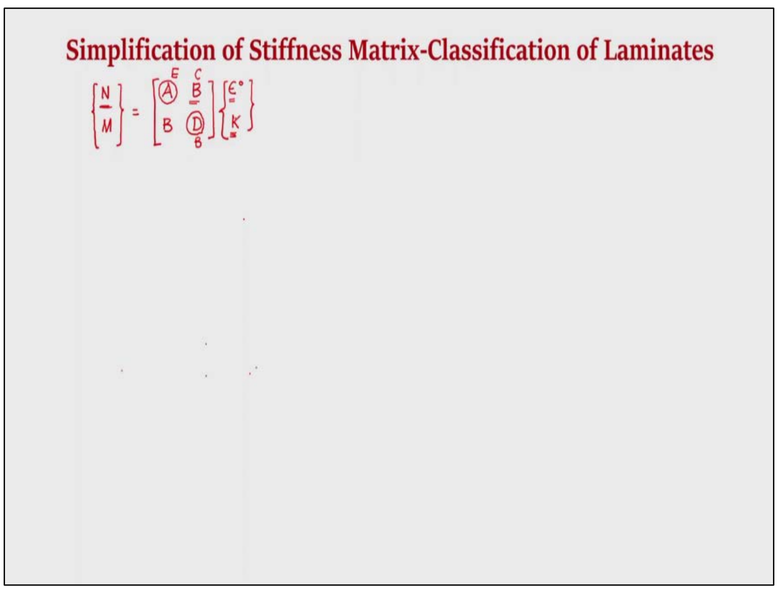
Mechanics of Fiber Reinforced Polymer Composite Structures
Prof. Debabrata Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology-Guwahati

Module-08
Elastic Behaviour of Laminates-II
Lecture-21
Special Classes of Laminates

In the last lecture, the constitutive relation for a laminate was developed starting with classical lamination theory. This constitutive equation for laminate defines the characteristics of laminate as in

(Refer Slide Time: 00:53)



$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \epsilon^{\circ} \\ K \end{Bmatrix} \qquad (1)$$

where

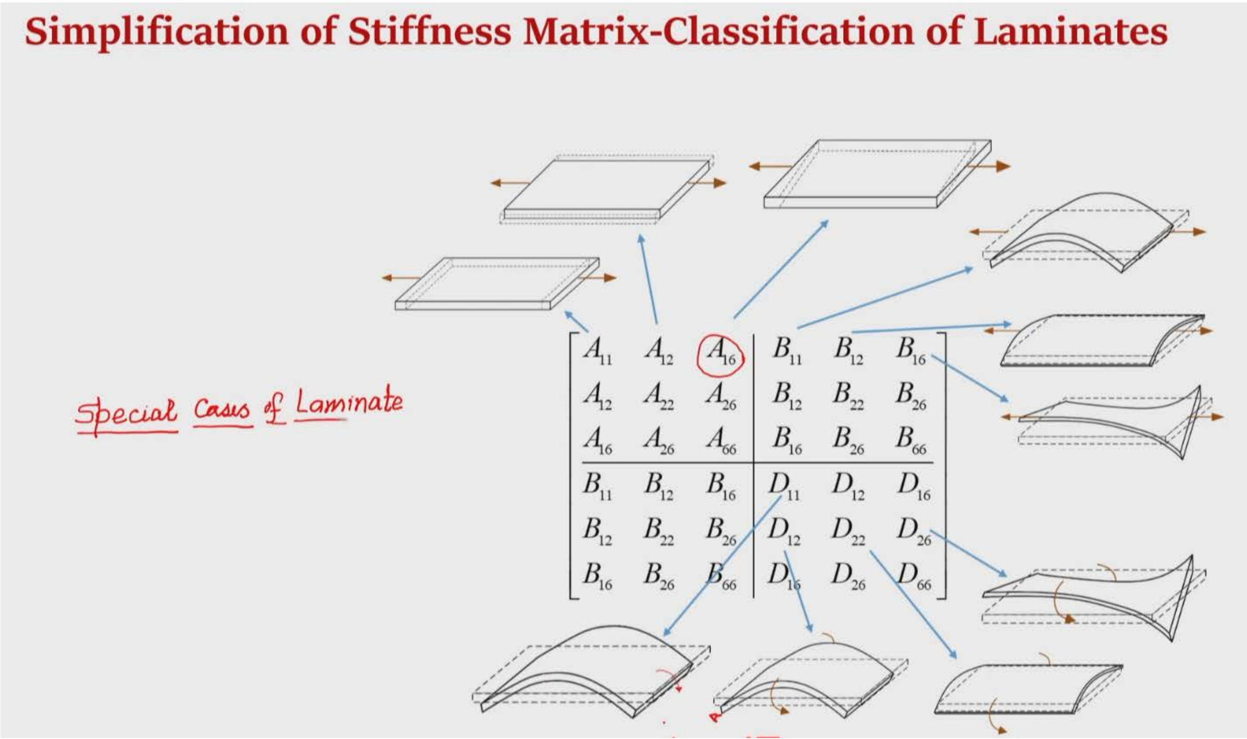
$$\{N\} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}; \{M\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}$$

and

$$[A]=\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}_{3\times3};[B]=\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}_{3\times3};[D]=\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}_{3\times3}$$

This relation (1) for a laminate which defines the characteristics of a laminate actually relates the in plane forces and moments resultants to the mid surface strains and curvature by means of the so called ABBD matrix. [A] is the extensional stiffness which relates the in plane forces to the corresponding mid surface strains and [D] is the bending stiffness which relates the moment resultants to the corresponding curvatures. [B] is the coupling matrix which actually couples the in plane force resultants to the curvatures and the moment resultants to the in plane strains. Therefore, it couples the force resultants to the curvatures and the moment resultants to the in plane strengths. The role played by each element of the matrix was also discussed in details in the last lecture.

(Refer Slide Time: 02:57)



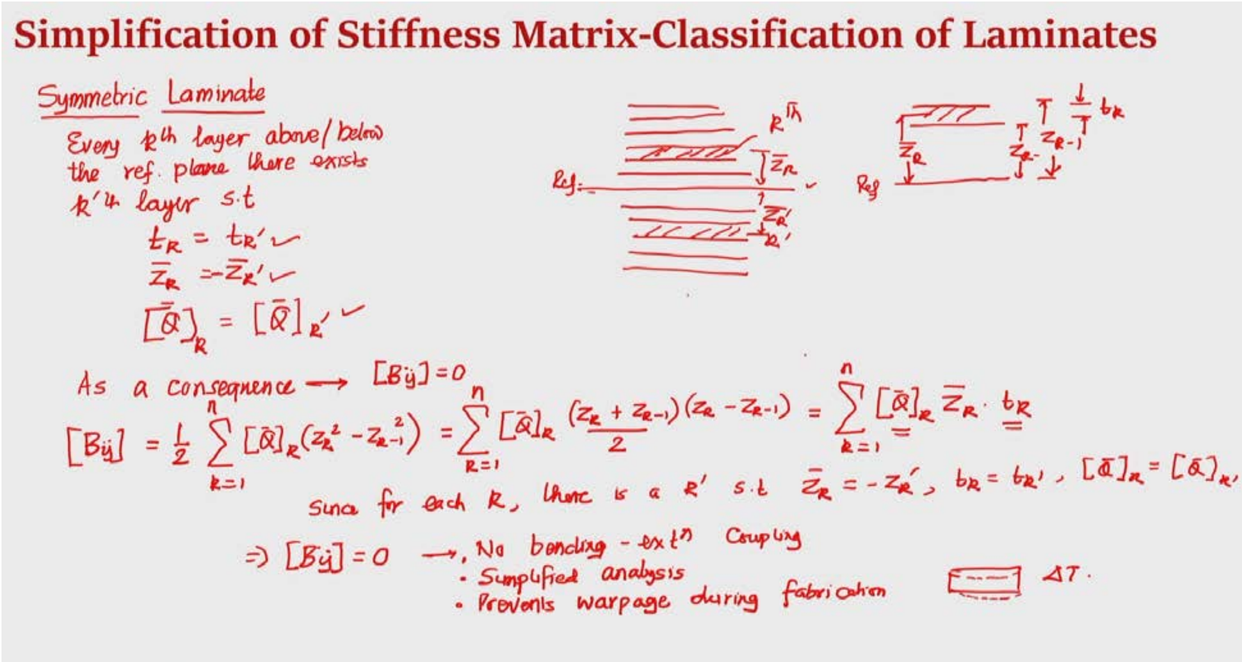
And more importantly we understood that each element of this ABBD matrix is actually decided by the geometrical properties and mechanical properties of the constituent laminae from which the laminate is made. Geometrical properties are the thickness of the lamina and the location with reference to the mid surface plane or the stacking sequence of laminae. Mechanical properties are the engineering constants for each lamina.

So, the geometrical properties and as well as the mechanical properties of each lamina actually decides what will be the value of each of these elements and therefore by choosing the geometrical and mechanical properties of the constituent laminae, desirable properties may be achieved or the undesirable characteristics may be eliminated. For example, if laminate is expected not to have any coupling between the in-plane load and the curvature, then we shall choose the geometrical and mechanical properties of the laminae is such a way that the elements of the [B] matrix are zero which also endures that subjected to moment it will only have curvatures and no implant strains. Similarly, if the terms A_{16} and A_{26} are zero, then there is no coupling between the in-plane normal force and the in-plane shear strain.

In this lecture, simplification of the stiffness matrix $\begin{bmatrix} A & B \\ B & D \end{bmatrix}$ leading to special cases of laminate stiffness will be discussed. For a laminate with arbitrary stacking sequence, all the elements of laminate stiffness matrix will be highly populated and non-zero.

But in many situations, there is a need for specific stacking sequence such that some of the elements laminate stiffness matrix will be zero, which simplifies the analysis avoiding undesirable responses like lamina and laminate level couplings.

(Refer Slide Time: 05:45)



Symmetric Laminate:

If a laminate is constructed by stacking several laminae such that for every k[th] lamina above/below reference plane there is a k[th] lamina below/above reference (refer Fig.) plane such that,

$(i)\quad z_k = -z_k'$ *ie.* same distance from the ref plane, one aboveand other below

$(ii)\ t_k = t_k'$ *ie.* they have the same thickness

$(iii)\left[\bar{Q}_{ij}\right]_k = \left[\bar{Q}_{ij}\right]_{k'}$ *ie.* they have the same material properties

The laminate is a Symmetric Laminate and it possesses symmetry with reference to the mid surface whatever is above the mid surface, below the mid surface is just the reflection of that. As a consequence of symmetry $[B_{ij}]$=0 ie each element of the coupling matrix is zero. This could be proved as follows.

$$\left[B_{ij}\right] = \frac{1}{2}\sum_{k=1}^{n}[\bar{Q}]_k\left(z_k^2 - z_{k-1}^2\right) = \sum_{k=1}^{n}\left[\bar{Q}\right]_k \frac{\left(z_k + z_{k-1}\right)}{2}\left(z_k - z_{k-1}\right) = \sum_{k=1}^{n}[\bar{Q}]_k \bar{z}_k \cdot t_k$$

Now, that for every k there is a k' for which $t_k = t_{k'}$, $\bar{z}_k = -\bar{z}_{k'}$, $[\bar{Q}]_k = [\bar{Q}]_{k'}$, and therefore B_{ij}=0.

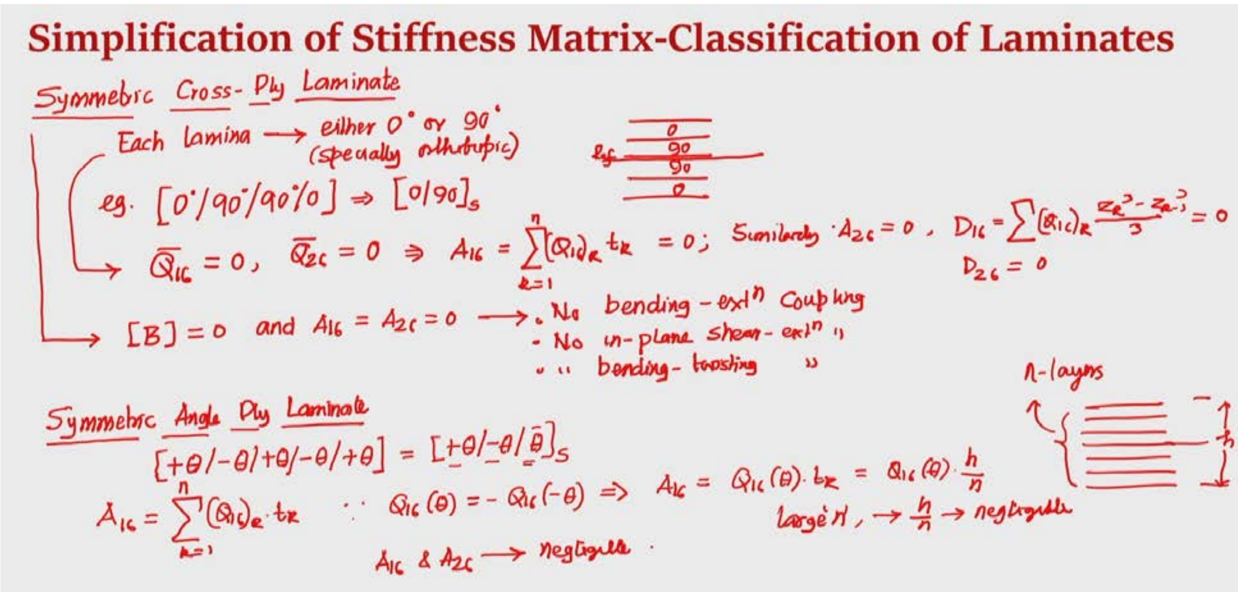
Thus, as a consequence of this symmetry (note that this is a symmetry of geometry as well as material properties) each element of [B] becomes zero and there is no bending extension coupling leading to simplified behavior and simplified analysis.

In the last lecture in the discussion on the analysis of laminate for a symmetric laminate the analysis could be decoupled as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{13} & A_{23} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} \text{ and } \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{23} & D_{66} \end{bmatrix} \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$$

In addition, this kind of symmetric laminates prevent warpage during fabrication. That means when a laminate during fabrication is allowed to cool to room temperature from the curing temperature and it experiences the temperature gradient of ΔT. If it is not symmetric it will bend leading to warpage.

(Refer Slide Time: 14:06)



Now symmetric laminates could be symmetric cross ply laminates. In a symmetric cross ply laminate, in addition to satisfying the conditions of symmetry, each layer is a specially orthotropic layer that means each layer of the laminate is either 0˚ or 90˚. Just to remind, in a specially orthotropic lamina, the material axes coincide with the analysis axes. For example [0/90/90/0] is a cross ply symmetric laminate which could be written as $[0/90]_S$. Because each layer is specially orthotropic, therefore $\bar{Q}_{16} = \bar{Q}_{26} = 0$ and that means that there is no shear extension coupling in a specially orthotropic lamina.

As a consequence of $\bar{Q}_{16} = \bar{Q}_{26} = 0$,

$A_{i6} = \sum_{k=1}^{n} \left[\bar{Q}_{i6}\right]_k (z_k - z_{k-1}) = 0 \rightarrow \boxed{A_{16} = A_{26} = 0}$ and $D_{i6} = \frac{1}{3}\sum_{k=1}^{n} \left[\bar{Q}_{i6}\right]_k (z^3{}_k - z^3{}_{k-1}) = 0 \rightarrow \boxed{D_{16} = D_{26} = 0}$. Also,

$\left[B_{ij}\right] = 0$ because it is symmetric.

Therefore the characteristics of a symmetric cross ply laminate are

$$\left[A_{ij}\right] = \begin{bmatrix} A_{11} & A_{12} & 0 \\ & A_{22} & 0 \\ & & A_{66} \end{bmatrix}$$ no in-plane shear-extension coupling

$\left[B_{ij}\right] = 0$ no laminate level bending-extension coupling

$$\left[D_{ij}\right] = \begin{bmatrix} D_{11} & D_{12} & 0 \\ & D_{22} & 0 \\ & & D_{66} \end{bmatrix}$$ no bending-twisting coupling

Similarly symmetric laminates could be symmetric angle ply laminate. An angle ply laminate is actually having its adjacent plies oriented at + θ and − θ. In such an angle ply laminate, if the number of layers are odd like say [+θ/– θ/+θ/– θ/+θ], then it is a symmetric angle ply laminate. This symmetric angle laminate could be written as $[+\theta/-\theta/+\bar{\theta}]_S$. For this

$$\begin{aligned}
\bar{Q}_{16} &= c^3 s(Q_{11} - Q_{12} - 2Q_{66}) + cs^3(Q_{12} - Q_{22} + 2Q_{66}) \\
\bar{Q}_{26} &= cs^3(Q_{11} - Q_{12} - 2Q_{66}) + c^3 s(Q_{12} - Q_{22} + 2Q_{66}) \\
\therefore \bar{Q}_{i6}(\theta) &= -\bar{Q}_{i6}(-\theta)
\end{aligned}$$

ie. the Q_{16} is an odd function of θ. As a consequence of this for a [+θ/– θ/+θ/– θ/] laminate

$$A_{i6} = \sum_{k=1}^{n} \left[\bar{Q}_{i6}\right]_k (z_k - z_{k-1}) = \sum_{k=1}^{n} \left[\bar{Q}_{i6}\right]_k t_k = 0$$

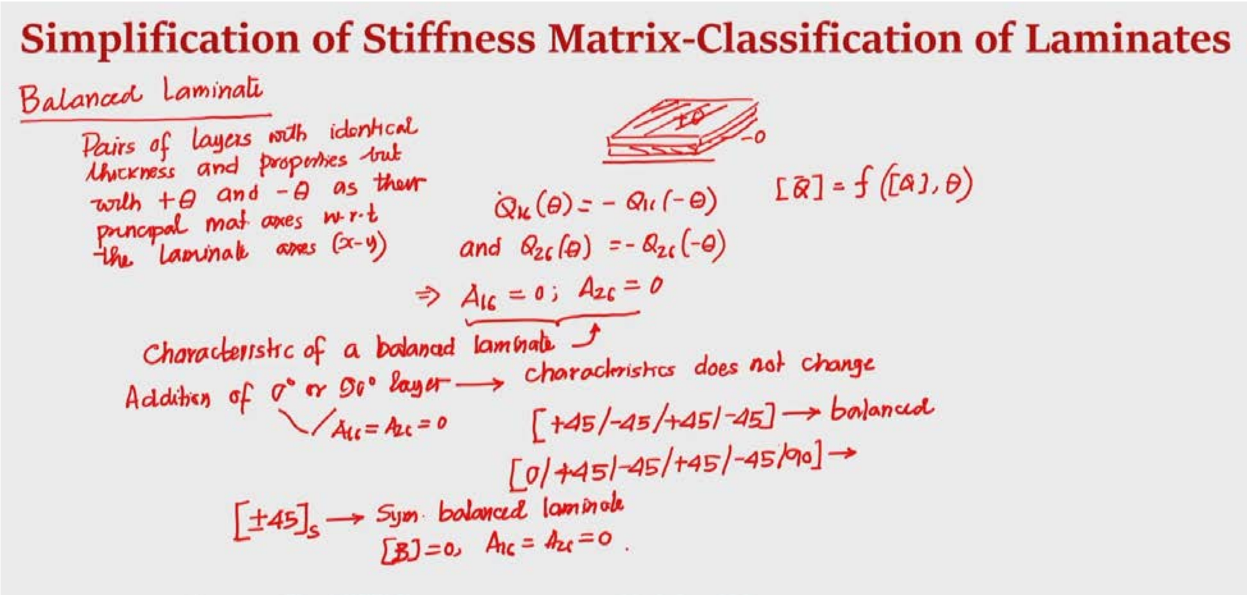
But for a symmetric angle ply laminate, say $[+\theta/-\theta/+\bar{\theta}]_S$, Q_{16} for the +θ and –θ pairs cancel out and only one Q_{16} (θ) remains. So, A_{16} is for only one layer. If a laminate is having *n* layers of thickness t_k and the total thickness is *h*, then

$$\Rightarrow \quad A_{16} = Q_{16}(\theta) \cdot t_k = Q_{16}(\theta) \cdot \frac{h}{n}$$

$$\text{large } 'n', \rightarrow \frac{h}{n} \rightarrow \textit{very small}$$

Therefore A_{16} and A_{26} are negligible. Therefore depending upon the number of layers, if the number of layers are large, A_{16} and A_{26} could be negligible.

(Refer Slide Time: 22:09)



Balanced Laminate:

Balance laminate consists of pairs of layers with identical thickness and properties but have +θ and – θ as their fiber orientation with reference to the laminate analysis axes x-y. In a balanced laminate

$$\bar{Q}_{16} = c^3 s(Q_{11} - Q_{12} - 2Q_{66}) + cs^3(Q_{12} - Q_{22} + 2Q_{66})$$
$$\bar{Q}_{26} = cs^3(Q_{11} - Q_{12} - 2Q_{66}) + c^3 s(Q_{12} - Q_{22} + 2Q_{66})$$
$$\therefore \bar{Q}_{i6}(\theta) = -\bar{Q}_{i6}(-\theta)$$

So, $Q_{16}(\theta) = -Q_{16}(-\theta)$ and $Q_{26}(\theta) = -Q_{26}(-\theta)$.

Since for every theta every +θ there is a – θ therefore it implies that for a balanced laminate $A_{16} = 0; A_{26} = 0$ Thus, the characteristics of a balanced laminate is that in plane shear extension coupling is eliminated.

Now, the significance of angle plies was already discussed. In a specially orthotropic laminate ie in a cross ply laminate (0˚ and 90˚), it will be strong and stiff in the longitudinal and transverse direction but poor when it comes to shear. To enhance the shear stiffness, say 45˚ layers are included in the layup. But as soon as one 45˚ layer is included, that immediately leads to the existence of A_{16} and A_{26} which is actually zero for 0˚ and 90˚. Therefore including a 45˚ layer will lead to shear extension coupling. But if along with that 45 ˚ layer there is another −45˚ layer that means if they come in pair, then $A_{16} = A_{26} = 0$ and the shear-extension be eliminated. So, that is how a balanced laminate is important while it provides the required shear stiffness at the same time, the shear extension coupling is eliminated. Also note that if 0˚ or 90˚ layer is added to a balanced laminate, this characteristic $A_{16} = A_{26} = 0$ does not change because for 0˚ or 90˚ layer anyway $A_{16} = A_{26} = 0$. Thus adding 0˚ or 90˚ to a balanced laminate like [0/+45/- 45/+45/-45/90], does not change the characteristics of $A_{16} = A_{26} = 0$ and is also considered to be balanced laminate. Then balance laminate could be symmetric also. Say $[\pm 45]_S$ is a symmetric balance laminate. Therefore, for a balanaced symmetric laminate, [B]=0, in addition to $A_{16} = A_{26} = 0$.

Now if a laminate does not have symmetry then it could be unsymmetric laminate or also called asymmetric or non-symmetric. The characteristics of unsymmetric laminate that means in ABBD matrix elements of [B] are non-zero unlike symmetric laminates where all the elements of [B] matrix are zero and hence bending extension coupling exists.

A laminate could also be anti-symmetric where the balanced pair of +θ and – θ are symmetrically placed with reference to the mid surface. For example, $[+\theta_1/+\theta_2/-\theta_1/-\theta_2]$ is an anti-symmetric laminate. So, in an anti-symmetric laminate, for each +θ layer there is a – θ layer of identical thickness and same material but they are placed symmetrically with respect to the middle surface. Anti-symmetric laminates are definitely balanced laminates. Because it is balanced, therefore the characteristics of balanced laminate $A_{16} = A_{26} = 0$ is also the characteristics of an anti-symmetric laminate. In addition, for anti-symmetric laminate $D_{16}= D_{26}0$ as shown below.

$$D_{i6} = \frac{1}{3}\sum_{k=1}^{n}\left[\bar{Q}_{i6}\right]_k (h^3{}_k - h^3{}_{k-1}) = 0 \rightarrow \boxed{D_{16} = D_{26} = 0}$$

Therefore the characteristics of an anti-symmetric laminate is that $A_{16} = A_{26} = 0$ and also $D_{16} = D_{26} = 0$. However because it is not symmetric therefore the bending extension coupling exist and $[Bij] \neq 0$.

(Refer Slide Time: 29:32)

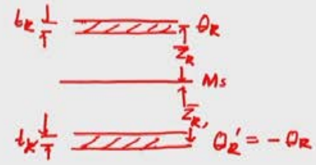
Simplification of Stiffness Matrix-Classification of Laminates

→ Unsymmetric or Asymmetric or non-symmetric Laminate

— $[B] \neq 0 \rightarrow$ Bending-Ext^{n} Coupling exists

→ Anti-Symmetric Laminate

— Balanced pair of $+\theta$ & $-\theta$ are symmetrically placed w.r.t mid-surface



$t_k = t_{k'}$

$\theta_k = -\theta_{k'}$

$\bar{z}_k = -\bar{z}_{k'}$

$\because$ Balanced $\longrightarrow A_{16} = A_{26} = 0$

Additionally, $D_{16} = \frac{1}{3} \sum_{k=1}^{n} (\bar{Q}_{16})_k (z_k^3 - z_{k-1}^3) = 0 \qquad (\bar{Q}_{16})_k = -(\bar{Q}_{16})_{k'} \qquad [\because Q_{16}(\theta) = -Q_{16}(-\theta)]$

and $D_{26} = 0$

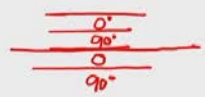
$\left.\begin{array}{l} A_{16} = A_{26} = 0 \\ D_{16} = D_{26} = 0 \end{array}\right\} [B] \neq 0.$

(Refer Slide Time: 35:22)

Simplification of Stiffness Matrix-Classification of Laminates

Anti symmetric Cross-Ply Laminate

$[0/90/0/90]$



$A_{16} = A_{26} = 0$

$D_{16} = D_{26} = 0$

$[B] \neq 0$

Quasi-Isotropic Laminate

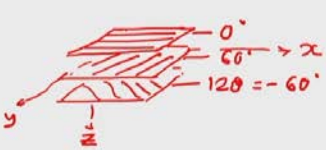
- Shows isotropic behaviour under in-plane loading ✓

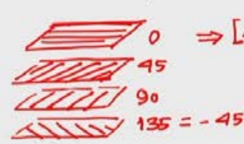
'n' ($n \geq 3$) layer laminate where the fibre orientation of the adjacent layer is incremented by $\frac{\pi}{n}$

$n = 3 \rightarrow$ three laminae Laminate

$[0/60/120] = [0/60/-60]$

$n = 4 \rightarrow [0/45/90/135] = [0/45/90/-45]$





$\Rightarrow [A] = \begin{bmatrix} A_{11} & A_{12} & 0 \\ & A_{11} & 0 \\ & & \frac{A_{11} - A_{12}}{2} \end{bmatrix}$

Isotropic

$A_{11} = A_{22}$;

$A_{16} = A_{26} = 0$

$A_{66} = \frac{A_{11} - A_{12}}{2}$;

Similarly there could be anti-symmetric cross ply laminate. Like symmetric cross ply here also each ply is a specially orthotropic plies but for each 0˚ ply above the mid surface there is 90˚ symmetrically placed below the mid surface. For example, [0/90/0/90] an anti-symmetric cross ply laminate. So,

$$A_{16} = A_{26} = 0$$
$$D_{16} = D_{26} = 0$$
$$[B] \neq 0$$

There is another special class of laminate which is called quasi isotropic laminate. Quasi- isotropic means it is isotropic in some sense but not fully isotropic. Therefore a quasi-isotropic laminate is a laminate which shows isotropic behaviour under in plane loading, and not in bending that is why it is quasi isotropic. This is defined as a n layer laminate and such that $n \geq 3$ and the fiber orientation of the adjacent layer or lamina is incremented by $\frac{\pi}{n}$ (for n=3, it incremented by $\frac{\pi}{3}$ = 60˚). The stacking sequence of a three lamina laminate will be like, the first is lamina is 0˚, the next lamina will be 60˚ and then incrementing again by 60˚, the next will be 120˚ and so the lamina will be [0/60/120]. Similarly for a four layer laminate, the laminate will be [0/45/90/135]. The fiber orientations are with reference to that laminate axis x-y and thus 120˚ with respect to x-y is same as –60˚ with respect to x-y. So, [0/60/120] could be written as [0/60/–60]. Similarly, 135˚ is nothing but –45˚. So, the laminate [0/45/90/135] could also be written as [0/45/90/–45]. These are quasi isotropic laminates. It could be shown that the [A] for this kind of quasi isotropic laminates becomes

$$[A] = \begin{bmatrix} A_{11} & A_{12} & 0 \\ & A_{11} & 0 \\ & & \dfrac{A_{11}-A_{12}}{2} \end{bmatrix}; \; A_{22} = A_{11} \quad \text{and } A_{66} = \frac{A_{11}-A_{12}}{2}$$

Note that [A] represents the in plane response of the laminate. So, this [A] is actually it behaves like an isotropic material. $A_{11} = A_{22}$ means in both the x and y directions the stiffness is same for the laminate in the under in plane loading also the relation between A_{66} and A_{11} and A_{12} follow that for isotropic materials and hence it shown isotropic behavior under in-plane loading and hence the name quasi-isotropic.

Now since [A] does not depend on stacking sequence, rather it is only decide by the number of layers, their properties and the thickness and not by how they are stacked. Therefore if [0/60/−60] is quasi isotropic then we can write [0/−60/60] is also quasi−isotropic. Going by the same logic [0/45/90/−45] and [0/+45/−45/90] does not have any difference as far as the [A] is concerned and both are quasi isotropic .

A quasi-isotropic laminate could also be symmetric, like $[0/\pm45/90]_S$. So, in this case in addition to [A] behaving as an isotropic, also [B] = 0.

In $[0/\pm45/90]$, 0˚ and 90˚ as usual provide stiffness in the longitudinal and transverse direction and presence of ±45˚ actually provides the shear stiffness. Therefore, it has both longitudinal transverse and shear stiffness at the same time under in plane load, it behaves isotropically.