Mechanics of Fiber Reinforced Polymer Composite Structures
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Module-08
Elastic Behaviour of Laminates-II
Lecture-24
Analysis of Laminates

Hello and welcome. We have been discussing the macro mechanical analysis of laminate and in last few lectures, the classical lamination theory has been discussed and the constitutive relation for a layered laminate has been obtained where the force and moment resultants were related to the mid surface strains and curvatures by the so called $\begin{bmatrix} A & B \\ B & D \end{bmatrix}$ matrix. Using this, the determination of strains and stresses in each lamina of the laminate was also discussed.

Depending upon the values of the elements of the $\begin{bmatrix} A & B \\ B & D \end{bmatrix}$ matrix, some special cases of laminate stiffnesses and some special types of laminates viz. symmetric laminates anti-symmetric laminates, balanced laminates, quasi-isotropic laminate etc and their significance in terms of achieving the desired stiffness and behavior of laminates under load have also been discussed. Then, when a laminate experiences a temperature change or a moisture concentration change the residual stresses induced in the laminate known as hygrothermal stresses have been discussed in the last lecture.
In today's lecture two problems will be solved with an objective to understand the steps in analysis of laminate subjected to load.

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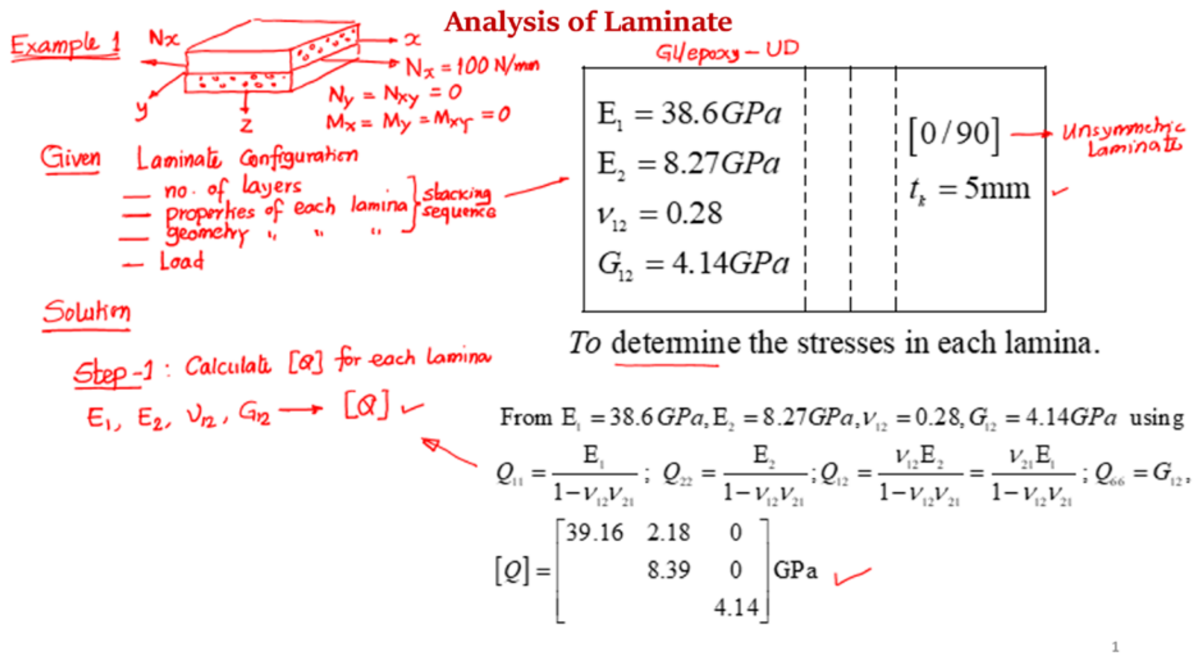

Example 1: In the first problem a Glass/epoxy [0/90] laminate as shown in the Fig. is subjected to only Nx ≠ 0 (Ny=Nxy=Mx=My=Mxy=0). Properties of the UD Glass/epoxy lamina are given as

$E_1 = 38.6GPa$ $E_2 = 8.27GPa$ $\nu_{12} = 0.28$ $G_{12} = 4.14GPa$	$[0/90]$ $t_k = 5\text{mm}$

Stresses in each lamina need to be determined.

Solution:

Here the laminate is a [0/90]. Glass/epoxy subjected to Nx = 100 N/mm and Nx = N_y= N_{xy}=0. Similarly M_x= M_y= M_{xy}=0. The laminate configuration is given that is the number of layers in the laminate, properties (E_1, E_2, ν_{12}, G_{12}) of each layer, geometry of each lamina meaning the thickness (5 mm) of the lamina as well as the positions with reference to the mid surface of the laminate that is the stacking sequence is known.

Note that this is not a symmetric laminate and is an unsymmetric laminate. In this problem, intentionally an unsymmetric laminate has been considered to keep the problem general in nature that is all the three matrices [A], [B] and [D] exist. In addition, only a two layer laminate is considered for the ease of analysis. However the procedure is same for a general n-layer laminate.

Step 1: Reduced stiffness matrix $[Q]$ for UD Glass/epoxy lamina has been calculated as

From $\mathrm{E}_1 = 38.6GPa, \mathrm{E}_2 = 8.27GPa, \nu_{12} = 0.28, G_{12} = 4.14GPa$ using

$$Q_{11} = \frac{\mathrm{E}_1}{1-\nu_{12}\nu_{21}};\; Q_{22} = \frac{\mathrm{E}_2}{1-\nu_{12}\nu_{21}}; Q_{12} = \frac{\nu_{12}\mathrm{E}_2}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}\mathrm{E}_1}{1-\nu_{12}\nu_{21}}; Q_{66} = G_{12},$$

$$[Q] = \begin{bmatrix} 39.16 & 2.18 & 0 \\ & 8.39 & 0 \\ & & 4.14 \end{bmatrix} \mathrm{GPa}$$

Step 2: From $[Q]$ and knowing fiber orientation angle θ for each lamina, elements of reduced transformed stiffness matrix $[\bar{Q}]$ for each lamina are calculated using the following which is discussed in details in macromechanics of lamina.

$$\begin{Bmatrix} \bar{Q}_{11} = Q_{11}c^4 + 2(Q_{12} + 2Q_{66})s^2c^2 + Q_{22}s^4 \\ \bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})c^2s^2 + Q_{12}(s^4 + c^4) \\ \bar{Q}_{22} = Q_{11}s^4 + 2(Q_{12} + 2Q_{66})c^2s^2 + Q_{22}c^4 \\ \bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})s\,c^3 - (Q_{22} - Q_{12} - 2Q_{66})s^3c \\ \bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})s^3c - (Q_{22} - Q_{12} - 2Q_{66})s\,c^3 \\ \bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})c^2s^2 + Q_{66}(s^4 + c^4) \end{Bmatrix}$$

And the reduced transformed stiffness matrices for each layer are

$$[\bar{Q}]_{0^\circ} = [\bar{Q}] = \begin{bmatrix} 39.16 & 2.18 & 0 \\ & 8.39 & 0 \\ & & 4.14 \end{bmatrix} \mathrm{GPa}\;;$$

$$[\bar{Q}]_{90^\circ} = \begin{bmatrix} 8.39 & 2.18 & 0 \\ & 39.16 & 0 \\ & & 4.14 \end{bmatrix} \mathrm{GPa}$$

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Even though in the present problem, we have only two layers viz. 0° and 90°, even for an n layer laminate the reduce stiffness matrix $[\underline{Q}]$ could be calculated using the same procedure.

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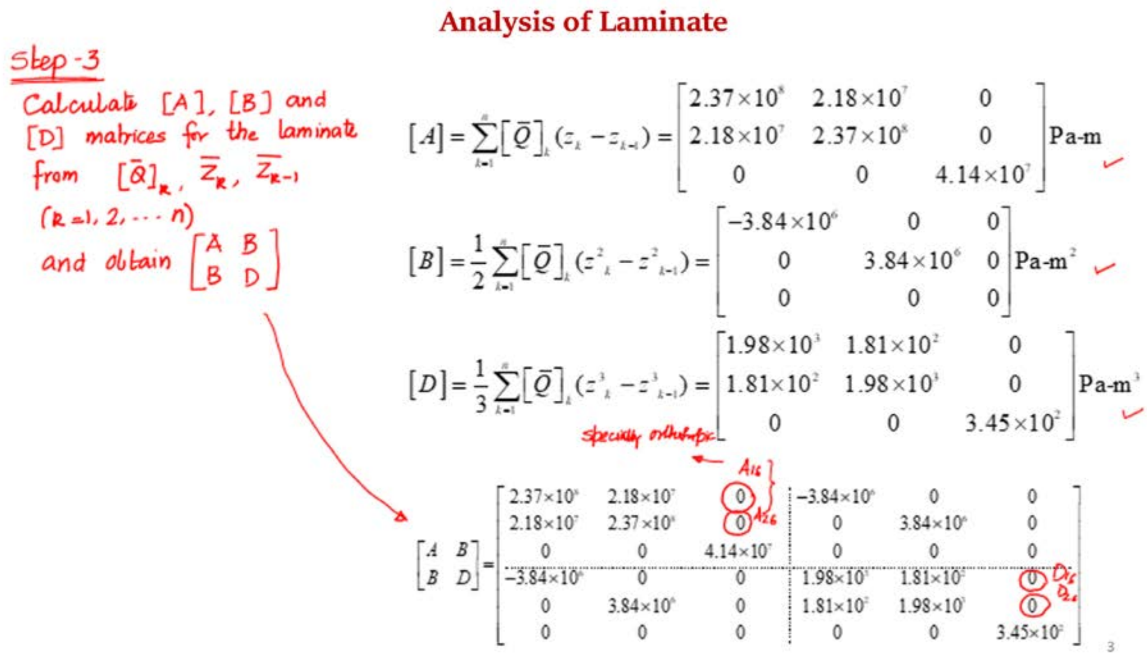

Step 3: Having calculated reduced transform stiffness matrix for each layer, in the next step [A], [B] and [D] matrices are obtained from the $[\underline{Q}]$ for each layer and their stacking sequence information as follows.

$$[A]=\sum_{k=1}^{n}\left[\bar{Q}\right]_k (z_k - z_{k-1}) = \begin{bmatrix} 2.37\times10^8 & 2.18\times10^7 & 0 \\ 2.18\times10^7 & 2.37\times10^8 & 0 \\ 0 & 0 & 4.14\times10^7 \end{bmatrix} \text{Pa-m}$$

$$[B]=\frac{1}{2}\sum_{k=1}^{n}\left[\bar{Q}\right]_k ({z^2}_k - {z^2}_{k-1}) = \begin{bmatrix} -3.84\times10^6 & 0 & 0 \\ 0 & 3.84\times10^6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{Pa-m}^2$$

$$[D]=\frac{1}{3}\sum_{k=1}^{n}\left[\bar{Q}\right]_k ({z^3}_k - {z^3}_{k-1}) = \begin{bmatrix} 1.98\times10^3 & 1.81\times10^2 & 0 \\ 1.81\times10^2 & 1.98\times10^3 & 0 \\ 0 & 0 & 3.45\times10^2 \end{bmatrix} \text{Pa-m}^3$$

With reference to the Fig., note that for the 0º layer, $z_{k-1} = -2.5$ mm and $z_k = 0$ and for the 90º layer, $z_{k-1} = 0$ and $z_k = -2.5$ mm. Therefore, the $\begin{bmatrix} A & B \\ B & D \end{bmatrix}$ matrix looks like (note that [A], [B] and [D] have different units)

$$
\begin{bmatrix} A & B \\ B & D \end{bmatrix} = \left[\begin{array}{ccc|ccc}
2.37\times10^{8} & 2.18\times10^{7} & 0 & -3.84\times10^{6} & 0 & 0 \\
2.18\times10^{7} & 2.37\times10^{8} & 0 & 0 & 3.84\times10^{6} & 0 \\
0 & 0 & 4.14\times10^{7} & 0 & 0 & 0 \\
\hline
-3.84\times10^{6} & 0 & 0 & 1.98\times10^{3} & 1.81\times10^{2} & 0 \\
0 & 3.84\times10^{6} & 0 & 1.81\times10^{2} & 1.98\times10^{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 3.45\times10^{2}
\end{array}\right]
$$

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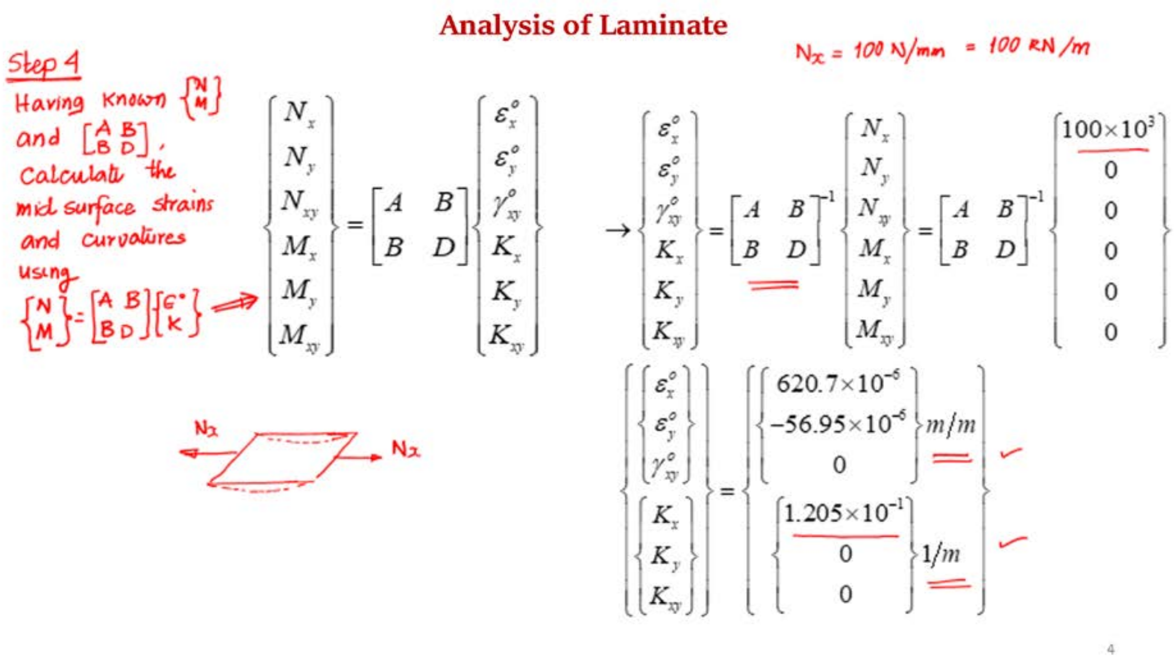

<u>**Step 4:**</u> Knowing the $\begin{bmatrix} A & B \\ B & D \end{bmatrix}$ matrix and using $\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix}$ the applied load vector the mid surface strains and curvatures are obtained as follows.

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1} \begin{Bmatrix} 100\times10^3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} \\ \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} \end{Bmatrix} = \begin{Bmatrix} \begin{Bmatrix} 620.7\times10^{-6} \\ -56.95\times10^{-6} \\ 0 \end{Bmatrix} m/m \\ \begin{Bmatrix} 1.205\times10^{-1} \\ 0 \\ 0 \end{Bmatrix} 1/m \end{Bmatrix}$$

Now in this case we have taken N_x = 100 N/mm = 100 kN/m because all other units are in Pascal (Pa).

Now the laminate considered is actually an unsymmetrical unsymmetric laminate. Therefore even though only Nx (uniaxial tensile loading along x) is applied, but besides producing in plane strains it also resulted to curvatures.

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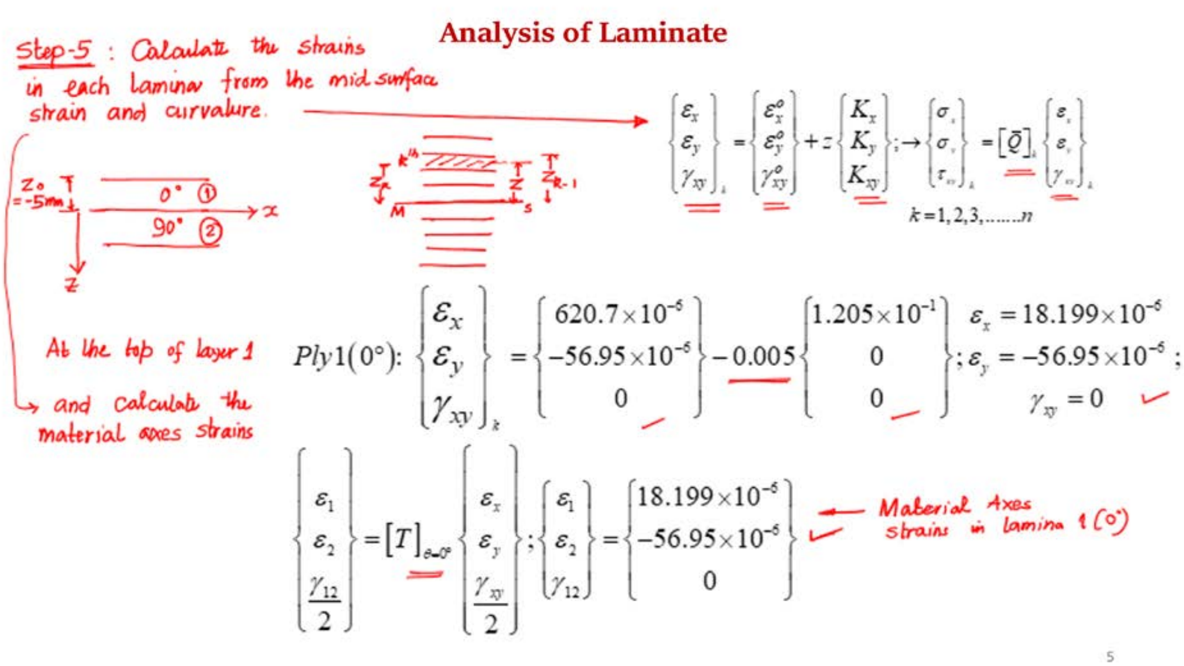


Step 5: Knowing the mid surface strains and curvatures, the strains in each (kth) layer (in x-y) could be obtained using the mid surface strains and curvature as

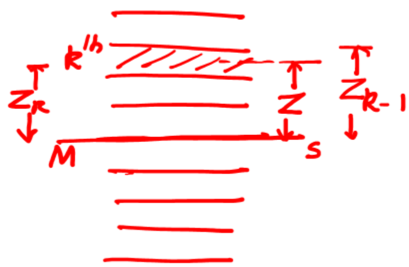
$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k = \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + z \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}; \rightarrow \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = [\bar{Q}]_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k$$

$$k = 1, 2, 3, \ldots\ldots n$$

Where, z is the distance of the middle surface of the kth layer from the laminate mid surface (ref Fig.).

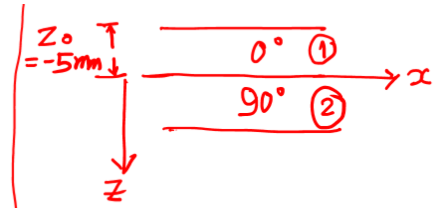
$$Ply1(0°):\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}_k=\begin{Bmatrix}620.7\times10^{-6}\\ -56.95\times10^{-6}\\ 0\end{Bmatrix}-0.005\begin{Bmatrix}1.205\times10^{-1}\\ 0\\ 0\end{Bmatrix};\begin{matrix}\varepsilon_x=18.199\times10^{-6}\\ \varepsilon_y=-56.95\times10^{-6}\\ \gamma_{xy}=0\end{matrix};$$

$$Ply\,2(90°):\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}_k=\begin{Bmatrix}620.7\times10^{-6}\\ -56.95\times10^{-6}\\ 0\end{Bmatrix}+0.005\begin{Bmatrix}1.205\times10^{-1}\\ 0\\ 0\end{Bmatrix};\begin{matrix}\varepsilon_x=1223.2\times10^{-6}\\ \varepsilon_y=-56.95\times10^{-6}\\ \gamma_{xy}=0\end{matrix}$$



So, knowing the strains is in the global axis (x-y) of a lamina, the stains in the local axes (1-2) for that lamina could be obtained using the strain transformation as follows.

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \frac{\gamma_{12}}{2}\end{Bmatrix}_0=[T]_{\theta=0°}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \frac{\gamma_{xy}}{2}\end{Bmatrix}_0;\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}_0=\begin{Bmatrix}18.199\times10^{-6}\\ -56.95\times10^{-6}\\ 0\end{Bmatrix}\text{ and }\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \frac{\gamma_{12}}{2}\end{Bmatrix}=[T]_{\theta=90°}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \frac{\gamma_{xy}}{2}\end{Bmatrix}_{90};\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}_{90}=\begin{Bmatrix}-56.95\times10^{-6}\\ 1223.2\times10^{-6}\\ 0\end{Bmatrix}$$



Even though the strains are calculated at the middle of the layer, strains could be calculated at the top of bottom of the layer also using the appropriate values of z.

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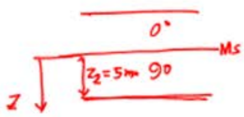
Analysis of Laminate

Similarly, in layer 2



At the bottom surface of layer 2

Strains in the material axes

$$Ply\,2(90°):\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}_k=\begin{Bmatrix}620.7\times10^{-6}\\ -56.95\times10^{-6}\\ 0\end{Bmatrix}+\underline{0.005}\begin{Bmatrix}1.205\times10^{-1}\\ 0\\ 0\end{Bmatrix};\ \begin{matrix}\varepsilon_x=1223.2\times10^{-6}\\ \varepsilon_y=-56.95\times10^{-6}\\ \gamma_{xy}=0\end{matrix}$$

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \dfrac{\gamma_{12}}{2}\end{Bmatrix}=[T]_{\theta=90°}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \dfrac{\gamma_{xy}}{2}\end{Bmatrix};\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}=\begin{Bmatrix}-56.95\times10^{-6}\\ 1223.2\times10^{-6}\\ 0\end{Bmatrix}$$

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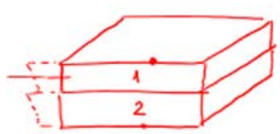
Analysis of Laminate

Step-6

Calculate the stresses wrt the material axes in each lamina

layer 1 (at the top surface)

$$Ply1(0°):\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix}_{0°}=[Q]\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}_{0°}=$$



layer 2 (at the bottom)

$$Ply2(90°):\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix}_{90°}=[Q]\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}_{90°}=$$

Step 6: Multiplying the material axis strains of each layer by the reduced stiffness matrix for the layer, the material axes (1-2) stresses for each layer could be calculated as

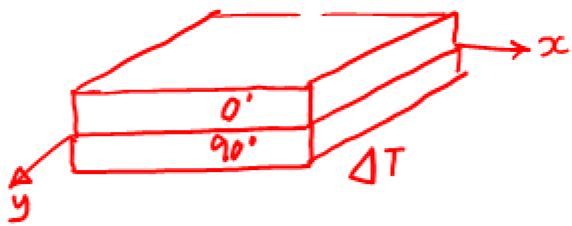
$$Ply1(0°):\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix}_{0°}=[Q]\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}_{0°}=$$

$$Ply2(90°):\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix}_{90°}=[Q]\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}_{90°}=$$

Note for a general laminate having n-layers that this kind of problems are actually not done manually because it will be tedious. However, to illustrate the steps, it was done manually considering a two layer laminate only.

Now, knowing the material axis stresses and strains in each lamina, appropriate failure theory could be applied to assess the failure or safety of that particular lamina and that will be discussed in details in failure of laminates. But here the steps that are involved in determination of the stresses in each lamina of a laminate has been discussed.

Example 2



Example 2: In the second example, the same [0/90] glass epoxy laminate is considered but it is only subjected to ΔT and the residual stresses in each lamina need to be determined.

$\Delta T = -75°\text{C}$ $\left.\begin{array}{l}\alpha_1 = 8.6\times10^{-6}\\ \alpha_2 = 22.1\times10^{-6}\end{array}\right\}\text{m/m/°C}$	$[0/90]\,Glass/Epoxy$ $t_k = 5\text{mm}$

Solution:

Here the laminate is a [0/90]. Glass/epoxy subjected to $\Delta T = -75°\text{C}$ and Nx = Nx = N_y= N_{xy}=0. Similarly M_x= M_y= M_{xy}=0. The laminate configuration is given that is the number of layers in the laminate, properties (E_1, E_2, ν_{12}, G_{12}) of each layer, geometry of each lamina meaning the thickness (5 mm) of the lamina as well as the positions with reference to the mid surface of the laminate that is the stacking sequence is known.

Note that this is not a symmetric laminate and is an unsymmetric laminate. In this problem, intentionally an unsymmetric laminate has been considered to keep the problem general in nature that is all the three matrices A, B and D exist. In addition, only a two layer laminate is considered for the ease of analysis. However the procedure is same for a general n-layer laminate.

Step 1: Reduced stiffness matrix $[Q]$ for UD Glass/epoxy lamina has been calculated as

From $E_1 = 38.6GPa, E_2 = 8.27GPa, \nu_{12} = 0.28, G_{12} = 4.14GPa$ using

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}};\ Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}; Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}; Q_{66} = G_{12},$$

$$[Q] = \begin{bmatrix} 39.16 & 2.18 & 0 \\ & 8.39 & 0 \\ & & 4.14 \end{bmatrix} \text{GPa}$$

Step 2: From $[Q]$ and knowing fiber orientation angle θ for each lamina, elements of reduced transformed stiffness matrix $[\bar{Q}]$ for each lamina are calculated using the following which is discussed in details in macromechanics of lamina.

$$\begin{Bmatrix} \bar{Q}_{11} = Q_{11}c^4 + 2(Q_{12} + 2Q_{66})s^2c^2 + Q_{22}s^4 \\ \bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})c^2s^2 + Q_{12}(s^4 + c^4) \\ \bar{Q}_{22} = Q_{11}s^4 + 2(Q_{12} + 2Q_{66})c^2s^2 + Q_{22}c^4 \\ \bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})s\,c^3 - (Q_{22} - Q_{12} - 2Q_{66})s^3c \\ \bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})s^3c - (Q_{22} - Q_{12} - 2Q_{66})s\,c^3 \\ \bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})c^2s^2 + Q_{66}(s^4 + c^4) \end{Bmatrix}$$

And the reduced transformed stiffness matrices for each layer are

$$[\bar{Q}]_{0^\circ} = [\bar{Q}] = \begin{bmatrix} 39.16 & 2.18 & 0 \\ & 8.39 & 0 \\ & & 4.14 \end{bmatrix} \text{GPa};$$

$$[\bar{Q}]_{90^\circ} = \begin{bmatrix} 8.39 & 2.18 & 0 \\ & 39.16 & 0 \\ & & 4.14 \end{bmatrix} \text{GPa}$$

Even though in the present problem, we have only two layers viz. 0º and 90º, even for an n layer laminate the reduce stiffness matrix $[\underline{Q}]$ could be calculated using the same procedure.

Step 3: Having calculated reduced transform stiffness matrix for each layer, in the next step [A], [B] and [D] matrices are obtained from the $[\underline{Q}]$ for each layer and their stacking sequence information as follows.

$$[A]=\sum_{k=1}^{n}\left[\bar{Q}\right]_k (z_k - z_{k-1}) = \begin{bmatrix} 2.37\times10^8 & 2.18\times10^7 & 0 \\ 2.18\times10^7 & 2.37\times10^8 & 0 \\ 0 & 0 & 4.14\times10^7 \end{bmatrix} \text{Pa-m}$$

$$[B]=\frac{1}{2}\sum_{k=1}^{n}\left[\bar{Q}\right]_k (z^2{}_k - z^2{}_{k-1}) = \begin{bmatrix} -3.84\times10^6 & 0 & 0 \\ 0 & 3.84\times10^6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{Pa-m}^2$$

$$[D]=\frac{1}{3}\sum_{k=1}^{n}\left[\bar{Q}\right]_k (z^3{}_k - z^3{}_{k-1}) = \begin{bmatrix} 1.98\times10^3 & 1.81\times10^2 & 0 \\ 1.81\times10^2 & 1.98\times10^3 & 0 \\ 0 & 0 & 3.45\times10^2 \end{bmatrix} \text{Pa-m}^3$$

With reference to the Fig., note that for the 0º layer, $z_{k-1} = -2.5$ mm and $z_k = 0$ and for the 90º layer, $z_{k-1} = 0$ and $z_k = -2.5$ mm. Therefore, the $\begin{bmatrix} A & B \\ B & D \end{bmatrix}$ matrix looks like (note that A, B and D have different units)

$$\begin{bmatrix} A & B \\ B & D \end{bmatrix} = \left[\begin{array}{ccc:ccc} 2.37\times10^8 & 2.18\times10^7 & 0 & -3.84\times10^6 & 0 & 0 \\ 2.18\times10^7 & 2.37\times10^8 & 0 & 0 & 3.84\times10^6 & 0 \\ 0 & 0 & 4.14\times10^7 & 0 & 0 & 0 \\ \hdashline -3.84\times10^6 & 0 & 0 & 1.98\times10^3 & 1.81\times10^2 & 0 \\ 0 & 3.84\times10^6 & 0 & 1.81\times10^2 & 1.98\times10^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.45\times10^2 \end{array}\right]$$

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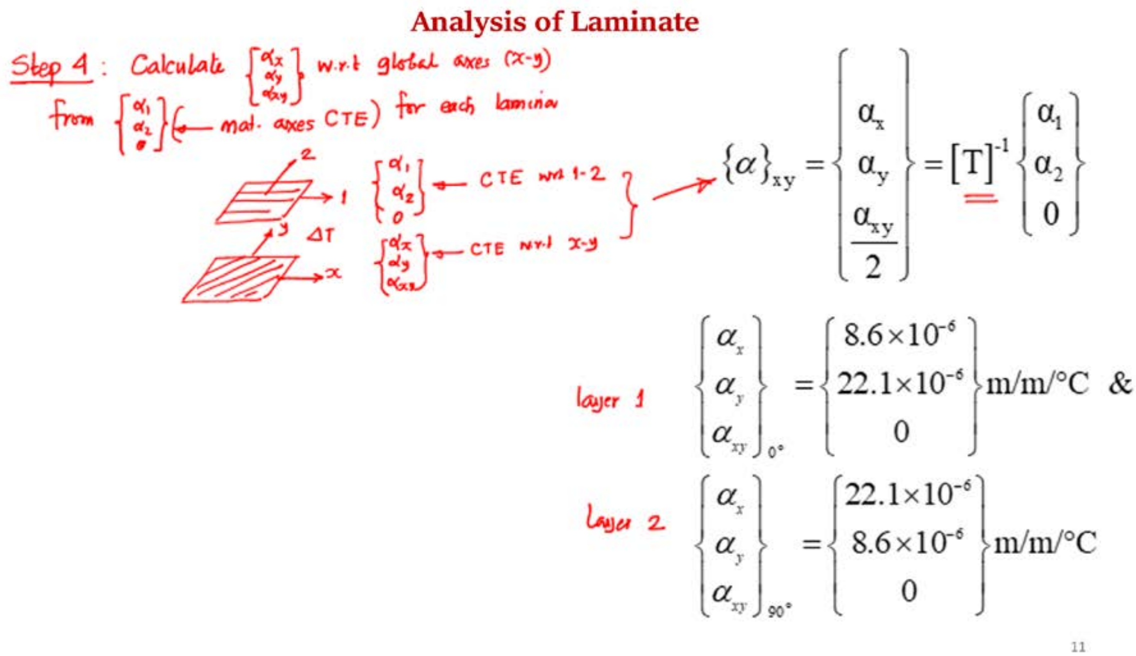

Step 4: Given, the CTEs of the lamina with reference to the materials axes as $\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 8.6\times10^{-6} \\ 22.1\times10^{-6} \\ 0 \end{Bmatrix}$, the CTEs in the global axes (x-y) could be evaluated as

$$\{\alpha\}_{XY} = \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \dfrac{\alpha_{xy}}{2} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{Bmatrix}$$

And then for each layer

$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{0^\circ} = \begin{Bmatrix} 8.6\times10^{-6} \\ 22.1\times10^{-6} \\ 0 \end{Bmatrix} \text{m/m/°C} \ \ \& \ \ \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{90^\circ} = \begin{Bmatrix} 22.1\times10^{-6} \\ 8.6\times10^{-6} \\ 0 \end{Bmatrix} \text{m/m/°C}$$

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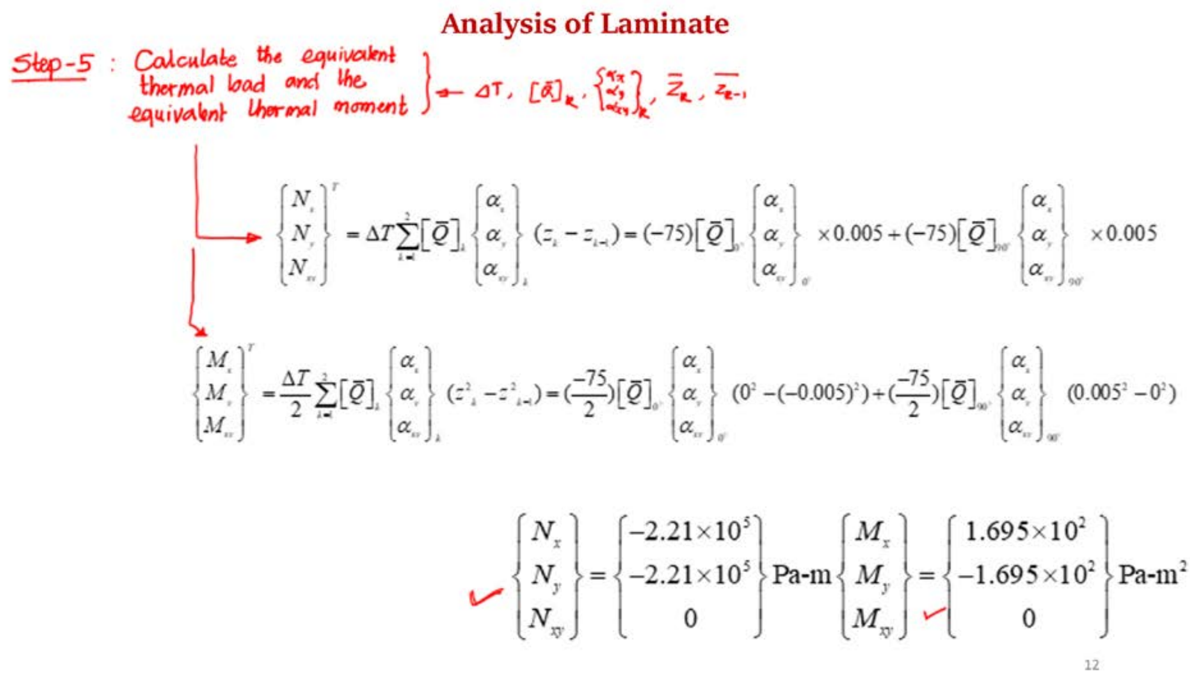

Step 5: Knowing for each layer (k), , equivalent thermal loads have been evaluated as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}^T = \Delta T \sum_{k=1}^{2} \left[\bar{Q}\right]_k \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_k (z_k - z_{k-1}) = (-75)\left[\bar{Q}\right]_{0^\circ} \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{0^\circ} \times 0.005 + (-75)\left[\bar{Q}\right]_{90^\circ} \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{90^\circ} \times 0.005$$

$$\Rightarrow \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{Bmatrix} -2.21\times 10^5 \\ -2.21\times 10^5 \\ 0 \end{Bmatrix} \text{Pa-m}$$

And the equivalent thermal moments are evaluated as

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}^T = \frac{\Delta T}{2} \sum_{k=1}^{2} \left[\bar{Q}\right]_k \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_k (z^2{}_k - z^2{}_{k-1}) = (\frac{-75}{2})\left[\bar{Q}\right]_{0^\circ} \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{0^\circ} (0^2 - (-0.005)^2) + (\frac{-75}{2})\left[\bar{Q}\right]_{90^\circ} \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{90^\circ} (0.005^2 - 0^2)$$

$$\Rightarrow \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{Bmatrix} 1.695\times 10^2 \\ -1.695\times 10^2 \\ 0 \end{Bmatrix} \text{Pa-m}^2$$

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Analysis of Laminate

Step- 6 :
Calculate the mid surface strains and curvatures
$\left[\text{from } \begin{Bmatrix} N \\ M \end{Bmatrix}^T = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \epsilon^{\circ} \\ K \end{Bmatrix}\right]$ ⟶
due to ΔT

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix}^T = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix} \rightarrow \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix}^T = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1} \begin{Bmatrix} -2.21\times10^5 \\ -2.21\times10^5 \\ 0 \\ 1.695\times10^2 \\ -1.695\times10^2 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} \\ \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} \end{Bmatrix} = \begin{Bmatrix} \begin{Bmatrix} -1.042\times10^{-3} \\ -1.042\times10^{-3} \\ 0 \end{Bmatrix} m/m \\ \begin{Bmatrix} -1.28\times10^{-1} \\ 1.28\times10^{-1} \\ 0 \end{Bmatrix} 1/m \end{Bmatrix}$$

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Step 6: Using the relation

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix}^T = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix} \rightarrow \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix}^T$$

the mid surface strains and curvatures due to ΔT are calculated as

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} 2.37\times10^{8} & 2.18\times10^{7} & 0 & -3.84\times10^{6} & 0 & 0 \\ 2.18\times10^{7} & 2.37\times10^{8} & 0 & 0 & 3.84\times10^{6} & 0 \\ 0 & 0 & 4.14\times10^{7} & 0 & 0 & 0 \\ \hline -3.84\times10^{6} & 0 & 0 & 1.98\times10^{3} & 1.81\times10^{2} & 0 \\ 0 & 3.84\times10^{6} & 0 & 1.81\times10^{2} & 1.98\times10^{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.45\times10^{2} \end{array}\right]^{-1} \begin{Bmatrix} -2.21\times10^{-4} \\ -2.21\times10^{-4} \\ 0 \\ 1.695\times10^{2} \\ -1.695\times10^{2} \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} \\ \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} \end{Bmatrix} = \begin{Bmatrix} \begin{Bmatrix} -1.042\times10^{-3} \\ -1.042\times10^{-3} \\ 0 \end{Bmatrix} m/m \\ \begin{Bmatrix} -1.28\times10^{-1} \\ 1.28\times10^{-1} \\ 0 \end{Bmatrix} 1/m \end{Bmatrix}$$

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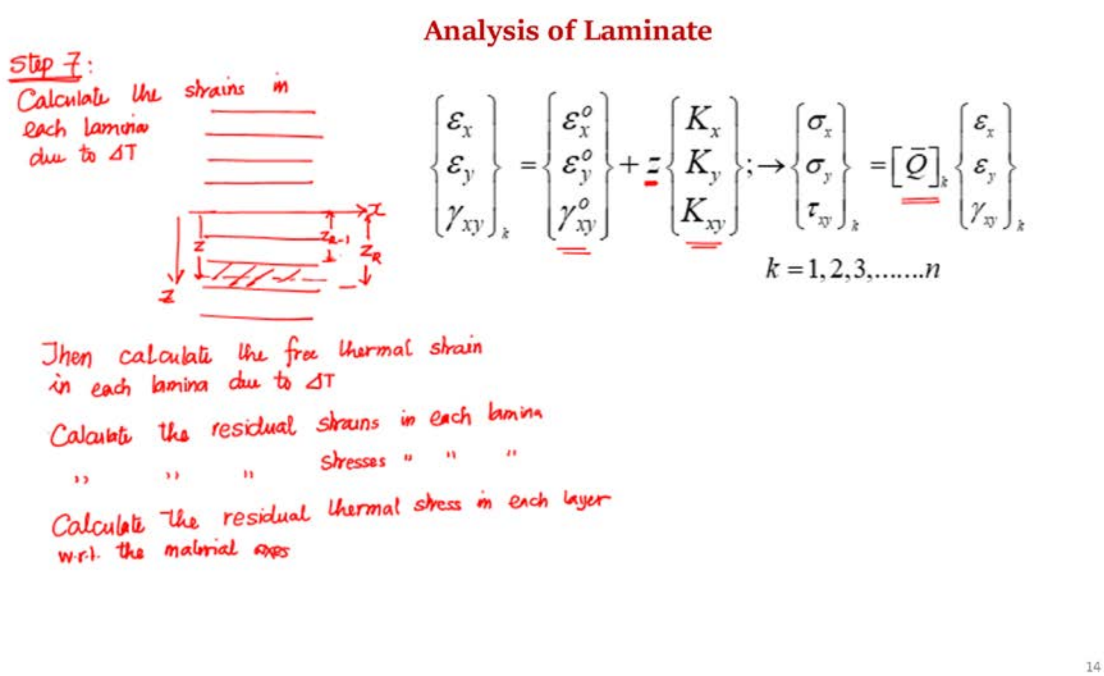

Step 7: From the mid surface strains and curvatures, the strains and stresses in global (x-y) axes in all the layers are calculated using

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k = \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + z \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \left[\bar{Q}\right]_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k \qquad (k = 1, 2, 3, ...n)$$

Now,

At Top of 0° (z=-0.005m)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_0 = \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + (-0.005) \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} = \begin{Bmatrix} -3.99\times10^{-4} \\ -1.68\times10^{-3} \\ 0 \end{Bmatrix}$$

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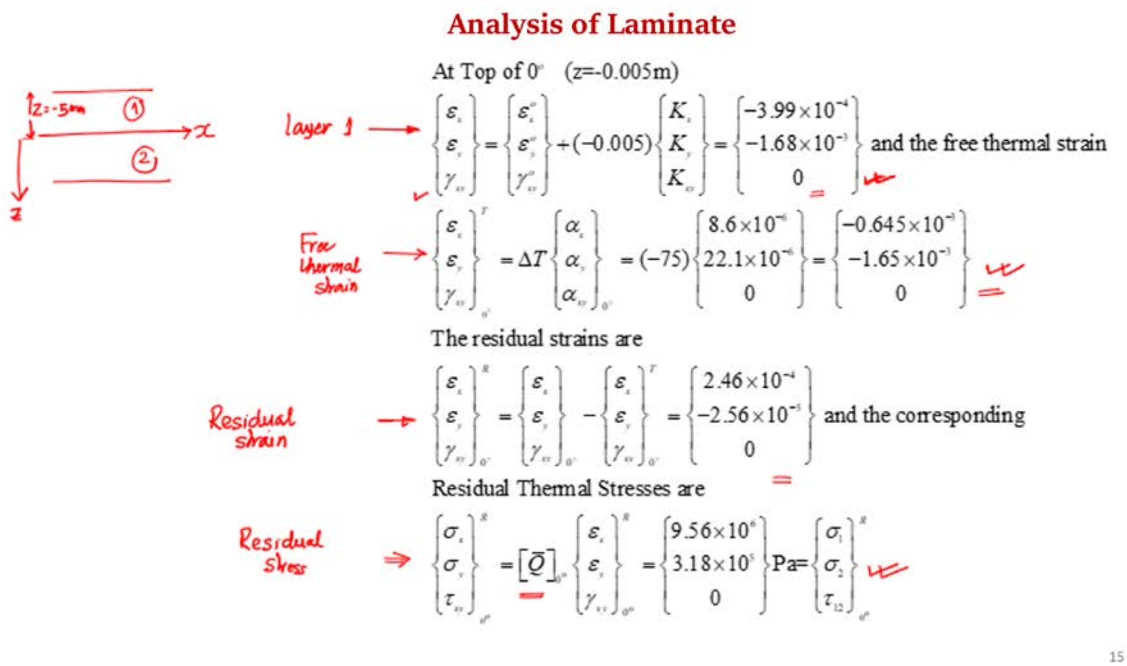

The free thermal strain in layer 1 (0° layer) is

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_{0^\circ}^T = \Delta T \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{0^\circ} = (-75) \begin{Bmatrix} 8.6\times10^{-6} \\ 22.1\times10^{-6} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.645\times10^{-3} \\ -1.65\times10^{-3} \\ 0 \end{Bmatrix}$$

Therefore, the residual strains in layer 1 is

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^R_{0^\circ} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_{0^\circ} - \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^T_{0^\circ} = \begin{Bmatrix} 2.46\times10^{-4} \\ -2.56\times10^{-5} \\ 0 \end{Bmatrix}$$

and the corresponding residual thermal stresses are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^R_{0^o} = \left[\bar{Q}\right]_{0^o} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^R_{0^o} = \begin{Bmatrix} 9.56\times10^{6} \\ 3.18\times10^{5} \\ 0 \end{Bmatrix} \text{Pa} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}^R_{0^o}$$

which also happened to be the material axes residual stresses for the 0^o layer.

Similarly for layer 2

Analysis of Laminate

Bottom of 90° $(z=+0.005\text{m})$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + (0.005)\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} = \begin{Bmatrix} -1.68\times10^{-3} \\ -3.99\times10^{-4} \\ 0 \end{Bmatrix}$$

and the free thermal strains

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^T_{90^\circ} = \Delta T \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{90^\circ} = (-75)\begin{Bmatrix} 22.1\times10^{-6} \\ 8.6\times10^{-6} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1.65\times10^{-3} \\ -0.645\times10^{-3} \\ 0 \end{Bmatrix}$$

The residual thermal strains are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^R_{90^\circ} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_{90^\circ} - \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^T_{90^\circ} = \begin{Bmatrix} -2.56\times10^{-5} \\ 2.46\times10^{-4} \\ 0 \end{Bmatrix}$$

and the corresponding

Residual Thermal Stresses
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^R_{90^o} = \left[\bar{Q}\right]_{90^o} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^R_{90^o} = \begin{Bmatrix} 3.18\times10^{5} \\ 9.56\times10^{6} \\ 0 \end{Bmatrix} \text{Pa}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}^R_{90^o} = [T]_{90^o} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^R_{90^o} = \begin{Bmatrix} 9.56\times10^{6} \\ 3.18\times10^{5} \\ 0 \end{Bmatrix} \text{Pa}$$

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Similarly, for the layer 2 (90^o layer), Bottom of 90° (z=+0.005m)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + (0.005) \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} = \begin{Bmatrix} -1.68\times10^{-3} \\ -3.99\times10^{-4} \\ 0 \end{Bmatrix}$$ and the free thermal strains

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_{90^\circ}^{T} = \Delta T \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{90^\circ} = (-75) \begin{Bmatrix} 22.1\times10^{-6} \\ 8.6\times10^{-6} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1.65\times10^{-3} \\ -0.645\times10^{-3} \\ 0 \end{Bmatrix}$$

The residual thermal strains are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_{90^\circ}^{R} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_{90^\circ} - \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_{90^\circ}^{T} = \begin{Bmatrix} -2.56\times10^{-5} \\ 2.46\times10^{-4} \\ 0 \end{Bmatrix}$$ and the corresponding

Residual Thermal Stresses $$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{90^o}^{R} = \left[\bar{Q}\right]_{90^o} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_{90^o}^{R} = \begin{Bmatrix} 3.18\times10^{5} \\ 9.56\times10^{6} \\ 0 \end{Bmatrix} \text{Pa}$$

The materials axes (1-2) residual stresses in layer 2 (90° layer) are obtained using the stress transformation as follows.

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}_{90^o}^{R} = \left[T\right]_{90^o} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{90^o}^{R} = \begin{Bmatrix} 9.56\times10^{6} \\ 3.18\times10^{5} \\ 0 \end{Bmatrix} \text{Pa}$$

Here, only ΔT has been considered. But, in the same way, the residual stresses in each layer due to ΔC, change in moisture concentration (hygroscopic) could also be evaluated.

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Analysis of Laminate

Following the similar steps we could calculate the residual hygroscopic stresses in each lamina by calculating the equivalent hygroscopic forces and equivalent hygroscopic moments $\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}^H$ & $\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}^H$

and using

$$\begin{Bmatrix} N \\ M \end{Bmatrix}^H = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \epsilon^\circ \\ k \end{Bmatrix}$$

After calculating the residual stresses in each lamina in a laminate is subjected to ΔT and due to ΔC individually, those stresses in the material axis could be superposed for a laminate which is simultaneously subjected to a mechanical load N_x, ΔT and ΔC.