Mechanics of Fiber Reinforced Polymer Composite Structures
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Module-08
Elastic Behaviour of Laminates-II
Lecture-22
Engineering Constants of Laminates

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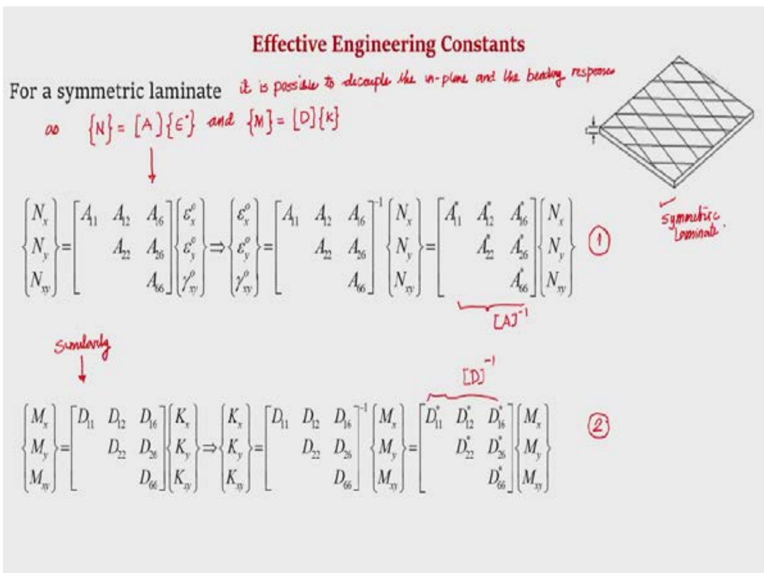

Hello and welcome. So, in our last class we have discussed some special cases of laminate stiffness where we understood the significance of making some of the elements of the ABBD matrix zero. For example by making [B]=0, we could eliminate the undesirable bending extension coupling. Similarly by making A_{16} =0 and A_{26} = 0, we could eliminate the shear extension coupling.

Therefore, we understood that by adjusting the geometrical as well as the mechanical properties of the constituent laminae in a laminate, some special classes of laminates have been defined where some of the desired stiffness characteristics could be achieved whereas some of the undesirable characteristics could be eliminated.

In today's lecture, we will discuss the effective engineering constants of a laminate similar to the engineering constants of lamina namely E_1, E_2, ν_{12} and G_{12} discussed in macro mechanics of lamina. Similarly for a laminate also we could evaluate the effective engineering constants even though the stiffness of a laminate is actually characterized by the ABBD matrix which we have discussed in details.

Each element of this ABBD matrix is actually function of the laminar properties constant laminar properties that means properties of each lamina namely E_1, E_2, ν_{12} and G_{12} and the fiber orientation theta for each lamina, thickness of each lamina and the stacking sequence that means order in which the laminae are laid up to make a laminate. But sometimes it becomes useful to know the average or effective engineering constants for a laminate to have a first-hand idea about the extension stiffness or the bending stiffness or maybe the shear stiffness or maybe to understand the Poisson's ratio of the laminate which gives a rough idea about the behaviour of the laminate as a whole. It is possible to determine the effective engineering constants of a laminate in terms of its ABBD matrix. But it is only possible for symmetric laminates where the bending and the extensional responses could be decoupled. For an unsymmetric laminate the bending response and the in-plane response cannot be decoupled, they are coupled. Therefore, it is not possible to independently determine the in-plane response or independently determine the out of plane response whereas for a symmetric laminate this could be decoupled because [B] = 0. So, for a symmetric laminate, we can write the force strain and the moment curvature relations as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ & A_{22} & A_{26} \\ & & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} \qquad (1)$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ & D_{22} & D_{26} \\ & & D_{66} \end{bmatrix} \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} \qquad (2)$$

where [A], [D] are the extensional stiffness and the bending stiffness of the laminate and $\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix}$ and $\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$ are the mid surface strains and curvatures respectively and $\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}$ and $\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}$ are the force and moment resultants respectively.

Taking inverse of [A] and [D] we could write the strains in terms of the force as

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ & A_{22} & A_{26} \\ & & A_{66} \end{bmatrix}^{-1} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ & A_{22}^* & A_{26}^* \\ & & A_{66}^* \end{bmatrix} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}$$

and the curvatures in terms of the moments as

$$\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ & D_{22} & D_{26} \\ & & D_{66} \end{bmatrix}^{-1} \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11}^* & D_{12}^* & D_{16}^* \\ & D_{22}^* & D_{26}^* \\ & & D_{66}^* \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}$$

Now for a symmetric laminate let us discuss how to determine the effective Young's modulus in extension as well as effective Young's modulus in in flexure.

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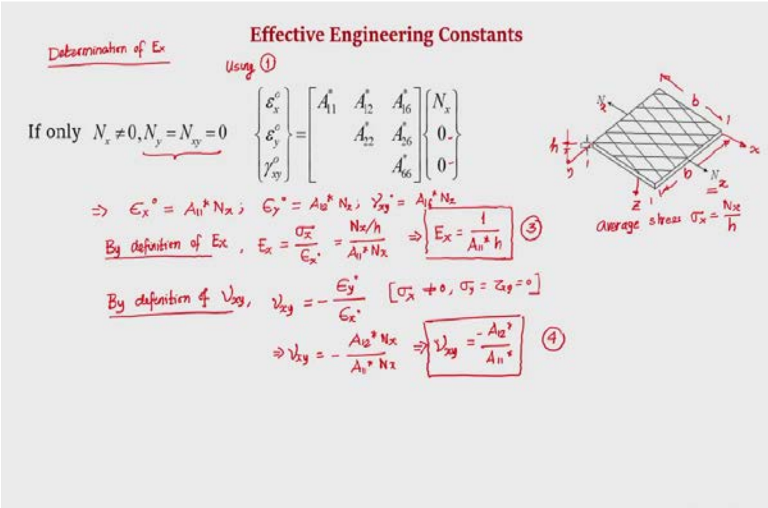

Considering a symmetric laminate subjected to only N_x (refer Fig.), the thickness of the laminate is h. So $N_x \neq 0$ $N_y = N_{xy} = 0$ and hence

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ & A_{22}^* & A_{26}^* \\ & & A_{66}^* \end{bmatrix} \begin{Bmatrix} N_x \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \varepsilon_x^o = A_{11}^* N_x ;\ \varepsilon_y^o = A_{12}^* N_x$$

Now by definition of Young's modulus by definition of Young's modulus along x,

$$E_x = \frac{\sigma_x}{\varepsilon_x^o} = \frac{N_x / h}{A_{11}^* N_x} \rightarrow \boxed{E_x = \frac{1}{h A_{11}^*}} \qquad (3)$$

This is the effective Young's modulus along x- under in plane loading that means the extensional Young's modulus or effective extensional Young's modulus along x.

Again, by definition of Poisson's ratio

$$\nu_{xy} = -\frac{\varepsilon_y^o}{\varepsilon_x^o} = -\frac{A_{12}^* N_x}{A_{11}^* N_x} \rightarrow \boxed{\nu_{xy} = -\frac{A_{12}^*}{A_{11}^*}} \qquad (4)$$

As each element of this A and B or D matrix is function of the laminate geometrical properties and the lamina mechanical properties. So, this is the expressions for effective Young's modulus along x- and the Poisson's ratio are also functions of the laminate geometrical properties and the lamina mechanical properties.

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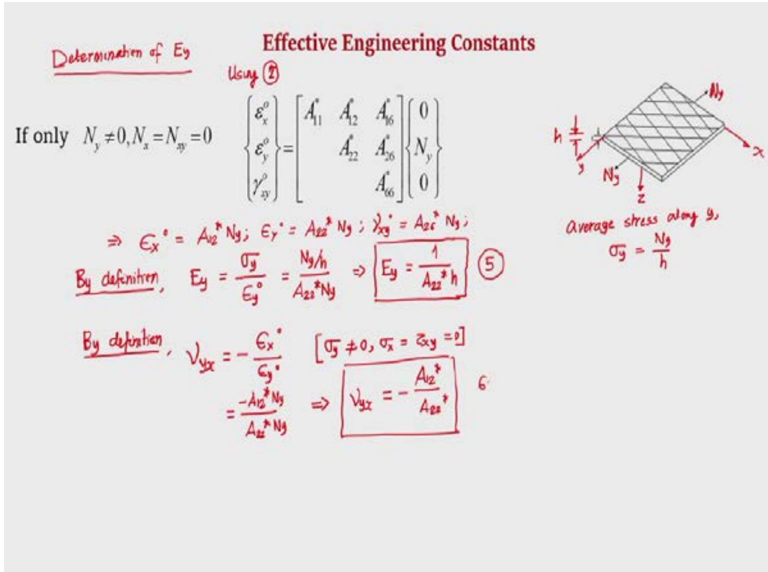

Similarly, we can also determine the effective Young's modulus E_y. Considering the laminate subjected to only $N_y \neq 0\ N_x = N_{xy} = 0$ and hence

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ & A_{22}^* & A_{26}^* \\ & & A_{66}^* \end{bmatrix} \begin{Bmatrix} 0 \\ N_y \\ 0 \end{Bmatrix}$$

$$\Rightarrow \varepsilon_y^o = A_{22}^* N_y;\ \varepsilon_x^o = A_{12}^* N_y$$

Therefore, again by definition of Young's modulus along y- direction,

$$E_y = \frac{\sigma_y}{\varepsilon_y^o} = \frac{N_y/h}{A_{22}^* N_y} \rightarrow \boxed{E_y = \frac{1}{hA_{22}^*}} \qquad (5)$$

This is the effective Young's modulus along y- under in plane loading that means the extensional Young's modulus or effective extensional Young's modulus along y.

By definition of Poisson’s ratio,

$$\nu_{yx} = -\frac{\varepsilon_x^o}{\varepsilon_y^o} = -\frac{A_{12}^* N_x}{A_{22}^* N_x} \rightarrow \boxed{\nu_{yx} = -\frac{A_{12}^*}{A_{22}^*}} \qquad (6)$$

So, we obtained the expressions for effective engineering constants for a laminate viz. E_x, E_y, ν_{xy} and ν_{yx}.

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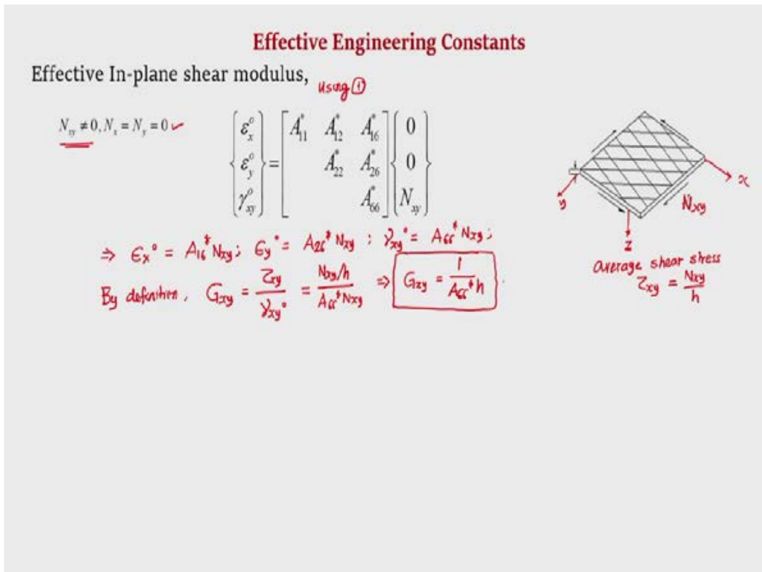

From (4) and (6), we get

$$\left.\begin{array}{l}\boxed{\nu_{xy} = -\dfrac{A_{12}^*}{A_{11}^*}} \\ \boxed{\nu_{yx} = -\dfrac{A_{12}^*}{A_{22}^*}}\end{array}\right\} \Rightarrow \boxed{\dfrac{\nu_{xy}}{\nu_{yx}} = \dfrac{A_{22}^*}{A_{11}^*} = \dfrac{E_x}{E_y}} \qquad (7)$$

Therefore, similar reciprocal relations between ν_{xy} and ν_{yx} what we obtained for lamina also exists for effective Poisson's ratio of laminate.

For determination of Gxy, considering the laminate subjected to only $N_{xy} \neq 0 \; N_x = N_y = 0$. So

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ & A_{22}^* & A_{26}^* \\ & & A_{66}^* \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ N_{xy} \end{Bmatrix}$$

$$\Rightarrow \varepsilon_x^o = A_{16}^* N_{xy};\; \varepsilon_y^o = A_{26}^* N_{xy}; \gamma_{xy}^o = A_{66}^* N_{xy}$$

Therefore, by definition of shear modulus G_{xy}

$$G_{xy} = \frac{\tau_{xy}}{\gamma_{xy}} = \frac{N_{xy}/h}{A_{66}^{*} N_y} \rightarrow \boxed{G_{xy} = \frac{1}{hA_{66}^{*}}} \quad (8)$$

We could thus obtain the expressions for effective engineering constants E_x E_y G_{xy} ν_{xy} for a laminate under in plane load.

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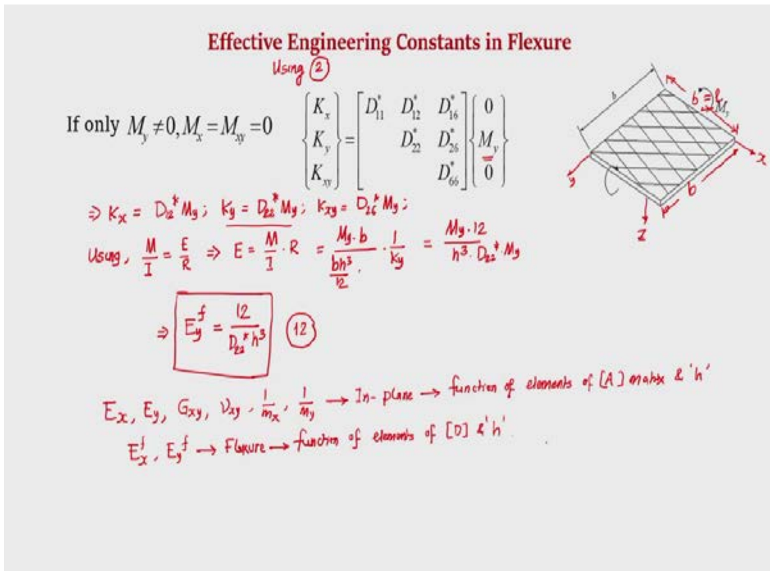
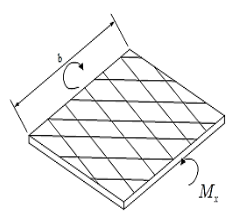


Similarly we could also obtain the Young's modulus under flexure. Considering a laminate subjected to only $M_x \neq 0$ $M_y = M_{xy} = 0$. So

$$\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11}^{*} & D_{12}^{*} & D_{16}^{*} \\ & D_{22}^{*} & D_{26}^{*} \\ & & D_{66}^{*} \end{bmatrix} \begin{Bmatrix} M_x \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow K_x = D_{11}^{*} M_x;$$



Now we know for pure bending $\frac{M}{I} = \frac{E}{R} \Rightarrow E = \frac{M}{I} R$ and by definition of $M = M_x \cdot b$, $I = \frac{bh^3}{12}$

and $K_x = \frac{1}{R}$ we can write

$$E_x^f = \frac{M_x \cdot b}{K_x \frac{bh^3}{12}} = \frac{12 M_x}{K_x h^3} = \frac{12 M_x}{D_{11}^{*} M_x h^3} \Rightarrow \boxed{E_x^f = \frac{12}{h^3 D_{11}^{*}}} \quad (9)$$

where suffix '*f*' stands for flexural.

Similarly we could also find the effective Young's modulus in flexure in the y- direction. Considering a laminate subjected to only $M_y \neq 0 \; M_x = M_{xy} = 0$. So

$$\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11}^* & D_{12}^* & D_{16}^* \\ & D_{22}^* & D_{26}^* \\ & & D_{66}^* \end{bmatrix} \begin{Bmatrix} 0 \\ M_y \\ 0 \end{Bmatrix} \rightarrow K_y = D_{22}^* M_y$$

Now we know for pure bending $\frac{M}{I} = \frac{E}{R} \Rightarrow E = \frac{M}{I} R$ and by definition of $M = M_y \cdot b$, $I = \frac{bh^3}{12}$ and $K_y = \frac{1}{R}$ we can write

$$E_y^f = \frac{M_y \cdot b}{K_y \frac{bh^3}{12}} = \frac{12 M_y}{K_y h^3} = \frac{12 M_y}{D_{22}^* M_y h^3} \Rightarrow \boxed{E_y^f = \frac{12}{h^3 D_{22}^*}} \quad (10)$$

So, we could express the effective engineering constants E_{x}, E_{y}, G_{xy}, ν_{xy} of a laminate in terms of the elements of the ABBD matrix and the laminate thickness. These effective engineering constants provide a firsthand idea about the extensional stiffness of the laminate, the shear stiffness of the laminate and the bending stiffness of the laminate. For a balanced symmetric laminate the shear extension coupling will not be there but for a general symmetric laminate there may be shear extension coupling and we could also obtain the effective shear extension coupling for a symmetric laminate.