Mechanics of Fiber Reinforced Polymer Composite Structures
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Module-10
Design Examples
Lecture-28
Design Example II

Welcome to the second lecture of 10th module, we have been discussing some design examples of design of components made of FRP composites. In the last lecture we have discussed the design of a thin cylindrical pressure vessel made of FRP composites. Today we will discuss two more examples where we will discuss the steps involved in the design of components made of laminated FRP composites.

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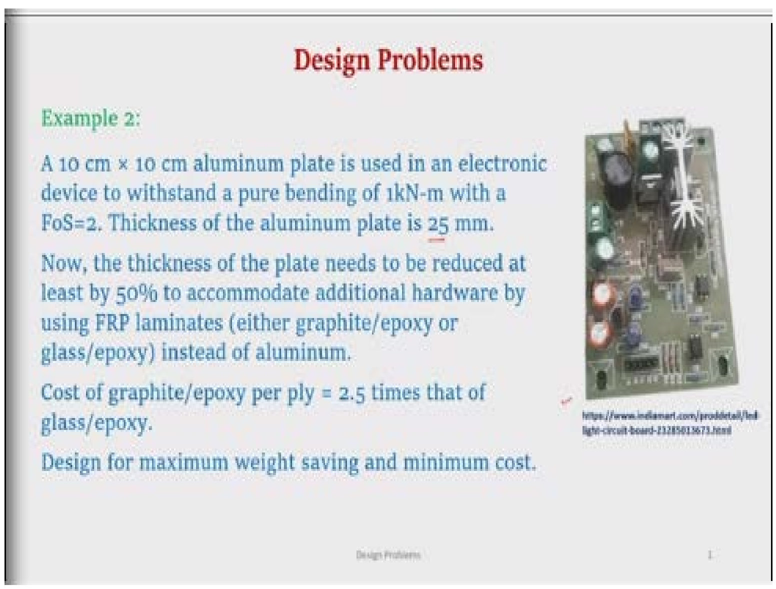

The first example we have already done in the last class. In this example, we have one 10 cm x 10 cm aluminum plate which is used in an electronic device to withstand a pure bending of 1 kN-m with a factor of safety (FoS) = 2; thickness of the aluminum plate is 25 mm. As shown in this figure this is a typical aluminum board, where some electronic components are actually mounted and therefore it experiences bending.

Now in this problem the thickness of the plates needs to be reduced at least by 50%. That means initially this is 25 mm; we need to reduce it by 50%. Why? –to accommodate some additional hardware. How we do this? Maybe instead of aluminum FRP laminates could be used. Suppose both graphite epoxy or glass epoxy are available. So we can design the same plate using either graphite epoxy or glass epoxy instead of aluminum and thereby reduce the thickness and see how much weight savings could be obtained. Now it is given that the cost of glass epoxy is less and cost of graphite epoxy is almost 2.5 times higher compared to that of glass epoxy. Say, we need to design this for maximum weight saving and minimizing the cost.

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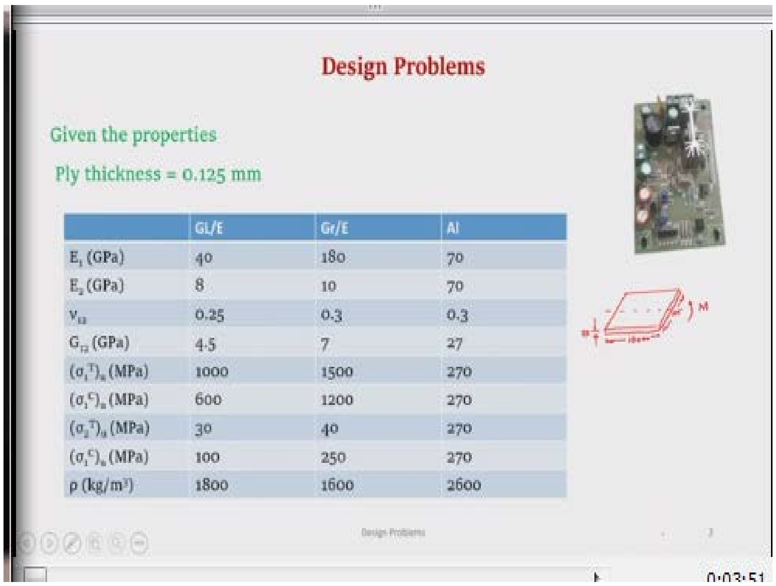

The properties = of unidirectional glass epoxy lamina, unidirectional graphite epoxy lamina and aluminum are given and the thickness of ply is 0.125 mm. Basically, this plate is made of aluminum, thickness is 25 mm and there are components in it and it actually experiences a bending moment. It is square plate with side 10 centimeters, that is 100 mm × 100 mm. So, we need to design this using FRP laminates.

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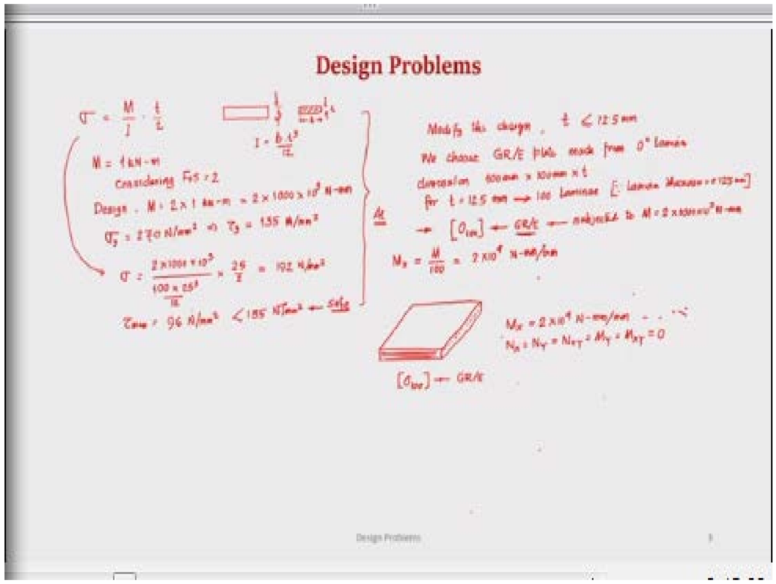

So, to start with let us first see that the aluminum plate. So, the properties of aluminum are given. We know that subjected to bending, that the maximum bending stress, $\sigma = \frac{M}{I}.\frac{t}{2}$. Referring to the Fig. the cross-section is (b × t). So, $I = \frac{bt^3}{12}$ and in this case**,** M=1kN-m and given that the FoS is 2.

Considering a FoS = 2, the design moment, M = (2 × 1) kN-m. So, we can write this in terms of millimeter also, so (2 × 1) kN-m= (2 × 1000 × 10^3) N-mm. Now given that allowable maximum stress or the strength of the aluminum, σ_y = 270 MPa = 270 N-mm^{-2}. Therefore, since it is a ductile material therefore yield point stress, τ_y =135 N-mm^{-2}.

Therefore, in this case, $\sigma = \frac{M}{I}.\frac{t}{2} = \frac{2 \text{ x } 1000 \text{ x } 10^3}{\left(\frac{100\text{x}25^3}{12}\right)} \text{x} \frac{25}{2} = 192 \text{ N-mm}^{-2}$. Therefore, now this is a one-dimensional bending stress. Therefore, the maximum shear stress, $\tau_{\max} = \frac{\sigma}{2} = 96 \text{ N-mm}^{-2}$.

Clearly, $\tau_{\max} \le \tau_y = 135 \text{ N-mm}^{-2}$, therefore this is safe, so this aluminum plate is safe.

Just to understand the design steps because we will be following the same thing when we go for design with FRP composites. We need to modify the design and the constraint is that the thickness, $t \le 12.5$ mm. Now suppose we choose graphite epoxy plate made from say all 0^0

lamina. And the lateral dimensions remain same 100 mm × 100 mm and we need to know what will be the thickness?

Now for t = 12.5 mm, we need 100 laminae, because each lamina thickness is 0.125 mm. Therefore the laminate is $[0_{100}]$ GR/E. Now this needs to withstand a bending moment of subjected to M = (2 × 1000 × 10^3) N-mm. Therefore, in classical lamination theory, the moment per unit width, $M_x = \dfrac{M}{100} = 2\times10^4$N-mm/mm = 2×10^4N.

Therefore the problem is now that we have a laminate $[0_{100}]$ GR/E and this is subjected to only $M_x = 2\times10^4$N-mm/mm , all other forces are 0. So, now we will have to analyze this laminate which is a $[0_{100}]$ layer GR/E laminate. We have to see that subjected to $M_x = 2\text{x}10^4\text{N}$ whether it is safe or not? Now how we do this?

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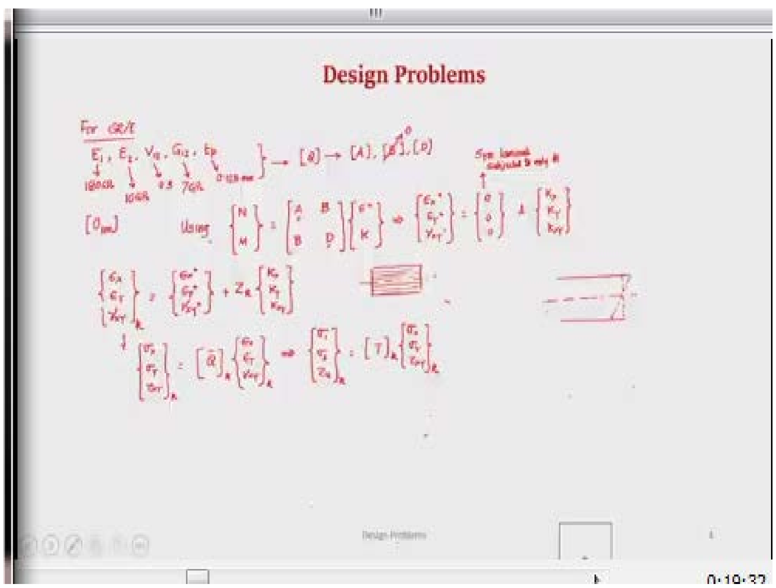

We have done it number of times using classical lamination theory. So, for graphite epoxy lamina now we have the properties are given E_1, E_2, ν_{12}, G_{12} as well as the thickness of each ply is given. For graphite epoxy this is E_1=180 GPa, E_2 is 10 GPa it is given in this table, $\nu_{12} = 0.3$, G_{12} = 7 GPA and ply thickness, t_p = 0.125 mm. So, for this we can immediately calculate [Q], and from the [Q] we can calculate [A], [B], [D]. Because this is what we need, we need [Q] for each layer; in this case all are 0^0 layers.

We need thickness of each ply and the stacking sequence. Of course, it is a symmetric laminate and therefore [B]= 0, so we have ABBD matrix. So, now $\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ K \end{Bmatrix} \Rightarrow \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$.

In this particular laminate subjected to only moment M_x, mid surface strains will be 0, because it is symmetric laminate. There will be curvatures K_x, K_y, K_{xy}, we can calculate the curvatures from

$$\{M\} = [D]\{K\} \Rightarrow \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} = [D]^{-1} \begin{Bmatrix} M_x \\ 0 \\ 0 \end{Bmatrix}.$$

Now knowing this we could calculate the strains in all the layers, We know that in any layer k (k=1,2,…100)

the strain $\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + Z_k \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$. Of course, in this case $\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_z^0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$.

For this laminate we have the stacking sequence, here all the layers are 0^0 layers, there are 50 layers above, 50 layers below the reference plane and therefore we can find out in all the layers, all the lamina what are the strains.

So, knowing these strains and z_k is the distance of that lamina from the reference plane or the mid surface. So, knowing these strains we can calculate what is thestress. In this case of course $[\overline{Q}] = [Q]$ because all the layers are 0^0 layers and from there we can find out what are the material axis stress in all the layers.

Again, in this case $\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}_k = [T]_k \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k$.

So, once we get the material axis stresses in each lamina, now we can assess using appropriate failure theories.

But here in this particular case you can see that it is subjected to pure bending and all the layers are 0^0 layers meaning that their properties are same. Therefore, with reference to the middle, the strain variation and the stress variation will follow the same trend. So, we can actually find out what are the stresses in the top and bottom layers that will be sufficient to analyze the failure.

Because the stress will be maximum at the top and the bottom layer, of course their signs will be different. In this particular case the first ply to fail will be either the top or the bottom layer.

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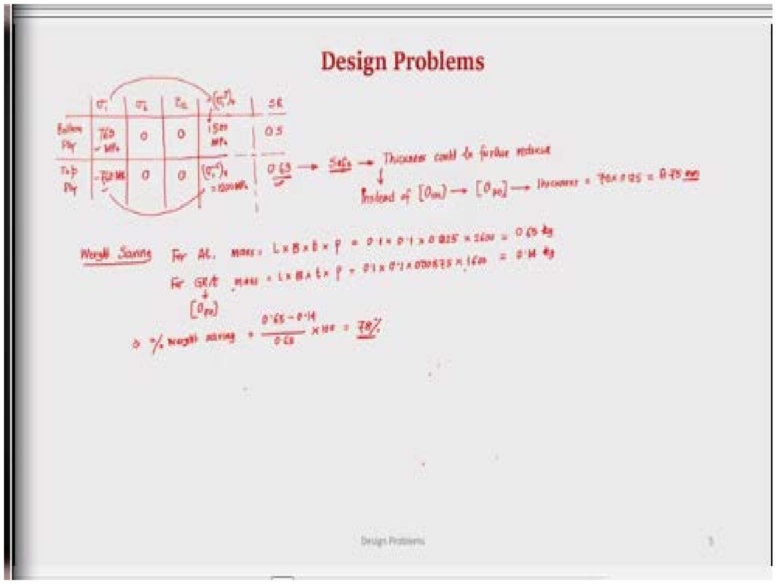

Therefore, we can calculate for all the 100 layers but the stresses (σ_1 , σ_2 , τ_{12}) only for the top and the bottom layer are shown. σ_1 in bottom ply is 760 MPa and that in the top ply is -760 MPa, σ_2 in both the plies are almost 0 and so is the shear stress, τ_{12}. Because all are 0^0 layers therefore there is no coupling subjected to M_X, there is no normal stress.

Now, what is the strength ratio? Corresponding to the positive, the strength is because it is positive $\left(\sigma_1^T\right) = 1500 \text{ MPa}$. Corresponding to the negative the corresponding strength $\left(\sigma_1^c\right) = 1200 \text{ MPa}$.

What is the strength ratio? Strength ratio (SR) for bottom ply is roughly 0.5 and for top ply is roughly 0.63. Therefore, the highest strength ratio is in the top ply = 0.63 that means if we keep on increasing the moment if at all it fails it will fail at the top ply (SR=0.63) first. But it is far away from the failure (SR=1), it is only 0.63. So, what does it imply? It implies that it is not only safe in fact thickness could be further reduced.

That means we can still reduce the number of plies of GR/E. So, ideally maybe suppose instead of 100 layers we can actually select maybe safely 70 and then if we take 70 layers that leads to thickness will be 70 × 0.125= 8.75 mm. So, we can see that 70 layers of GR/E which translates to a thickness of 8.75 mm which is much less than half of the thickness of the aluminum plate would suffice.

You can still do the analysis and see that it is safe. Or else what you can also do is because instead of reducing the thickness further safe maybe we can use glass epoxy. Because glass epoxy the strength is less compared to GR/E therefore thickness will be slightly more compared to GR/E. But then in terms of cost, the cost may be less. Therefore, we will have to take a decision whether we should go with GR/E or glass epoxy.

So, there could be number of solutions to this kind of problem and we will have to really look for the optimum one depending upon different constraints. Maybe in this case if you have to minimize the cost, we will have to take a decision whether we should go with 70 layers of GR/E or maybe 80 layers of glass epoxy and then take a decision. However, from the strength point of view this is also safe.

So, now let us see what is the weight savings? What is the weight saving of this modified design? Now for aluminum what is the mass of the aluminum plate? Mass of aluminium plate = Length (L) × breadth (B) × thickness (t) × density (ρ). So, L = 100 cm = 0.1 m, B = 0.1 m, t = 0.025 m. And the density (ρ) of aluminum was as per the table it was 2600 kg mm^{-3}.

Therefore, this gives us a mass of 0.65 kg for aluminum. Similarly for GR/E, length and breadth remains same. For $[0_{70}]$ graphite epoxy, mass = 0.1 × 0.1 × 0.00875 × 1600 = 0.14 kg. Therefore, % weight saving = $\frac{0.65-0.14}{0.65}\times 100 \approx 78\%$. So, this is how the modified design not only reduces the thickness, it also reduces the weight by 78%.

But again, this may not be the optimum design because the cost is also involved, so we will have to maybe next we can check with glass epoxy and come out with the solution which is also safe, which will also lead to some weight savings but maybe the cost will be even less. So, we will have to really look for the optimum solution.

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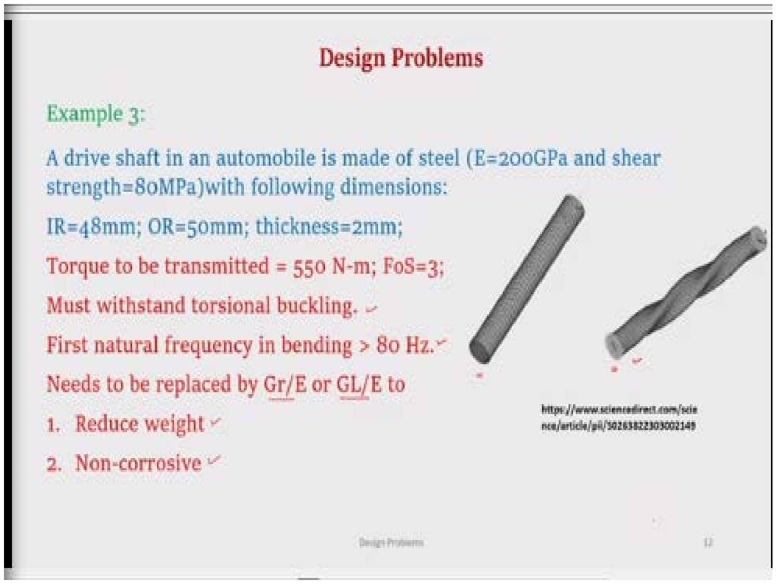

Next, we will take another problem say this a drive shaft all of you may be knowing in automobile the drive shaft actually transfers power from the front engine to the rear wheel differential. Generally made of steel and has the following dimensions, it is a hollow shaft with inner radius of 48 mm, outer radius 50 mm and thickness is 2 mm. Now the torque to be transmitted is 550 N-m and it has to operate with a factor of safety of 3.

Then it must withstand torsional buckling, it also it is first natural frequency in bending must be more than 80 Hz, I think you understand these two implications. Torsional buckling many of you may not be knowing that is why I have taken this figure. Referring to the figure, suppose this is a hollow shaft if you subject it to torsion there is local buckling like this. As you know that the pure shear could actually be looked at as equal and opposite normal stresses at an angle of 45^0.

Therefore, there is the compression stress actually causes buckling and if it is thin and subjected to torsion it might buckle. Now whether it will buckle or not that is decided by it is stiffness, how strong it is, it can resist buckling or not? Also for the shaft, its bending vibration the first natural frequency in this case should be greater than 80 Hz.

That means the forcing frequency should not be near to the first natural frequency in order to avoid resonance. So, here it is given that it should be more than 80 Hz, so this should be checked whenever we design this drive shaft. Now this drive shaft is already there made of steel, now it is required to be replaced by GR/E or glass epoxy, why? The reason is that one is of course the there is a need to reduce the weight and because of it is non-corrosiveness. If we make it with a GR/E or glass epoxy it is resistant to corrosion. So, let us see that how to go about it.

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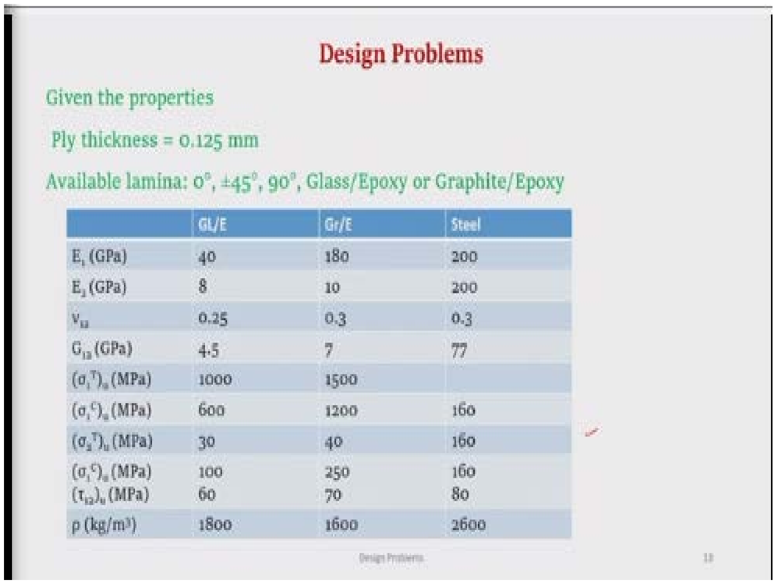

Design Problems

Given the properties

Ply thickness = 0.125 mm

Available lamina: 0°, $\pm45^\circ$, 90°, Glass/Epoxy or Graphite/Epoxy

	GL/E	Gr/E	Steel
E_1 (GPa)	40	180	200
E_2 (GPa)	8	10	200
ν_{12}	0.25	0.3	0.3
G_{12} (GPa)	4.5	7	77
$(\sigma_1^T)_u$ (MPa)	1000	1500	
$(\sigma_1^C)_u$ (MPa)	600	1200	160
$(\sigma_2^T)_u$ (MPa)	30	40	160
$(\sigma_2^C)_u$ (MPa)	100	250	160
$(\tau_{12})_u$ (MPa)	60	70	80
ρ (kg/m³)	1800	1600	2600

Now given that you have graphite epoxy and glass epoxy with lamina is available with 0^0, $\pm 45^0$ and 90^0 and the ply thickness is 0.125 mm. Again, as usual the defined properties for GR/E, glass epoxy and steel are given here.

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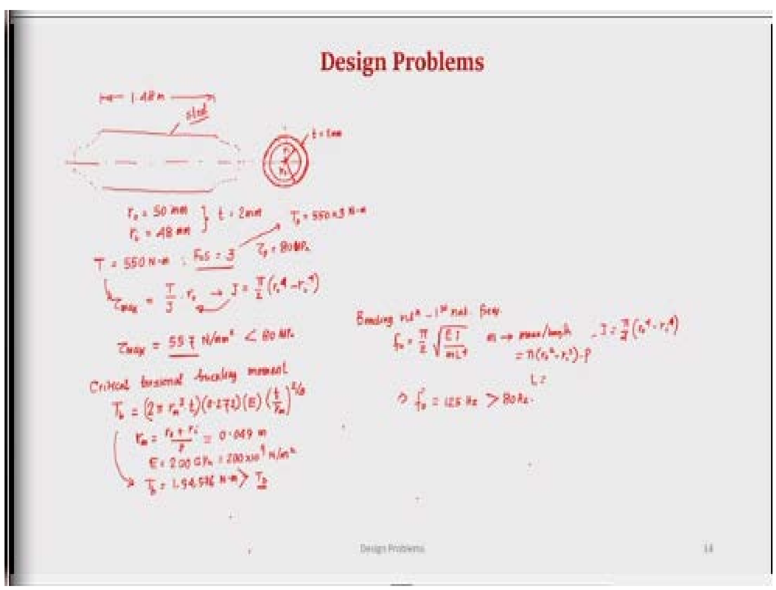

So, to start with let us see first this drive shaft. Suppose this is a drive shaft suppose say the inner radius r_i, the outer radius is r_0, given this r_0 = 50 mm and r_i = 48 mm, it is given and therefore thickness (t) = 2 mm. Now the torque to be transmitted is 550 N-m but it has to operate with a FoS of 3. Given that the shear strength of steel is 80 MPa and it has to withstand the torsional buckling as well as it is first natural frequency in bending must be more than 80 Hz.

Now properties are given, now let us see that for steel let us first check whether these dimensions are safe with steel or not. So, we know that maximum shear stress induced due to torsional moment T is given by, $\tau_{\max} = \frac{T}{J}.r_0$ for a hollow shaft, where J is the polar moment of inertia which is $J = \frac{\pi}{2}\left(r_0^4 - r_i^4\right)$. And in this case r_0 and r_i we know and then T is also known, so considering a FoS of 3 that means the design torque is actually 3 times that of 550 N-m.

So, we can find out that $\tau_{\max} = 55.7 \text{ N-mm}^{-2} = 55.7 \text{ MPa} < 80 \text{ MPa}$ and hence it is safe. So, therefore the design torque, $T_D = (550 \text{ x } 3) \text{ N-m}$. Say, if you put this here, we get that the maximum stress induced in the material is much less than the shear strength and therefore it is safe, from the strain point of view it is safe.

Then it has to withstand the torsional buckling. Now, we know critical buckling load for Euler column buckling. Similarly there are expressions for torsional buckling. Now critical torsional buckling moment say T_b is given by for a hollow shaft

$$T_b = (2\pi r_m^2 t)(0.272)(E)\left(\frac{t}{r_m}\right)^{2/3}, \text{ where } r_m = \left(\frac{r_0 + r_i}{2}\right) = 0.049\text{m}$$

and $E = 200 \text{ GPa} = 200 \text{ x } 10^9 \text{ N-m}^{-2}$

$\therefore\ T_b = 194576 \text{ N-m} > T_D$

Therefore, that it is safe with critical, I mean it will not undergo torsional buckling. We can also check the bending vibration. Bending vibration first natural frequency is given by $f_0 = \frac{\pi}{2}\sqrt{\frac{EI}{mL^4}}$, where m is the mass per unit length. In this case what is the mass per unit length?

Mass, $m = \pi\left(r_0^2 - r_i^2\right).\rho$. So, we can calculate the mass per unit length and I is the second moment, $I = \frac{\pi}{4}\left(r_0^4 - r_i^4\right)$. So, when you put all these here and L is the length of the shaft is given which is 1.48 m. When you put all these, we get f_0 = 125 Hz > 80 Hz therefore it is also say safe from the bending frequency point of view, that means the chances of resonance is avoided.

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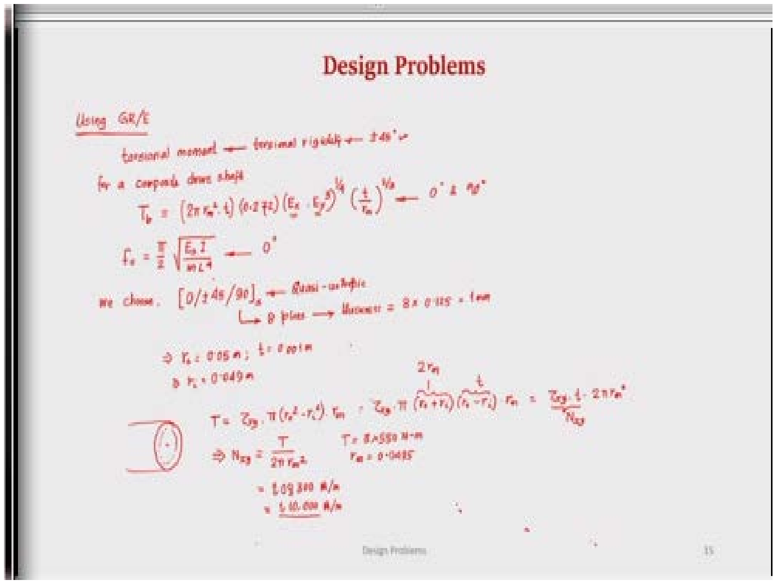

Now coming to the design of the shafts by GR/E. So, let us take GR/E as the material instead of steel. We have the choice of GR/E or glass epoxy let us start with GR/E. Now the question is what will be the stacking sequence? We have with us 0^0, 90^0 and $\pm 45^0$. Now if you look at what are the constraints? One thing is that it has to withstand torsional moment therefore it must have good torsional rigidity.

That means torsional rigidity must be that means it must have good shear modulus and we know that the shear modulus is maximum for $\pm 45^0$ therefore will resist the torsional moment. Now for a composite drive shaft, for a composite hollow shaft or drive shaft that critical torsional buckling moment is again this is an empirical there are many such relations, this

$$\mathrm{T}_b = (2\pi \mathrm{r}_\mathrm{m}^2 t)(0.272)(E_x.E_y{}^3)^{1/4}\left(\frac{t}{\mathrm{r}_\mathrm{m}}\right)^{2/3}$$

Now you can clearly see that the critical buckling moment actually depends on E_x and more significantly on E_y. Therefore, there must be lamina which could resist stiffness in the Y and X direction. So, it comes from 0^0 and 90^0 lamina, so we must have 0 and 90 also, for shear we must have $\pm 45^0$, for this critical torsional buckling we must have 0 and 90, more importantly 90.

Also, that for a composite drive shaft the first natural frequency of bending vibration, $f_0 = \frac{\pi}{2}\sqrt{\frac{E_x I}{mL^4}}$. Therefore, you can clearly see that the first natural frequency depends upon E_x, therefore we must have some 0^0 plies. So, in this when we decide the stacking sequence, we have to have ± 45^0, we must have 0^0, we must also have 90^0 ply. So, what we do is, we choose based on this discussion, so we choose $[0/\pm 45/90]_s$ symmetric laminate.

This is a quasi-isotropic laminate that means under in-plane load it behaves like isotropic. So, how many plies are there? It has 8 plies, therefore what is the thickness? Thickness (t) = 8 × 0.125 = 1 mm, this is the thickness. So, r_0 is fixed, r_0 = 0.05 m, t = 0.001 m therefore r_i = 0.049 m. Now we have an 8-ply laminate and what is the load? It is subjected to N_{xy}**.**

Because this is subjected to torsional shear, now you will see that this torsional moment T,

$$\mathrm{T} = \tau_{xy}.\pi\left(\mathrm{r}_0^2 - \mathrm{r}_\mathrm{i}^2\right).\mathrm{r}_\mathrm{m} = \tau_{xy}.\pi\left(\mathrm{r}_0 + \mathrm{r}_\mathrm{i}\right)\left(\mathrm{r}_0 - \mathrm{r}_\mathrm{i}\right).\mathrm{r}_\mathrm{m} = \tau_{xy}.t.2\pi\mathrm{r}_\mathrm{m}^2$$

$$\Rightarrow \mathrm{N}_{xy} = \frac{T}{2\pi\mathrm{r}_\mathrm{m}^2}$$

So, when we put these values, we have now what is the value of r_m? Now T is known, design torque T_D = (3 × 550) N-m, so r_m = 0.0495 m. And this comes out to be then N_x comes out to be roughly 109300 N-m^{-1} we take this as say 110000 N-m^{-1}.

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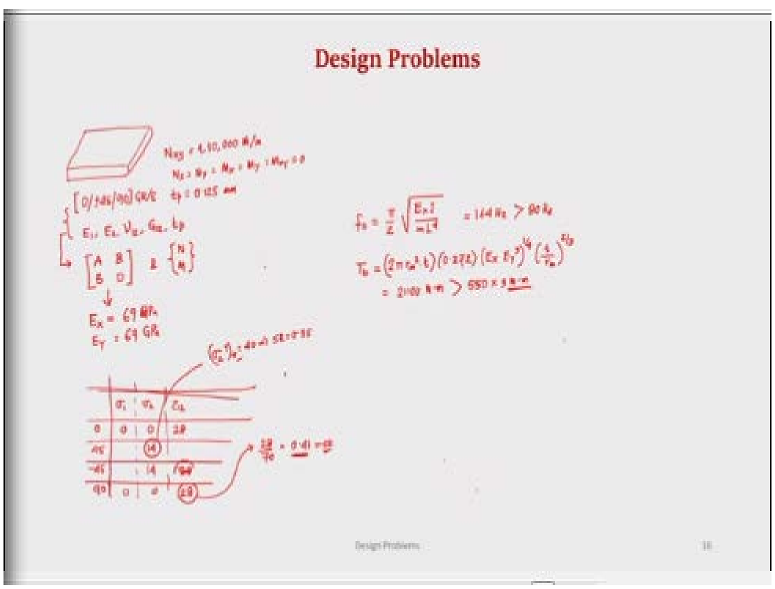

So, the problem is now we have a laminate which is subjected to only, what is this laminate? $[0/\pm 45/90]_s$ GR/E which is subjected to only N_{xy} =110000 N-m^{-1} all other N_x, N_y, M_x, M_y, M_{xy} = 0. So, we can analyze this now, where the ply thickness, t_p = 0.125 mm, we have the properties

E_1, E_2, ν_{12}, G_{12}, t_p and along with this the stacking sequence therefore we can compute $\begin{bmatrix} A & B \\ B & D \end{bmatrix}$

and we have $\begin{Bmatrix} N \\ M \end{Bmatrix}$

So, from this ABBD matrix we can find out what is the equivalent extensional or flexural Young's modulus, in this case it is 69 GPa. You can see why E_x = E_y = 69 GPa are same here because it is a quasi-isotropic laminate.

Therefore, E_x and E_y are same; it behaves like an isotropic under in-plane loading. So, once we analyze this, we can now list the stresses after doing the analysis we can find out like as usual we can now, we say 0, 45, -45, 90 and we can list all the stresses σ_1, σ_2, τ_{12}. And you do this and you can see that for 0^0 lamina, σ_1= 0, σ_2= 0, τ_{12} = 28 MPa.

θ	σ_1	σ_2	τ_{12}
0	0	0	28
+45		14	
-45		14	
+90	0	0	28

What is the strength ratio (SR)? For σ_2=14 MPa, the corresponding strength $\left(\sigma_2^T\right)_u = 40$ MPa $\Rightarrow$ S.R = $\dfrac{14}{40} = 0.35$. Similarly, $\left(\tau_{12}\right)_u = 70$ MPa $\Rightarrow$ S.R = $\dfrac{28}{70} = 0.41$.

Therefore, this is the strength ratio; maximum strength ratio is at 90^0 lamina.

So, we see that this is 0.41 that means it is far away from the failure, it will not fail, no ply will fail under this. Therefore, we can still reduce the number of plies or because the strength ratio is very less, we can actually go for glass epoxy laminate and check with that. But for the time being what we can do is suppose we continue with if we look at this laminate it is safe from the strength point of view but whether it satisfies the other condition.

That means what is the first natural frequency in bending? Let us see this is $f_0 = \dfrac{\pi}{2}\sqrt{\dfrac{E_x I}{mL^4}}$, so we can find out the values of I and we can find out the value of m like the earlier case and we can see that this comes out to be 164 Hz > 80 Hz and therefore this condition is also satisfied and what about the buckling? So, the critical torsional buckling moment is $T_b = (2\pi r_m^2 t)(0.272)(E_x.E_y{}^3)^{1/4}\left(\dfrac{t}{r_m}\right)^{2/3}$.

As I told these are empirical relations, [from Kaw's book]. So, you know all these values now, therefore if you put this you get $T_b = 21000\text{ N-m} > T_D = 550 \text{ x } 3\text{ N-m}$, therefore it is also safe against torsional buckling. Now, so we can say that this is a $[0/\pm 45/90]_s$ symmetric graphite epoxy is a feasible solution, which actually reduces the weight.

We could now try to see what is the weight saving like the earlier case. But what we have seen here is that this is far stronger than required. So, maybe we can try with glass epoxy and do the same exercise and then we can take a decision whether we should go for glassy epoxy or graphite epoxy. Because of the fact that the glass epoxy the cost will be much less compared to that of the graphite epoxy.

So, we can actually try with glass epoxy with the same quasi-isotropic laminate and check that is also safe or not. In that case the cost of glass epoxy is almost 2.5 times less than that of graphite epoxy and we can compare the cost. So, what we will have to compare? We will have to see that it is safe from the strength point of view; it satisfies that it does not buckle under torsional buckling. The first natural frequency of bending vibration must be more than 80 Hz and its cost should be minimum and there should be sufficient weight savings. So, we can compare all these parameters and then take a decision to arrive at an optimum design. In this lecture, we have discussed the steps and understood how to go about it.