Mechanics of Fiber Reinforced Polymer Composite Structures
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Module-10
Design Examples
Lecture-27
Design Example I

Hello and welcome to the 10^{th} module of this course mechanics of fiber reinforced polymer composite structures. In the last module, failure of laminate has been discussed in details. Starting with classical lamination theory, stresses in all the laminae are calculated followed by the calculation of the first ply failure load and the last ply failure load using appropriate failure criterion (maximum stress criterion or Tsai–Hill failure criterion).

With this background, in this module, design and analysis of some common components made from FRP laminated composites will be discussed. In today's lecture specifically design of a thin cylindrical pressure vessel will be understood by solving one problem where the weight savings by using this fiber reinforced composite materials will also be understood.

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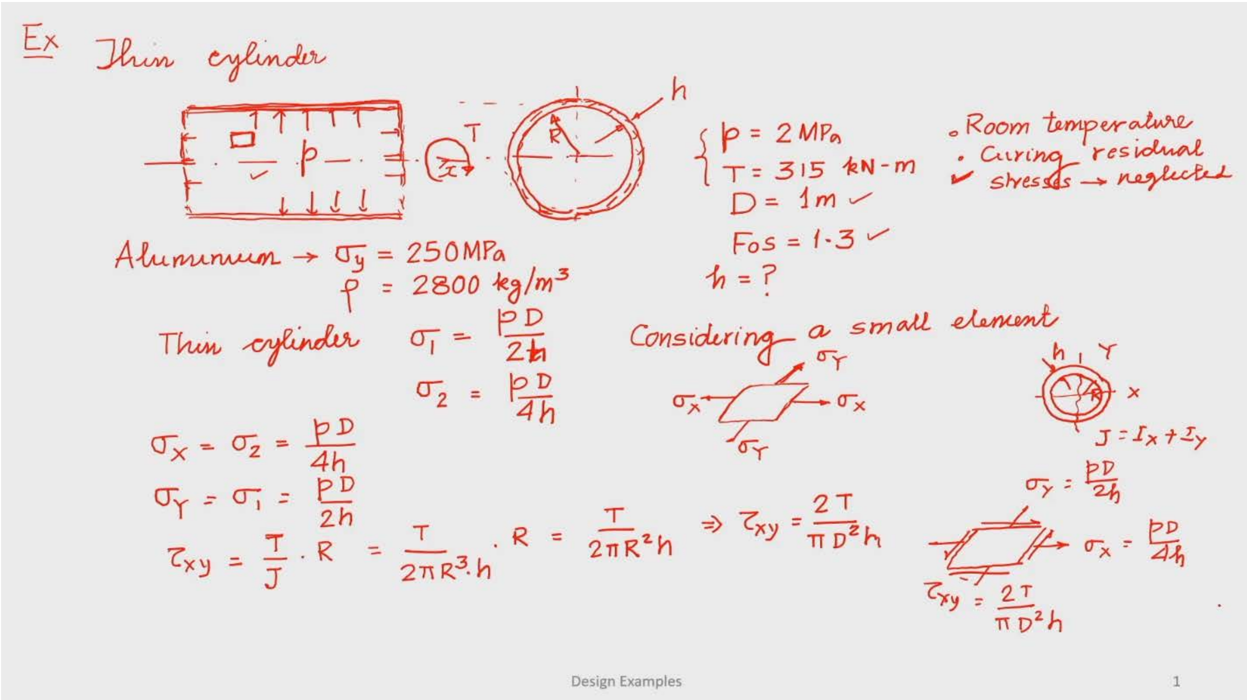

Considering a thin cylindrical pressure vessel, closed at both the ends. The wall of the cylinder has a thickness, of *h*. The cylinder contains a fluid with internal pressure *p* and is also subjected to a twisting moment *T*. This is made of aluminum having ultimate yield point stress = 250 MPa and density of aluminum = 2800 kg/m^3. Suppose the intensity of internal pressure *p* = 2 MPa and the applied twisting moment *T* = 315 kN-m. The internal diameter of the cylinder is =1 m. It is a thin cylinder meaning *h* is very small compared to *D*. It is operating at room temperature and the curing residual stresses due to curing may be neglected. First, the cylindrical pressure vessel is analyzed when it is made of aluminum then the amount of weight saving by replacing aluminum by means of a fiber composite? Given, the operating factor of safety FoS=1.3.

We need to first find out the minimum thickness for the pressure vessel to be safe. We consider this as a thin cylinder as the ratio of the mean radius to the thickness is more than 10. Therefore, the Hoop stress or circumferential stress is $\sigma_Y = \sigma_1 = \dfrac{PD}{2h}$ and the longitudinal stress $\sigma_X = \sigma_2 = \dfrac{PD}{4h}$, where *h* is the thickness of the cylinder.

Now if we take a small element from the cylinder as shown in Fig., it experiences

$$\sigma_X = \sigma_2 = \frac{PD}{4h}; \sigma_Y = \sigma_1 = \frac{PD}{2h}$$

In addition to that because it is experiencing a twisting moment, so the torsional shear stress,

$\tau_{xy} = \frac{T}{J} \cdot R$, where T is the twisting moment, J is the polar moment of inertia and R = $D/2$.

Now for a thin circular cylinder, the polar moment of inertia is given by $J = 2\pi R^3 h$.

[$J = I_X + I_Y$ and $I_X = \pi R^3 h = I_Y$ and hence $J = 2\pi R^3 h$]

$$\text{So, } \tau_{xy} = \frac{T}{J} \cdot R = \frac{T}{2\pi R^3 h} \cdot R = \frac{T}{2\pi R^2 h}$$

We can write τ_{xy} in terms of D as

$$\Rightarrow \tau_{xy} = \frac{2T}{\pi D^2 h}$$

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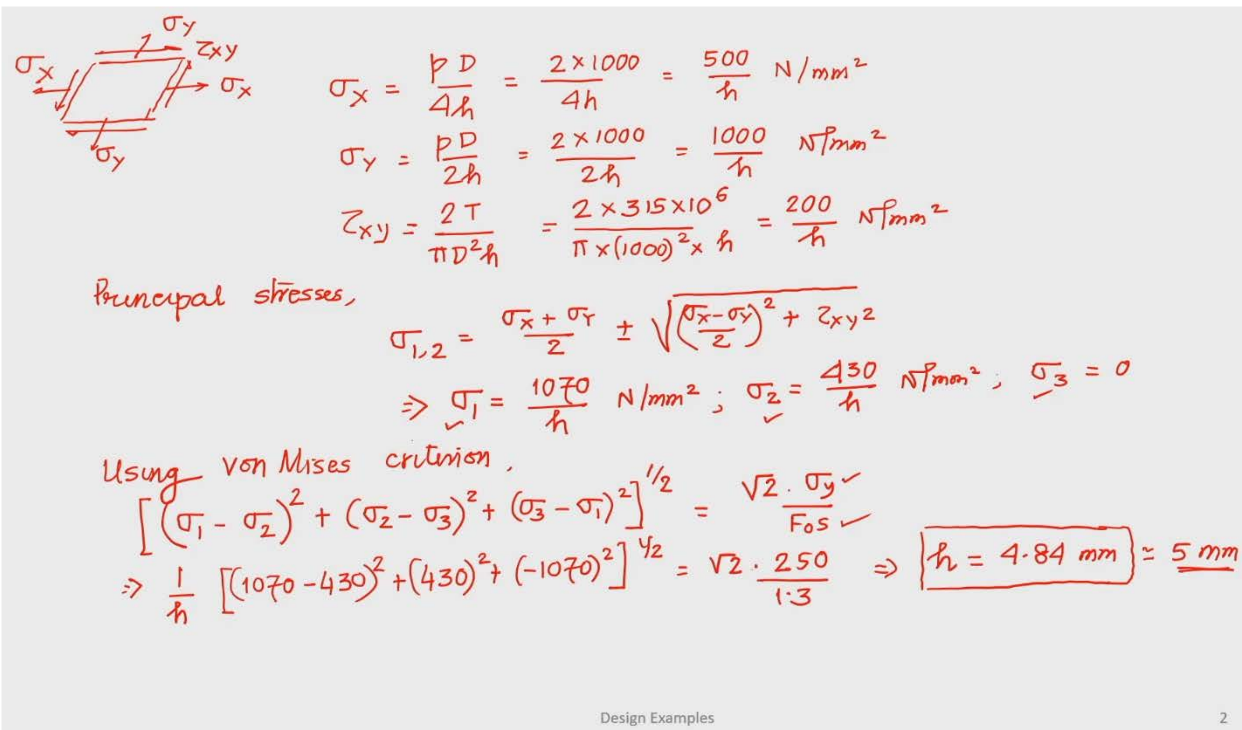

Putting the values of P, D and h, we get the following:

$$\sigma_x = \frac{PD}{4h} = \frac{2\times1000}{4h} = \frac{500}{h}\,\text{N/mm}^2$$

$$\sigma_y = \frac{PD}{2h} = \frac{2\times1000}{2h} = \frac{1000}{h}\,\text{N/mm}^2 \qquad \text{as the state of stress.}$$

$$\tau_{xy} = \frac{2T}{\pi D^2 h} = \frac{2\times315\times10^6}{\pi\times(1000)^2\times h} = \frac{200}{h}\,\text{N/mm}^2$$

Now the principal stresses could be determined as follows.

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\Rightarrow \sigma_1 = \frac{1070}{h}\,\mathrm{N/mm^2}; \sigma_2 = \frac{430}{h}\,N/\mathrm{mm}^2, \sigma_3 = 0$$

So, this is a two dimensional state of stress with $\sigma_3 = 0$.

The cylinder is made of aluminum which is a ductile material, therefore the von Mises stress criteria or the distortion energy theory will be used. So, using this value of σ_1 and σ_2 and ($\sigma_3 = 0$) and considering a factor of safety as 1.3, therefore

$$\left[\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right]^{1/2} = \frac{\sqrt{2}\cdot\sigma_y}{FoS}$$

$$\Rightarrow \frac{1}{h}\left[(1070 - 430)^2 + (430)^2 + (-1070)^2\right]^{1/2} = \sqrt{2}\cdot\frac{250}{1.3}$$

This gives h = 4.84 mm.

So, using distortion energy theory or von Mises criterion the minimum thickness of the aluminum cylinder is 4.84 mm so that it does not fail and this could be taken as 5 mm. Now the objective of solving this problem has been that we want to replace aluminum as the material of the cylinder by means of fiber composites to reduce the weight.

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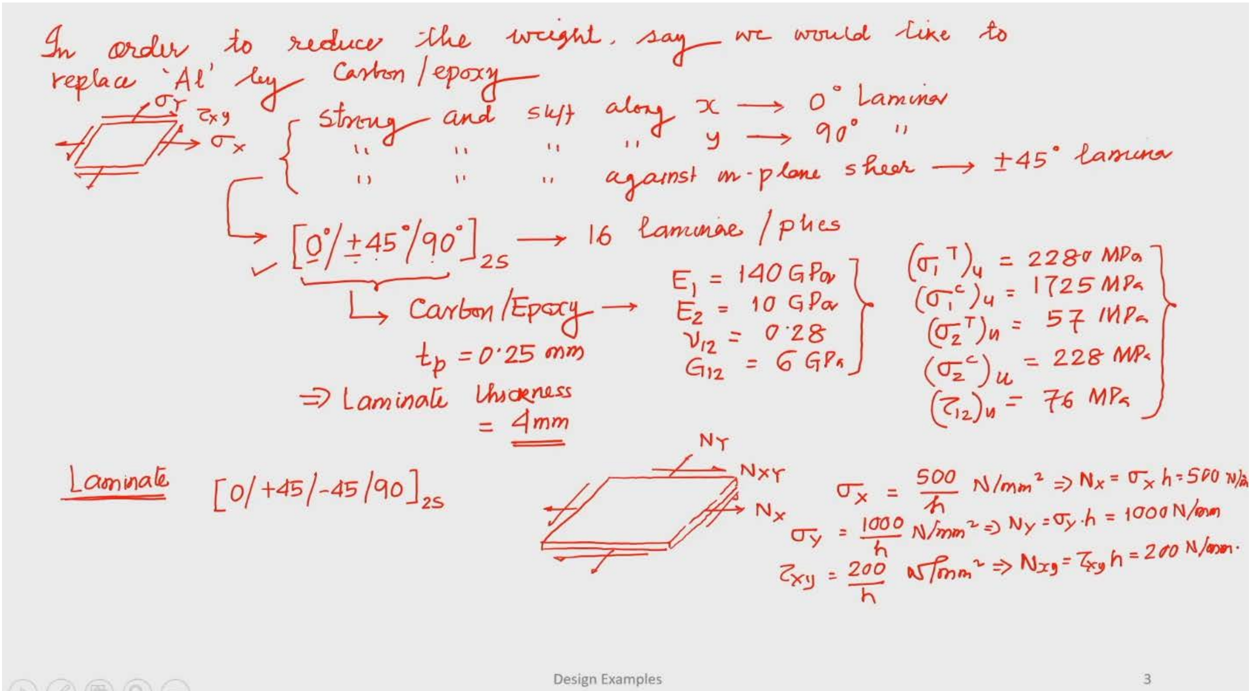

Therefore in order to reduce the weight we would like to replace aluminum by carbon epoxy. The state of stress being σ_x, σ_y and τ_{xy}, it must be strong and stiff along x, y and also possess shear stiffness. Therefore the stacking sequence must have some 0° laminae for withstanding σ_x. It must be strong and stiff along y- direction to withstand σ_y and therefore must have some 90° laminae. It is also experiencing shear in the x-y plane, therefore to make it strong and stiff against in-plane shear it must have some $\pm$ 45° lamina.

Therefore it is decided to choose a laminate which is $[0/+45/-45/90]_{2S}$ meaning that there are 16 lamina or 16 plies made of high strength carbon epoxy with properties of unidirectional carbon epoxy as E_1 = 140GPa, E_2 = 10GPa, ν_{12} = 0.28 and G_{12} = 6GPa which is an orthotropic lamina. The strength properties are as follows:

Longitudinal tensile strength = 2280MPa,

Longitudinal compression strength = 1725MPa,

Transverse tensile strength = 57MPa

Transverse compression strength = 228MPa and

In-plane shear strength = 76MPa.

Thickness of each ply is 0.25 mm.

Total thickness of the lamina is 16 × 0.25 mm = 4 mm.

It is decided to replace aluminum by means of this 4 mm thick laminate. Then we would like to see whether this is safe, and if it is safe whether it is within the operating desired factor of safety of 1.3 or not. Considering a small piece from the cylinder (as shown in Fig.), it is subjected to σ_x, σ_y and τ_{xy}. Following the convention of classical lamination theory it is subjected to in-plane force resultants N_x, N_y and N_{xy}.

Now, $\sigma_x = 500/h$, $\sigma_y = 1000/h$, $\tau_{xy} = 200/h$ (all in N/mm^2) or MPa

So, $N_x = \sigma_x h = 500$ N/mm or MPa-mm.

$N_y = \sigma_y h = 1000$ N/ mm or MPa-mm and

$N_{xy} = \tau_{xy} h = 200$ N/mm or MPa-mm.

Now we will have to assess whether this is safe under this loading or not.

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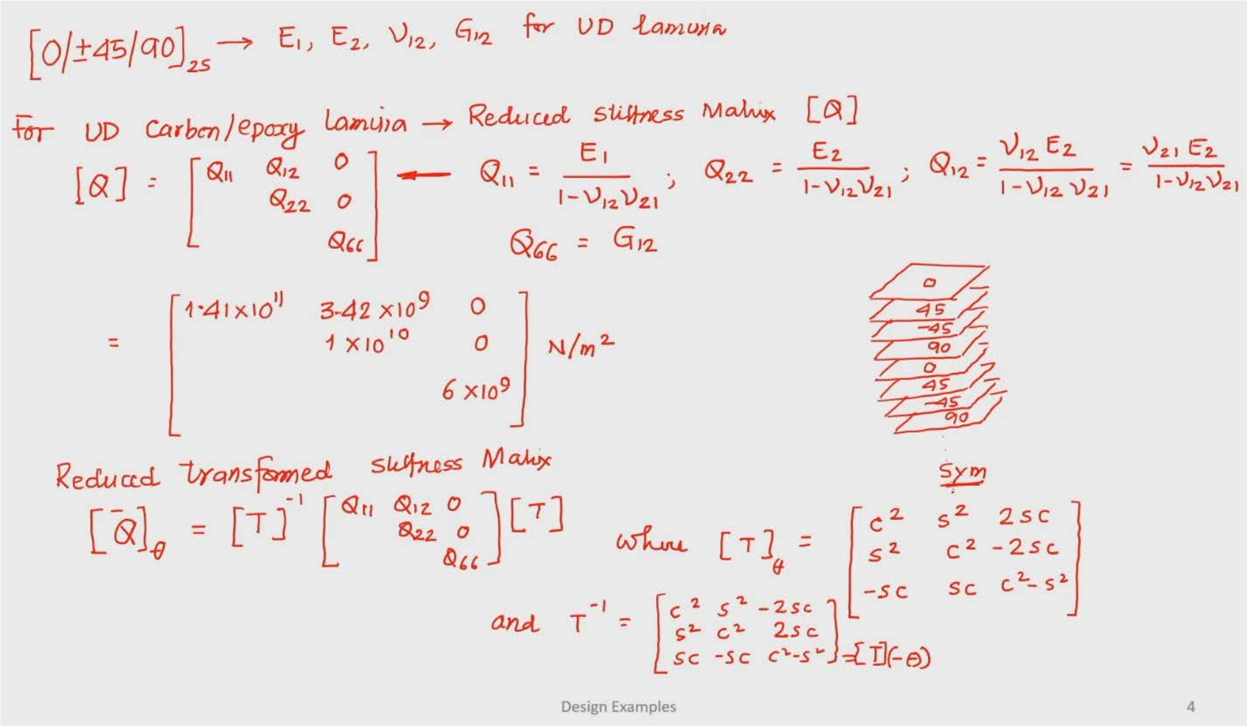

Using E_1, E_2, ν_{12}, G_{12} for unidirectional lamina we evaluate Q_{11}, Q_{12}, Q_{16}, Q_{22}, Q_{26} and Q_{66}, as

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}; Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}; Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_2}{1-\nu_{12}\nu_{21}}$$

$$Q_{66} = G_{12}; Q_{16} = Q_{26} = 0$$

and thus

$$[Q]=\begin{bmatrix} 1.41\times10^{11} & 3.42\times10^{9} & 0 \\ & 1\times10^{10} & 0 \\ & & 6\times10^{9} \end{bmatrix} \mathrm{N/m^2}$$

So, from this $[Q]$ we can actually find out the reduced transform stiffness matrix for each of the lamina depending upon it is fiber orientation as

$$[\bar{Q}]_\theta = [T]^{-1} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ & Q_{22} & 0 \\ & & Q_{66} \end{bmatrix} [T]$$

[T] is the stress transformation matrix given by

$$[T]=\begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2-s^2 \end{bmatrix}$$ and $[T]^{-1}$ is

$$[T]^{-1}=\begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2-s^2 \end{bmatrix}$$

So, for each lamina we obtain the transformed reduced stiffness matrix as shown in slide 3 as follows:

$$[\bar{Q}]_{0^\circ}=\begin{bmatrix} 1.41\times10^{11} & 3.42\times10^{9} & 0 \\ & 1\times10^{10} & 0 \\ & & 6\times10^{9} \end{bmatrix} N/m^2; \quad [\bar{Q}]_{45^\circ}=\begin{bmatrix} 4.5\times10^{10} & 3.35\times10^{10} & 3.27\times10^{10} \\ & 4.5\times10^{10} & 3.27\times10^{10} \\ & & 3.6\times10^{10} \end{bmatrix} N/m^2$$

$$[\bar{Q}]_{90^\circ}=\begin{bmatrix} 1\times10^{10} & 3.42\times10^{9} & 0 \\ & 1.41\times10^{11} & 0 \\ & & 6\times10^{9} \end{bmatrix} N/m^2; \quad [\bar{Q}]_{-45^{10}}=\begin{bmatrix} 4.5\times10^{10} & 3.35\times10^{10} & -3.27\times10^{10} \\ & 4.5\times10^{10} & -3.27\times10^{10} \\ & 3.6\times10^{10} & \end{bmatrix} N/m^2$$

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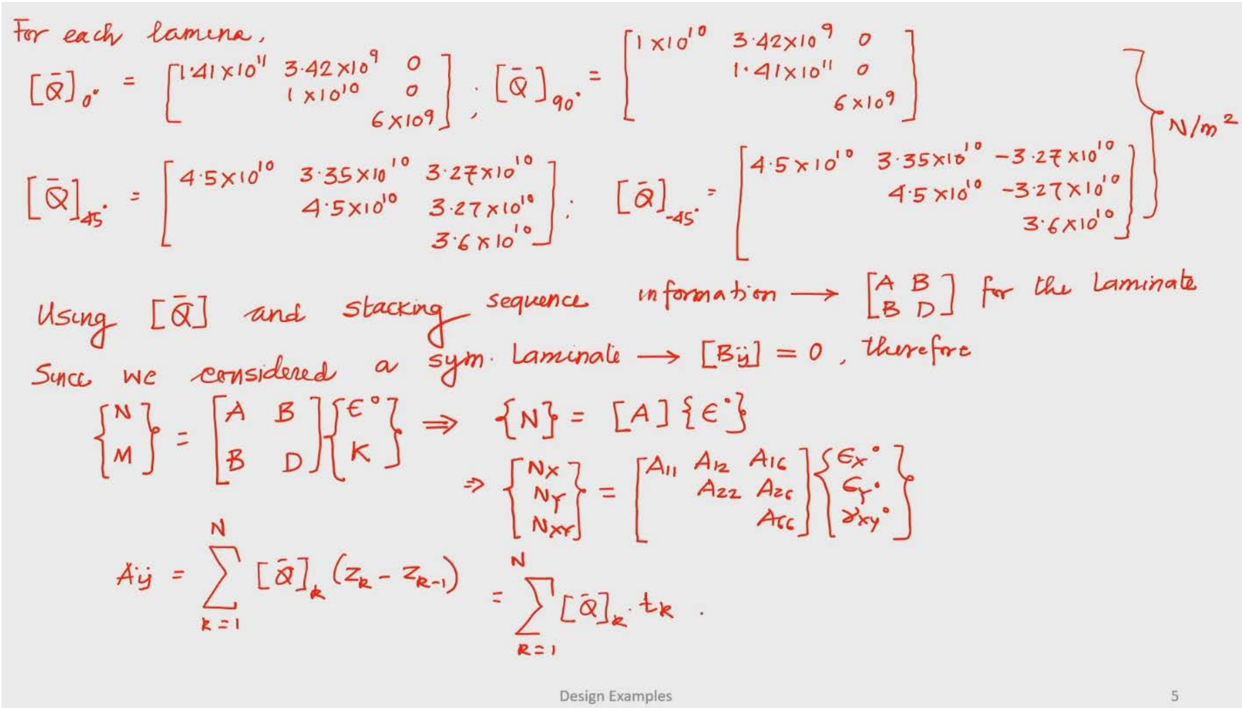

Using $[\bar{Q}]$ for each layer and stacking sequence information (location of each lamina and their thickness), the laminate ABBD matrix is evaluated. Being a symmetric laminate, the elements of [B] are all 0. So,

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^\circ \\ K \end{Bmatrix} \Rightarrow \{N\} = [A]\{\varepsilon\}$$

$$\Rightarrow \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ & A_{22} & A_{26} \\ & & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix}$$

We could decouple because it is a symmetric laminate. Now we can find out the elements of ABBD matrix using this $[\bar{Q}]$ for each layer and stacking sequence information. Now because it is only subjected to in-plane load, therefore we need only [A] and from classical lamination theory [A] could be calculated as

$$[A] = \sum_{k=1}^{N} [\bar{Q}]_k \left(z_k - z_{k-1} \right) = \sum_{k=1}^{N} [\bar{Q}]_k \cdot t_k$$

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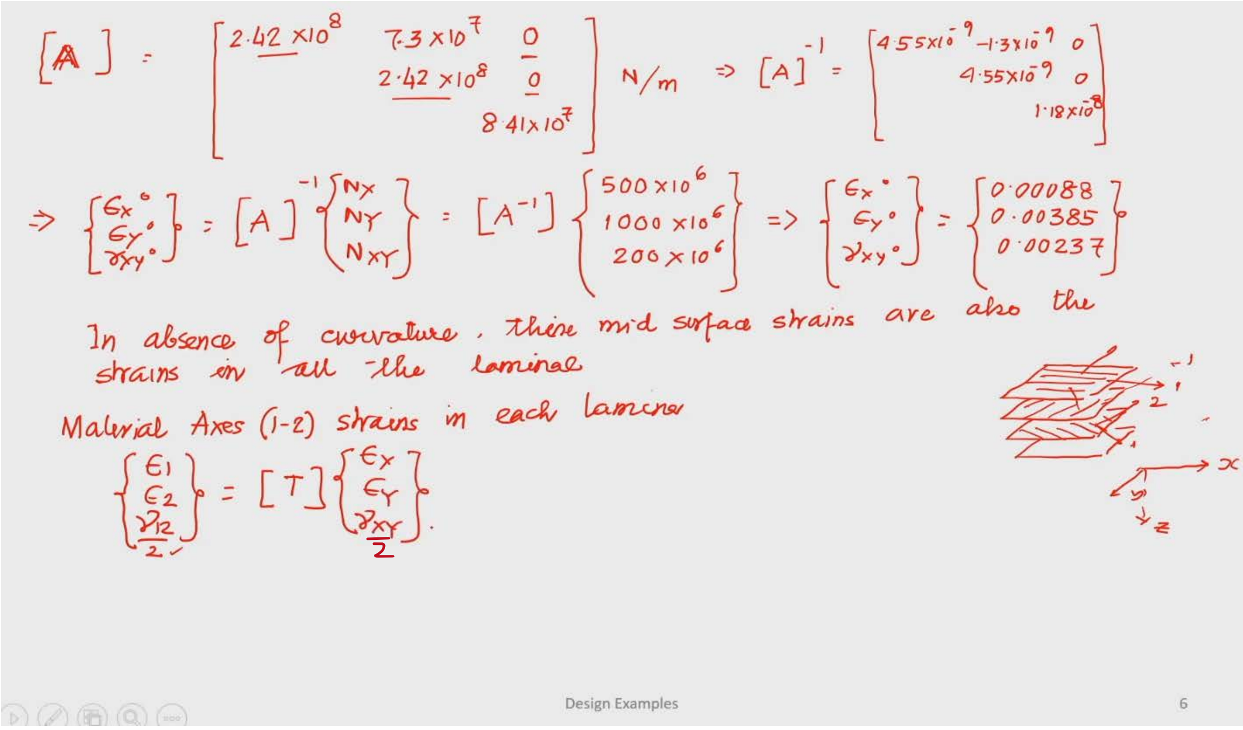

Therefore using this we get the *[A]* and $[A]^{-1}$ as

$$[A] = \begin{bmatrix} 2.42\times10^{8} & 7.3\times10^{7} & 0 \\ & 2.42\times10^{8} & 0 \\ & & 8.41\times10^{7} \end{bmatrix} \mathrm{N/m} \Rightarrow [A]^{-1} = \begin{bmatrix} 4.55\times10^{-9} & -1.3\times10^{-9} & 0 \\ & 4.55\times10^{-9} & 0 \\ & & 1.18\times10^{-8} \end{bmatrix} m/N$$

Now note that in this *[A]*, $A_{11} = A_{22}$, $A_{16} = A_{26} = 0$. Because $[0/+45/-45/90]_S$ is a symmetric quasi-isotropic laminate which behaves like isotropic under in-plane load and therefore there is no shear extension coupling. Using N_x = 500 N/mm, N_y =1000 N/mm, N_{xy} = 200 N/mm and

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ & A_{22} & A_{26} \\ & & A_{66} \end{bmatrix}^{-1} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}$$

We get

$$\begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = \begin{Bmatrix} 0.00088 \\ 0.00385 \\ 0.00237 \end{Bmatrix}$$

as the mid surface strains of the laminate. Now in absence of curvature this is also the strains (with respect to x-y) in all the layers. Now we determine the material axis strains in each lamina using strain transformation as

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \dfrac{\gamma_{12}}{2} \end{Bmatrix} = [T] \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \dfrac{\gamma_{xy}}{2} \end{Bmatrix}.$$

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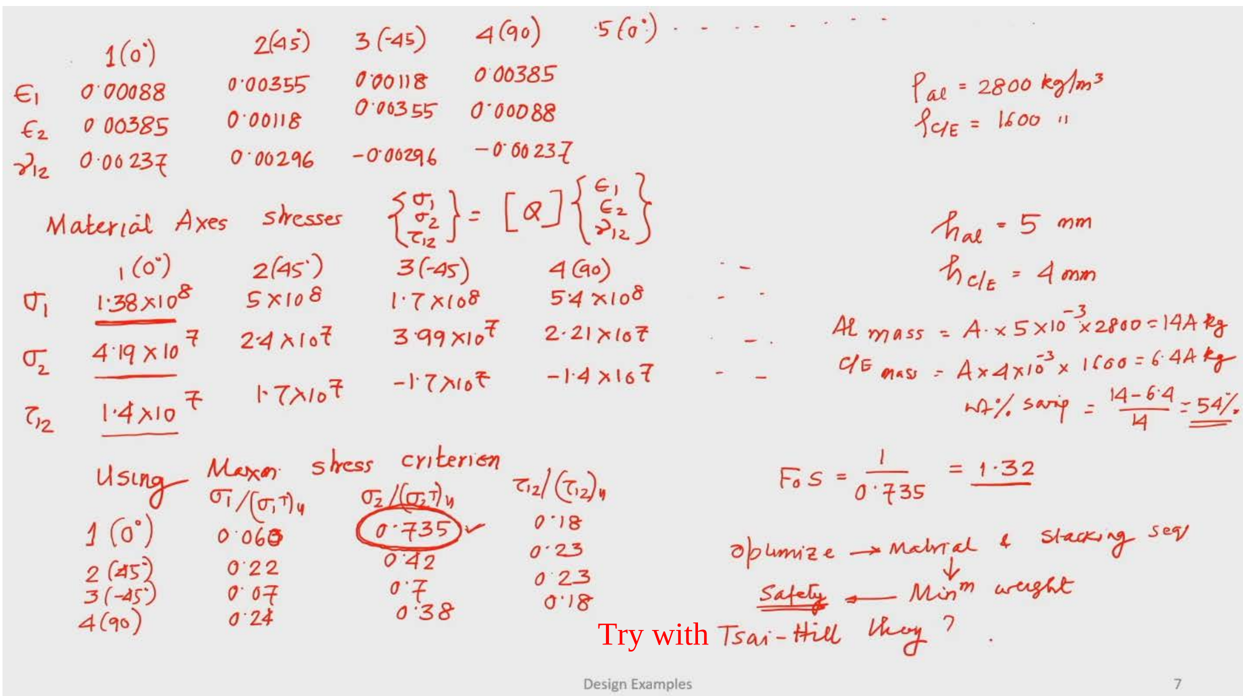

So, the material axis strains in each layer are tabulated as

	$1(0^\circ)$	$2(45^\circ)$	$3(-45)$	$4(90)$
ϵ_1	0.00088	0.00355	0.00118	0.00385
ϵ_2	0.00385	0.00118	0.00355	0.00088
γ_{12}	0.00237	0.00296	–0.00296	–0.00237

We can write for all the 16 layers, but it will be repetition only. Then we calculate the material axis stresses, from this as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [Q] \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} \text{ and are tabulated as}$$

	$1(0^\circ)$	$2(45^\circ)$	$3(-45)$	$4(90)$	
σ_1	1.38×10^8	5×10^8	1.7×10^8	5.4×10^8	in N/m^2
σ_2	4.19×10^7	2.4×10^7	3.99×10^7	2.21×10^7	
τ_{12}	1.4×10^7	1.7×10^7	-1.7×10^7	-1.4×10^7	

Note that unlike in the case of aluminum, σ_1, σ_2 are not principal stresses but they are material axes stresses in the case of FRP composites.Now using appropriate failure criterion, say using maximum stress criterion, we compare the material axis stresses to the corresponding strengths and the ratios of the stress to the corresponding strength (failure index) are tabulated as given below

	$\sigma_1/\left(\sigma_1^T\right)_u$	$\sigma_2/\left(\sigma_2^T\right)_u$	$\tau_{12}/\left(\tau_{12}\right)_u$
$1(0^\circ)$	0.060	$\boxed{0.735}$	0.18
$2(45^\circ)$	0.22	0.42	0.23
$3(-45^\circ)$	0.07	0.7	0.23
4(90)	0.24	0.38	0.18

We could see that the ratio is highest (0.735) in lamina 1 in TT mode. That means if we increase the load then this lamina 1 in the transverse direction will first reach it is corresponding strength, and when this is equal to 1, it fails. Now under the present loading it does not fail. That means the factor of safety is (1/ 0.735) = 1.32. The desired factor of safety was 1.3 that means it is still within the operating desired factor of safety and is safe.

Therefore this carbon epoxy laminate which we have chosen as a material to replace the aluminum will serve the purpose and the thin cylinder will not fail under this, internal pressure intensity and the twisting moment and it is operating factor of safety is 1.32. Note that the thickness of the aluminum cylinder was 5 or 4.84 mm. So, the *h* for aluminum was 5 mm and *h* for carbon epoxy is *4* mm that means it is even thinner. In addition to that the density of aluminum is 2800 kg /m^3, whereas, that for carbon epoxy is 1600 kg /m^3, therefore for a given area, *A*, the mass are

$$Al_{\text{mass}} = A\times5\times10^{-3}\times2800 = 14A \text{ kg}$$
$$C/E_{\text{mass}} = A\times4\times10^{-3}\times1600 = 6.4A \text{ kg}$$
$$\therefore \text{weight\% saving} = \frac{14-6.4}{4} = 54\%$$

Therefore the percentage weight saving is 54%.

That means by replacing aluminum with carbon epoxy we could save 54% of the weight. That means the weight of that cylinder made of $[0/+45/-45/90]_{2S}$ carbon/epoxy will be 54% less than that of the aluminum cylinder and it is equally strong.

However, this may not be the optimum solution and one may have a better laminate which will give rise to even further weight minimization. Therefore this is actually an optimization problem where you can optimize the stacking sequence and the material for minimizing the weight.

In addition, this analysis has been done using maximum stress theory. Instead of maximum stress theory, Tsai-Hill theory could also be used.

One may also use glass epoxy, definitely the density of glass box is more compared to that of carbon epoxy and one may try that for the same stacking sequence whether any weight savings could be achieved using glass epoxy or not.