilon_x^{/} & \gamma_{xy}^{/} & \gamma_{xz}^{/} \\ \gamma_{xy}^{/} & \varepsilon_y^{/} & \gamma_{yz}^{/} \\ \gamma_{xz}^{/} & \gamma_{yz}^{/} & \varepsilon_z^{/} \end{bmatrix} = \begin{bmatrix} \varepsilon_x & \gamma_{xy} & -\gamma_{xz} \\ \gamma_{xy} & \varepsilon_y & -\gamma_{yz} \\ -\gamma_{xz} & -\gamma_{yz} & \varepsilon_z \end{bmatrix}$$

So we could write the transformed stresses in terms of the untransformed stresses by multiplying it twice by the direction cosine matrix. Similarly, we performed the strain transformation and we could write the transformed strains in terms of the untransformed strains as

$$\begin{bmatrix} \varepsilon_x' & \gamma_{xy}' & \gamma_{xz}' \\ \gamma_{xy}' & \varepsilon_y' & \gamma_{yz}' \\ \gamma_{xz}' & \gamma_{yz}' & \varepsilon_z' \end{bmatrix} = \begin{bmatrix} \varepsilon_x & \gamma_{xy} & -\gamma_{xz} \\ \gamma_{xy} & \varepsilon_y & -\gamma_{yz} \\ -\gamma_{xz} & -\gamma_{yz} & \varepsilon_z \end{bmatrix}$$

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Planes of Material Property Symmetry - Monoclinic Materials

$\sigma_x = C_{11}\varepsilon_x + C_{12}\varepsilon_Y + C_{13}\varepsilon_Z + C_{14}\gamma_{YZ} + C_{15}\gamma_{ZX} + C_{16}\gamma_{XY}$

$\sigma_x' = C_{11}\varepsilon_x' + C_{12}\varepsilon_Y' + C_{13}\varepsilon_Z' + C_{14}\gamma_{YZ}' + C_{15}\gamma_{ZX}' + C_{16}\gamma_{XY}'$

$\gamma_{YZ}' = -\gamma_{YZ}$

$\gamma_{ZX}' = -\gamma_{ZX}$

- **Since,** $\sigma_x = \sigma_x'$

$$\Rightarrow C_{11}\varepsilon_x + C_{12}\varepsilon_y + C_{13}\varepsilon_z + C_{14}\gamma_{yz} + C_{15}\gamma_{zx} + C_{16}\gamma_{xy}$$
$$= C_{11}\varepsilon_x + C_{12}\varepsilon_y + C_{13}\varepsilon_z - C_{14}\gamma_{yz} - C_{15}\gamma_{zx} + C_{16}\gamma_{xy}$$
$$\Rightarrow \boxed{C_{15} = C_{14} = 0}$$

- **Similarly,** $\sigma_y = \sigma_y' \Rightarrow \boxed{C_{24} = C_{25} = 0}$

  and $\sigma_z = \sigma_z' \Rightarrow \boxed{C_{34} = C_{35} = 0}$

  $\tau_{xy} = \tau_{xy}' \Rightarrow \boxed{C_{46} = C_{56} = 0}$

- This could also be proved using $C_{m'n'o'p'} = Q_{m'i}Q_{n'j}Q_{o'k}Q_{p'l}C_{ijkl}$ under inversion @ x_3-axis.

Now comparing the expressions for the untransformed and transformed stresses written in terms of elements of stiffness matrix and the corresponding strains, we could clearly see that 8 out of 21 elements of the stiffness matrix becomes zero. That is as a consequence of existence of one plane of material property symmetry (where the stiffness matrix is invariant under the inversion of the Z- plane), 8 elements of the stiffness matrix become zero.

This is a simple way of showing that the existence of one plane of material property symmetry leads to 8 of the 21 independent elastic constant to become zero. This could also be done by a tensor stiffness transformation.

So for a monoclinic material having one plane of material property symmetry, the stiffness matrix is

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}$$

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Planes of Material Property Symmetry - Monoclinic Materials

- For ***MONOCLINIC*** Materials with **one** plane of material property symmetry C_{ij} is

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}$$

- ***MONOCLINIC*** Materials and have ***13*** independent *elastic constants*
- Example—Feldspar

The stiffness matrix consists of only 13 terms or 13 independent elastic constants. Similarly, the compliance matrix will also have 13 independent terms. So, for monoclinic materials, 13 independent elastic constants are required to characterize whereas in a fully anisotropic material (no plane of material property symmetry), 21 independent elastic constants are required to characterize.

An example of a monoclinic material is feldspar.

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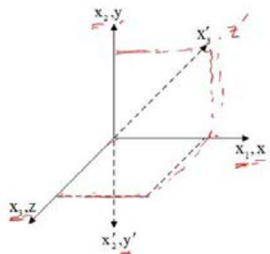
Planes of Material Property Symmetry - Orthotropic Materials

- Existence of two mutually perpendicular planes of material property symmetry, automatically guarantees 3rd plane also to be a plane of material property symmetry
- Such materials having three orthogonal planes of material property symmetry – ***ORTHOTROPIC*** materials



- In addition to X_1X_2 (plane of material property symmetry for monoclinic materials), we consider X_1X_3 as the another plane of symmetry

$$[Q] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Two mutually perpendicular planes of material property symmetry

Referring to Fig…, suppose, in addition to one plane of material property symmetry (discussed in the previous section), there is another plane of material property symmetry which is perpendicular to that first plane that means, there are 2 mutually perpendicular planes of material property symmetry. In such a case, it could be shown that it existence of two mutually perpendicular planes of material property symmetry automatically leads to the existence of the third perpendicular plane also to be a plane of material property symmetry. That is if Z-plane and Y- plane are the two mutually perpendicular planes of material property symmetry, then X- plane will also be a plane of material property symmetry. Such materials where there are 3 mutually perpendicular planes of material property symmetry are called Orthotropic materials. In fact, there are 2 mutually perpendicular planes of material property symmetry automatically leads to the third plane also to be a plane of material property symmetry and it is an orthotropic material.

Now, considering a monoclinic material with Z- plane as the plane of symmetry, we already obtained the stiffness matrix as

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}$$

Referring to Fig.,.., suppose Y− plane is also a plane of material property symmetry. Then the stiffness matrix as shown above will remain unchanged if we do the inversion of Y−axis ie with respect to X′, Y′, Z′ (where X′=X, Y′=−Y, Z′= Z). The direction cosine matrix for the coordinate transformation from X, Y, Z to X′, Y′, Z′ is

$$a_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, if the stresses and strains with reference to X, Y, Z and X′, Y′, Z′ are

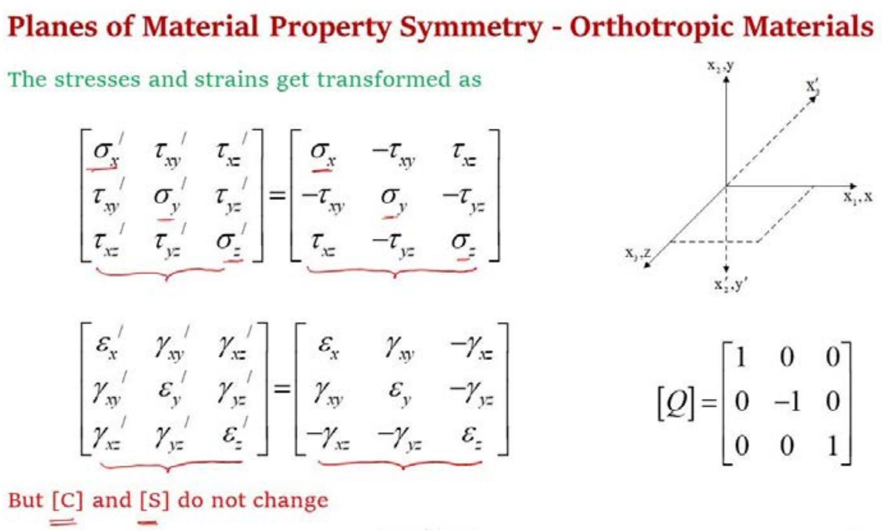
$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \text{and} \begin{bmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_y & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_z \end{bmatrix}$$ stress and strains with respect to XYZ or $X_1X_2X_3$ and

$$\begin{bmatrix} \sigma_x^{/} & \tau_{xy}^{/} & \tau_{xz}^{/} \\ \tau_{xy}^{/} & \sigma_y^{/} & \tau_{yz}^{/} \\ \tau_{xz}^{/} & \tau_{yz}^{/} & \sigma_z^{/} \end{bmatrix} \text{and} \begin{bmatrix} \varepsilon_x^{/} & \gamma_{xy}^{/} & \gamma_{xz}^{/} \\ \gamma_{xy}^{/} & \varepsilon_y^{/} & \gamma_{yz}^{/} \\ \gamma_{xz}^{/} & \gamma_{yz}^{/} & \varepsilon_z^{/} \end{bmatrix}$$ stress and strains with respect to $X^{/}Y^{/}Z^{/}$ or $X_1X_2X^{/}_3$

Then performing stress and strain transformations, we could relate the transformed stresses and strains with the untransformed stresses and strains as

$$\begin{bmatrix} \sigma_x^{/} & \tau_{xy}^{/} & \tau_{xz}^{/} \\ \tau_{xy}^{/} & \sigma_y^{/} & \tau_{yz}^{/} \\ \tau_{xz}^{/} & \tau_{yz}^{/} & \sigma_z^{/} \end{bmatrix} = \begin{bmatrix} \sigma_x & -\tau_{xy} & \tau_{xz} \\ -\tau_{xy} & \sigma_y & -\tau_{yz} \\ \tau_{xz} & -\tau_{yz} & \sigma_z \end{bmatrix}$$

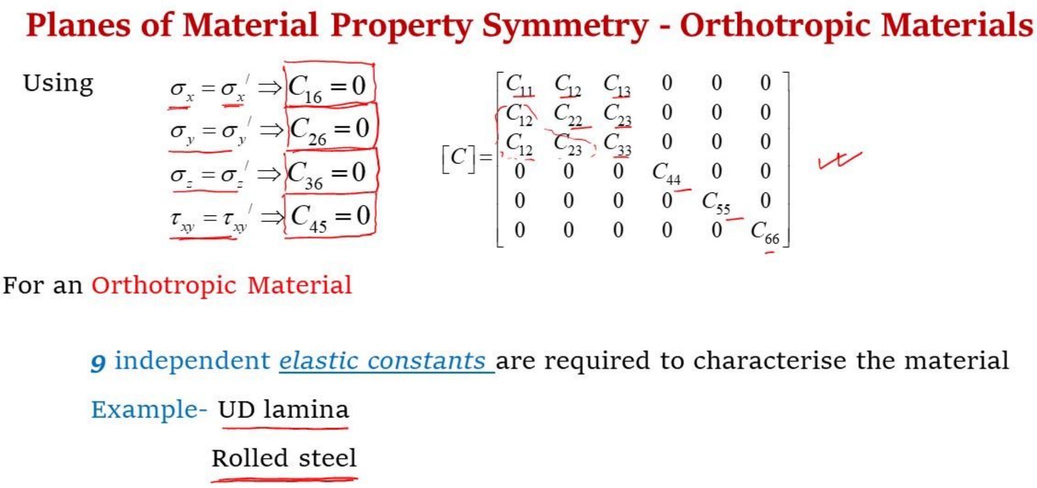
$$\begin{bmatrix} \varepsilon_x^{/} & \gamma_{xy}^{/} & \gamma_{xz}^{/} \\ \gamma_{xy}^{/} & \varepsilon_y^{/} & \gamma_{yz}^{/} \\ \gamma_{xz}^{/} & \gamma_{yz}^{/} & \varepsilon_z^{/} \end{bmatrix} = \begin{bmatrix} \varepsilon_x & \gamma_{xy} & -\gamma_{xz} \\ \gamma_{xy} & \varepsilon_y & -\gamma_{yz} \\ -\gamma_{xz} & -\gamma_{yz} & \varepsilon_z \end{bmatrix}$$



Again because the stiffness matrix does not change under this transformation, we could relate the stresses and strains in the transformed and untransformed coordinates by the same stiffness matrix as

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}$$

Again by comparing the transformed and untransformed stresses expresses in terms of corresponding strains and the elements of stiffness matrix, leads to four more elements of the stiffness matrix becoming zero as follows.



$$\sigma_x = \sigma_x{}' \Rightarrow C_{16} = 0$$
$$\sigma_y = \sigma_y{}' \Rightarrow C_{26} = 0$$
$$\sigma_z = \sigma_z{}' \Rightarrow C_{36} = 0$$
$$\tau_{xy} = \tau_{xy}{}' \Rightarrow C_{45} = 0$$

The stiffness matrix of such a material with two mutually perpendicular planes of material property symmetry is

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

Now, if we consider the third plane ie X-plane also to be a plane of material property symmetry and perform the same exercise, it will yield the same results meaning that the existence of two mutually perpendicular planes of material property symmetry automatically leads to the third perpendicular plane also to be a plane of material property symmetry and such a material is called orthotropic material.

As a consequence of three mutually perpendicular planes of material property symmetry, total 12 elements of the stiffness matrix become zero which means that to characterize an orthotropic material, only 9 (compared to 21 for fully anisotropic) independent elastic constants are required.

Examples of orthotropic materials are unidirectional lamina, rolled steel

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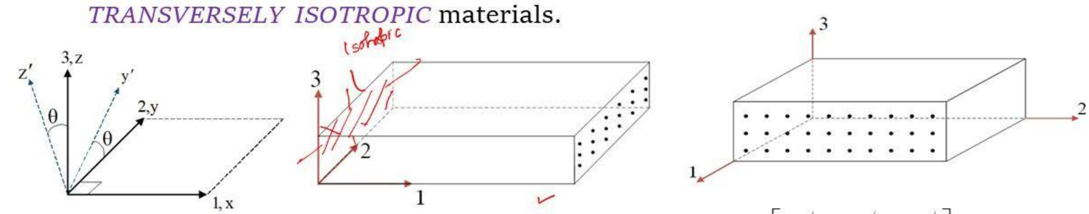
Planes of Material Property Symmetry - Transversely Isotropic Materials

- Orthotropic materials with one transverse plane of symmetry where material properties are independent of direction (a plane of isotropy) is called *TRANSVERSELY ISOTROPIC* materials.



For example: If plane 2-3 is a plane of isotropy, then $\left[\sigma_{XYZ'}\right] = \begin{bmatrix} \sigma_x' & \tau_{xy}' & \tau_{xz}' \\ \tau_{xy}' & \sigma_y' & \tau_{yz}' \\ \tau_{xz}' & \tau_{yz}' & \sigma_z' \end{bmatrix} = [R]^T \left[\sigma_{xyz}\right][R]$

Now, suppose in an orthotropic material one of the transverse plane is such that, in that plane the properties are independent of direction. Referring to Fig…, suppose in an orthotropic lamina, plane X-Y, plane Y-Z and plane X-Z are the three mutually perpendicular planes of material property symmetry.

Now, suppose the plane Y-Z is isotropic ie., in this the material properties are independent of directions. That is whether it is Y-Z or Y′-Z′ (rotated by an angle θ about X-axis) the properties do not change. In a more simple term to understand this in Y-Z plane, Young's modulus and the Poisson's ratio are same in all directions and shear modulus could be expressed in terms of Young’s modulus and Poisson’s ratio.

Thus, if in an orthotropic material, one of the transverse planes the material properties are independent of directions, that means isotropic, these are called transversely isotropic materials. Now, example in this case, say Y-Z is the plane of isotropy, the transformed stresses for a rotation in this Y-Z plane could be determined by multiplying by the proper rotation matrix.

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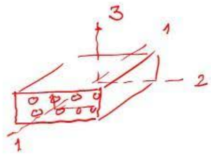
As a consequence of this four more elements of the stiffness matrix of the orthotropic materials become zero thus reducing the number of independent elastic constants to 5. The stiffness matrix for a a transversely isotropic materials is

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{22}-C_{23}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix}$$

Planes of Material Property Symmetry - Transversely Isotropic Materials

Leads to $\Rightarrow C_{12}=C_{13},\ C_{22}=C_{33},\ C_{44}=\frac{C_{22}-C_{23}}{2}=\frac{C_{33}-C_{23}}{2}$

$$\Rightarrow[C]=\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{22}-C_{23}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix}$$



5 independent elastic constants– to characterise Transversely Isotropic materials

Example– UD lamina where the fibers are arranged in a square array

Therefore, as a consequence of the fact that one of the three mutually perpendicular planes of material property symmetry is a plane of transverse isotropy, the number of independent elastic constants required to characterize the material is 5. Example of transversely isotropic materials is unidirectional lamina where the fibers are arranged in a square array.

Referring to the figure if in a lamina the fibers are arranged in a squared regularly spaced array, then naturally, in the 2-3 plane, it will have same properties in the direction 2 or direction 3 or any other direction. Therefore, this 2-3 plane happens to be a plane of transverse isotropy and this is an example of transversely isotropic material. But if the fibers are not really arranged regularly, it may not be the plane of transverse isotropy.

If all the planes are planes of material property symmetry, then it becomes an isotropic material. This could be again proved considering that all the planes are planes of material property symmetry and a consequence only 2 independent elastic constants remain all other become dependent and the stiffness matrix of an isotropic material looks like

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix}$$

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Planes of Material Property Symmetry - Isotropic Materials

If all the planes are planes of material property symmetry— ***ISOTROPIC*** material

As a result the stiffness matrix becomes

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix}$$

- 2 independent *elastic constants* – to characterise ***ISOTROPIC*** materials

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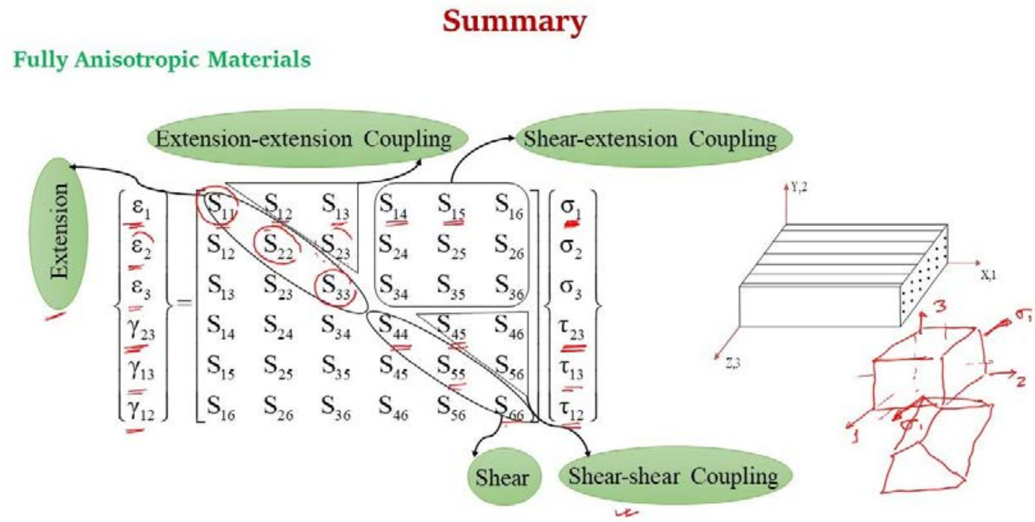

Having understood the stiffness and compliance matrix and the effect of existence of planes of material property symmetry, let us have a closer look at each term of the stiffness and the compliance matrix in terms of deciding the response of a deformable solid subjected to load. Referring to the Fig. … and writing the strains in terms of stresses in a loaded object as

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

Suppose, load is applied along 1 only. Now for a fully anisotropic material all the 21 components are non-zero and applying say only σ_1 let us see how an object responds ie. what are the different strains leading to the deformation. It is quite clear that subjected to only σ_1 (all others are stresses are zero), because S_{11}, S_{12}, S_{13},… all these are non-zero, therefore will lead to all the normal strains ε_1, ε_2, ε_3 and shear strains γ_{23}, γ_{13}, γ_{12}. That means, even if we have applied only a normal stress along 1, that will of course lead to normal strain along 1 ε_1, normal strains ε_2 along 2, and ε_3 along 3 (Poisson's effect). In addition, it will also lead to shear strains γ_{23}, γ_{13}, γ_{12} in 2-3, 3-1 and 1-2 planes respectively. Similarly, applying only σ_2 or σ_3 or τ_{23} or τ_{31} or τ_{12} will also lead to all the six strains meaning that all the stresses and strains are coupled and subjected to loading, an object made of anisotropic material will experience not only normal strains in all the three directions but also shear strains in all the planes.

Now, let us see what is S_{11}? S_{11} actually tells us, if we apply σ_1, what ε_1 is. That means, for a stress along 1, what is the stain along 1? Similarly, S_{22} tells us, if we apply a stress along 2, what is the strain along 2 and S_{33} tells us, if we apply a stress along 3, what the strain along 3 is. So, these three are actually the **extensional stiffness**.

Now, what is S_{12}? If we apply a stress σ_1 along 1, S_{12} decides what the strain along 2 is or if we apply a stress σ_2 along 2, S_{12} decides what the strain along 1 is.

Similarly, if we apply a stress, normal stress along 1, what the normal strain along 3 is decided by S_{13} and S_{23} is the coupling between normal stress along 2 and the strain along 3. Therefore, these S_{12}, S_{23} and S_{13} are called **extension-extension coupling**.

Now, what is S_{44}? If we apply a shear stress τ_{23} in plane 2-3 what the corresponding shear strain γ_{23} is in the plane 2-3 is decided by S_{44}. Similarly, if we apply a shear stress τ_{31} in plane 3-1 what the corresponding shear strain γ_{31} is in the plane 3-1 is decided by S_{55} and applying a shear stress τ_{12} in plane 1-2 what the corresponding shear strain γ_{12} is in the plane 1-2 is decided by S_{66}. So, these are called **shear stiffness**.

What do non-zero S_{45} S_{46} and S_{56} mean? It means that even if we apply a shear stress in the plane 2-3, that will also lead to a shear strain on the other planes. If we apply a shear stress along 1-2, that will not only lead to shear strain along 1-2, it will also lead to shear strains along 2-3 and 3-1. Similarly, for other planes also. Therefore, these S_{45} S_{46} and S_{56} are called **shear-shear coupling**, like extension-extension coupling.

In addition to that, the fact that S_{14} , S_{15} , S_{16} , S_{24} , S_{25} , S_{26} S_{46} and S_{34} , S_{35} , S_{36} are non-zero leads to the fact that application on normal stress leads to shear strains and application of shear stresses leads to normal strains and vice versa. Therefore, these terms are called **shear-extension coupling**. Therefore, in a fully anisotropic material, all the stresses and strains are coupled.

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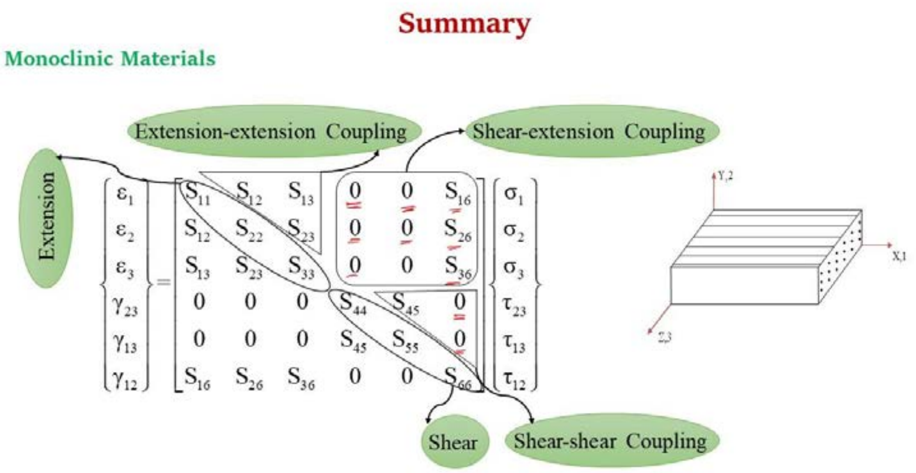

In a monoclinic material with one plane of material property symmetry, as shown in the stiffness matrix some of the coupling terms becomes zero.

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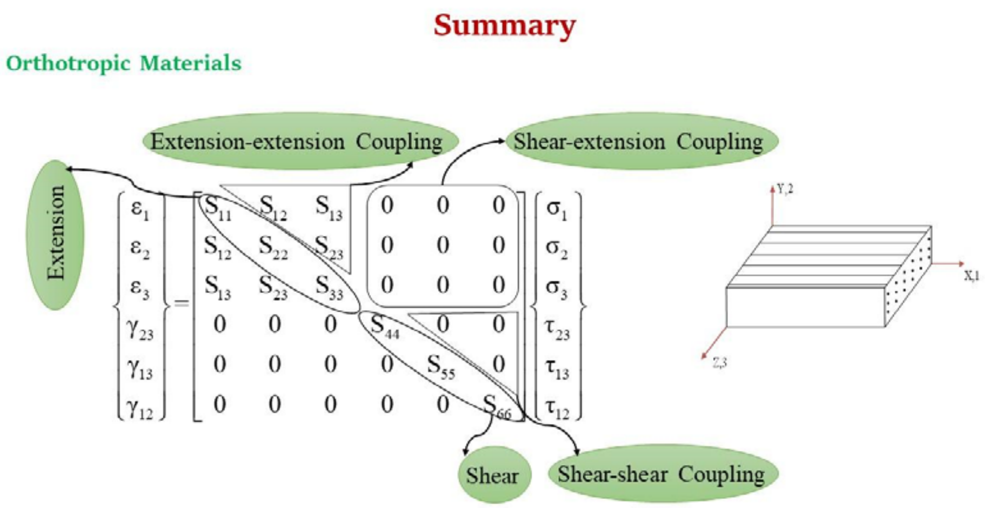

Similarly, in an orthotropic material with three mutually perpendicular planes of material property symmetry as shown in the stiffness matrix. All the shear-extension coupling terms as well as the shear-shear coupling terms become zero. That means, if we apply normal stress, that will lead to only normal strains and no shear strains; if we apply shear stresses, that will lead to only shear strains and no normal strains. Also there is no shear-shear coupling. If we apply shear stress in 1-2 plane, that will lead to shear strain in 1-2 plane only and it will not lead to shear strain in 2-3 or 3-1 plane. So, in an orthotropic material, shear-extension coupling is not there, shear-shear coupling is not there, but extension-extension coupling is there; that is because of the Poisson's effect.

Now, natural question comes that the same is also true for isotropic material. Even in isotropic material, if we apply normal stresses, that leads to only normal strains and there is no shear strain and vice-versa. Then, what is the difference? Difference is that, in isotropic materials, because of the normal stress is direction 1, the normal strains in the transverse directions (2 and 3) are same meaning, $S_{12}= S_{13}$, but these are not same for an orthotropic material where $S_{12}= S_{13}$.

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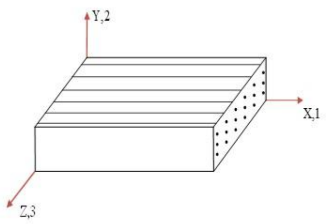
Summary

Transversely Isotropic Materials

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{S_{22}-S_{23}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{55} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$



In a transversely isotropic material, one plane actually behaves as isotropic and therefore, in that plane the relationship between the elastic constants exists like the modulus of rigidity is actually a function of Young's modulus and Poisson's ratio.