Mechanics of Fiber Reinforced Polymer Composite Structures
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Module-07
Elastic Behaviour of Laminates-I
Lecture-20
Response of Laminate-Significance of ABBD

Hello and welcome to the 4th lecture of this module and we have been discussing classical lamination theory.

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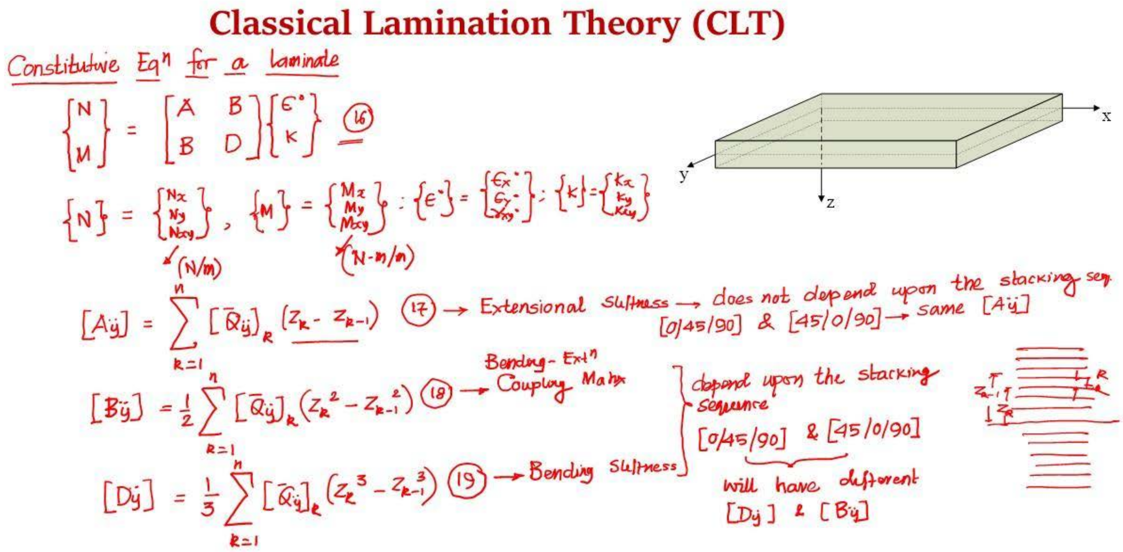

In the last lecture, the assumptions made in classical lamination theory and their significance was discussed in details. Key assumptions were assumptions on the nature of the displacement field, assumptions on strain displacement relationship, assumption on stress-strain relationship. Based on those assumptions the laminate constitutive equation was developed, where the force resultants $\{N\}=\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}$ (N/m) and moment resultants $\{M\}=\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}$ (N-m/m) were related to the mid surface strains $\{\varepsilon^o\}=\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix}$ and curvatures $\{K\}=\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$ by the $\left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right]$ matrix as

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right] \begin{Bmatrix} \varepsilon^o \\ K \end{Bmatrix} \quad (16)$$

Important to note that the force resultants are actually force per unit length maybe N/m and the moment resultants are moment per unit length maybe N-m/m. [A], [B] and [D] matrices are defines as

$$A_{ij} = \sum_{k=1}^{n} \left[\bar{Q}_{ij}\right]_k (z_k - z_{k-1}) \qquad (17)$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left[\bar{Q}_{ij}\right]_k ({z^2}_k - {z^2}_{k-1}) \quad (18)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left[\bar{Q}_{ij}\right]_k ({z^3}_k - {z^3}_{k-1}) \quad (19)$$

where z_k and z_{k-1} are the distance of the top surface and the bottom surface of the kth layer from the mid surface (refer Fig.) and $\left[\bar{Q}_{ij}\right]_k$ is the reduced transformed stiffness matrix for the kth lamina. Therefore the elements of these matrices depend upon the material properties of each lamina in the laminate and on the geometrical parameters of the laminate like the layup and the stacking sequence of the laminae in the laminate.

A closer look into the expressions for A matrix, shows that the elements of [A] matrix depend upon the $\left[\bar{Q}_{ij}\right]_k$ for each layer the thickness of each layer and do not depend on the stacking sequence. That means two glass/epoxy laminates say [0/45/90] and [90/0/45] having identical lamina thickness will have the same [A] irrespective of how the laminae are stacked. On the other hand the same is not true for [B] and [D] matrixes whose elements depend on $\left[\bar{Q}_{ij}\right]_k$ for each layer and the location of each layer relative to the mid surface since the values of $({z^2}_k - {z^2}_{k-1})$ and $({z^3}_k - {z^3}_{k-1})$ depend upon the locations of the layers. Therefore the [B] matrix and [D] matrix for the two laminates say [0/45/90] and [90/0/45] made of same materials say glass epoxy will be different because their stacking sequences are different.

Now this $\left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right]$ matrix actually defines the characteristics of the laminate. Therefore, this is called constitutive matrix or the laminar stiffness.

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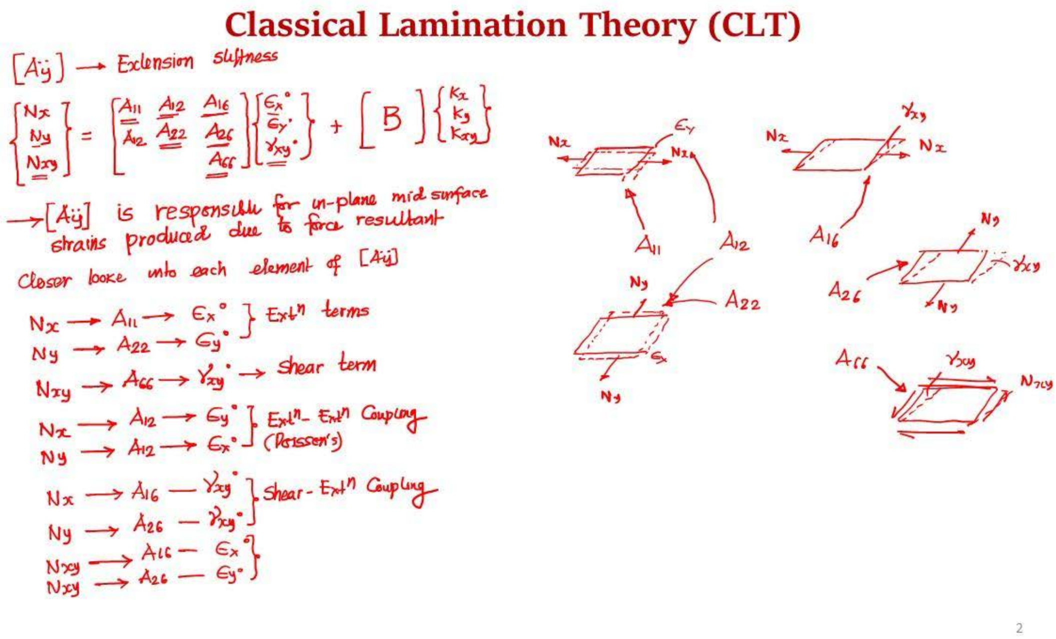

Let us see what this $\left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right]$ matrix means for the laminate in terms of the load deformation behavior with a closer look at the each of these matrices. The 3 × 3 [A_{ij}] matrix which relates the in-plane force resultants to the in-plane mid surface strains. Therefore, [A_{ij}] is responsible for in-plane mid surface strains produced due to force resultant and is termed as extensional stiffness. Now [A_{ij}] matrix has nine elements but since its symmetric, it has only six different elements viz. A_{11}, A_{22}, A_{66}, A_{12}, A_{16}, A_{26}.

Similarly, the 3 × 3 [D_{ij}] matrix which relates moment resultants to the mid surface curvatures. Therefore, [D_{ij}] is responsible for curvatures produced due to moment resultants and is termed as bending. Now his [D_{ij}] matrix has nine elements but since its symmetric, it has only six different elements viz. D_{11}, D_{22}, D_{66}, D_{12}, D_{16}, D_{26}.

The 3 × 3 [B_{ij}] matrix relates force resultants to the mid surface curvatures or the moment resultants to the in-plane mid surface strains. Therefore, [B_{ij}] actually couples the bending and extension and is responsible for curvatures produced due to force resultants and mid surface strains produced due to moment resultants and is termed as bending-extension coupling stiffness matrix. Now his [B_{ij}] matrix has nine elements but since its symmetric, it has only six different elements viz. B_{11}, B_{22}, B_{66}, B_{12}, B_{16}, B_{26}.

Now let us have a closer look at each element of this [A_{ij}] matrix.

Cause / Effect	N_x	N_y	N_{xy}	M_x	M_y	M_{xy}
ε_x^o	A_{11}	A_{12}	A_{16}	B_{11}	B_{12}	B_{16}
ε_y^o	A_{12}	A_{22}	A_{26}	B_{12}	B_{22}	B_{26}
γ_{xy}^o	A_{16}	A_{26}	A_{66}	B_{16}	B_{26}	B_{66}
K_x	B_{11}	B_{12}	B_{16}	D_{11}	D_{12}	D_{16}
K_y	B_{12}	B_{22}	B_{26}	D_{12}	D_{22}	D_{26}
K_{xy}	B_{16}	B_{26}	B_{66}	D_{16}	D_{26}	D_{66}

Table above shows the cause (Force/Moment) and effect (strain/curvature) and the elements responsible for the cause and effect. For example A_{11} actually decides that if the laminate is subjected to N_x what will be the mid surface strain ε_x^o. Similarly, A_{22} decides that if the laminate is subjected to N_y what will be the mid surface strain ε_y^o. Therefore A_{11} and A_{22} are the in-plane extensional stiffness terms. Now it could be seen that A_{12} decides that if the laminate is subjected to N_x what will be the mid surface strain ε_y^o or if the laminate is subjected to N_y what will be the mid surface strain ε_x^o. In a sense this A_{12} couples the extension in x- direction to that in the y- direction and hence it is extension-extension coupling. Similarly, A_{66} decides that if the laminate is subjected to N_{xy} what will be the mid surface strain γ_{xy}^o and hence it is in-plane shear stiffness. Now it could be seen that A_{16} decides that if the laminate is subjected to N_x what will be the mid surface shear strain γ_{xy}^o or if the laminate is subjected to N_{xy} what will be the mid surface strain ε_x^o. Then A_{26} decides that if the laminate is subjected to N_y what will be the mid surface shear strain γ_{xy}^o or if the laminate is subjected to N_{xy} what will be the mid surface strain ε_y^o. Therefore these A_{16} and A_{26} actually couples the in-plane shear force to the in-plane normal strains or the in-plane normal forces to the in-plane shear strains and hence these are in-plane shear-extension coupling terms. Overall this [A] matrix is known as extensional stiffness. However, within this A matrix there are coupling between extension-extension coupling and there are shear extension coupling also.

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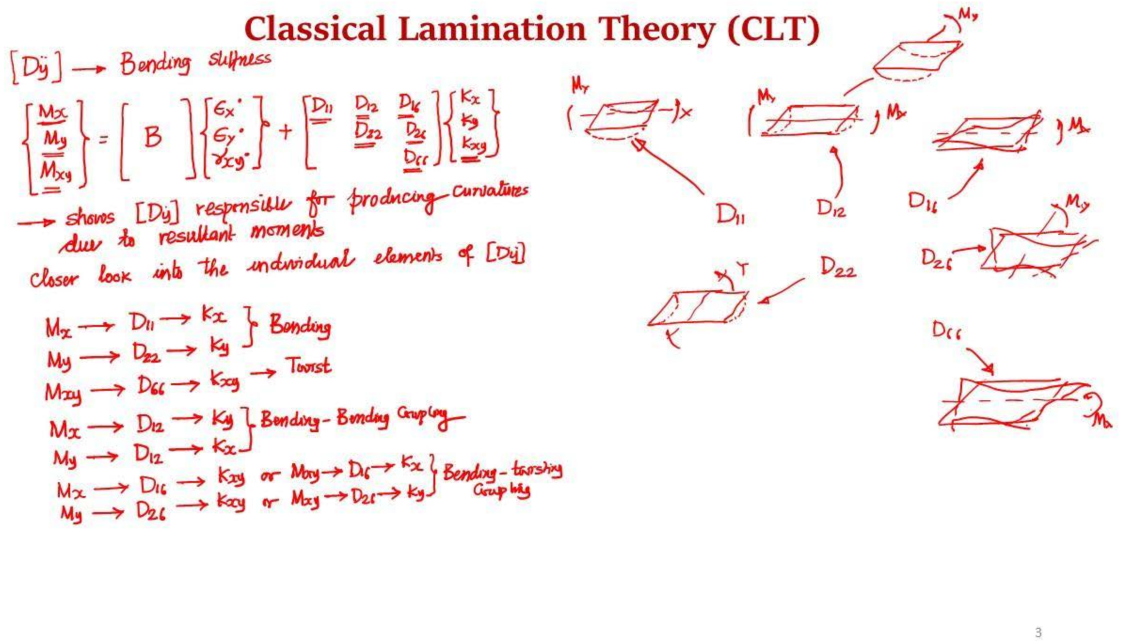

Now let us have a closer look at each element of this [D_{ij}] matrix.

For example D_{11} actually decides that if the laminate is subjected to M_x what will be the corresponding mid surface curvature K_x. Similarly, D_{22} decides that if the laminate is subjected to M_y what will be the corresponding mid surface curvature K_y. Therefore D_{11} and D_{22} are the bending stiffness terms. Now it could be seen that D_{12} decides that if the laminate is subjected to M_x what will be the mid surface curvature K_y or if the laminate is subjected to M_y what will be the mid surface curvature K_x. In a sense this D_{12} couples the bending in xz plane to that in the yz plane and hence it is bending-bending coupling. Similarly, D_{66} decides that if the laminate is subjected to twisting moment M_{xy} what will be the corresponding twisting curvature K_{xy} and hence it is twist stiffness. Now it could be seen that D_{16} decides that if the laminate is subjected to M_x what will be the twisting curvature K_{xy} or if the laminate is subjected to twisting moment M_{xy} what will be the mid surface bending curvature K_y. Then D_{26} decides that if the laminate is subjected to M_y what will be the mid surface twisting curvature K_{xy} or if the laminate is subjected to twisting moment M_{xy} what will be the mid surface bending curvature K_y. Therefore these D_{16} and D_{26} actually couples the bending and twisting and hence these are bending twisting coupling terms. Overall this [D] matrix is known as bending stiffness. However, within this D matrix there are bending-bending coupling and there are bending-twisting coupling also.

Now let us have a closer look at each element of this [**B**ij] matrix.

For example B_{11} actually decides that if the laminate is subjected to M_x what will be the corresponding mid surface strain ε_x^o or if the laminate is subjected to N_x what will be the corresponding mid surface curvature K_x. Similarly, B_{22} decides that if the laminate is subjected to M_y what will be the corresponding mid surface strain ε_y^o or if the laminate is subjected to N_y what will be the corresponding mid surface curvature K_y. Now it could be seen that B_{12} decides that if the laminate is subjected to M_x what will be the mid surface strain ε_y^o or if the laminate is subjected to M_y what will be the surface strain ε_x^o. B_{12} also decides that if the laminate is subjected to N_x what will be the mid surface curvature K_y or if the laminate is subjected to N_y what will be the surface curvature K_x. In a sense this B_{12} couples the bending in xz plane to that in the yz plane and hence it is bending-bending coupling. Similarly, B_{66} decides that if the laminate is subjected to twisting moment M_{xy} what will be the mid surface shear strain γ_{xy}^o and B_{66} also decides that if the laminate is subjected to twisting moment N_{xy} what will be the mid surface twisting curvature K_{xy}. Now it could be seen that B_{16} decides that if the laminate is subjected to M_{xy} what will be the mid surface strain ε_x^o or if the laminate is subjected to twisting moment M_x what will be the mid surface shear strain γ_{xy}^o. B_{16} also decides that if the laminate is subjected to N_{xy} what will be the mid surface curvature K_x or if the laminate is subjected to N_x what will be the mid surface curvature K_{xy}. Then B_{26} decides that if the laminate is subjected to M_{xy} what will be the mid surface strain ε_y^o or if the laminate is subjected to twisting moment M_y what will be the mid surface shear strain γ_{xy}^o. B_{26} also decides that if the laminate is subjected to N_{xy} what will be the mid surface curvature K_y or if the laminate is subjected to N_y what will be the mid surface curvature K_{xy}. Overall this [**B**ij] matrix is known as bending-extension coupling stiffness. However, within this [**B**ij] matrix there are couplings.

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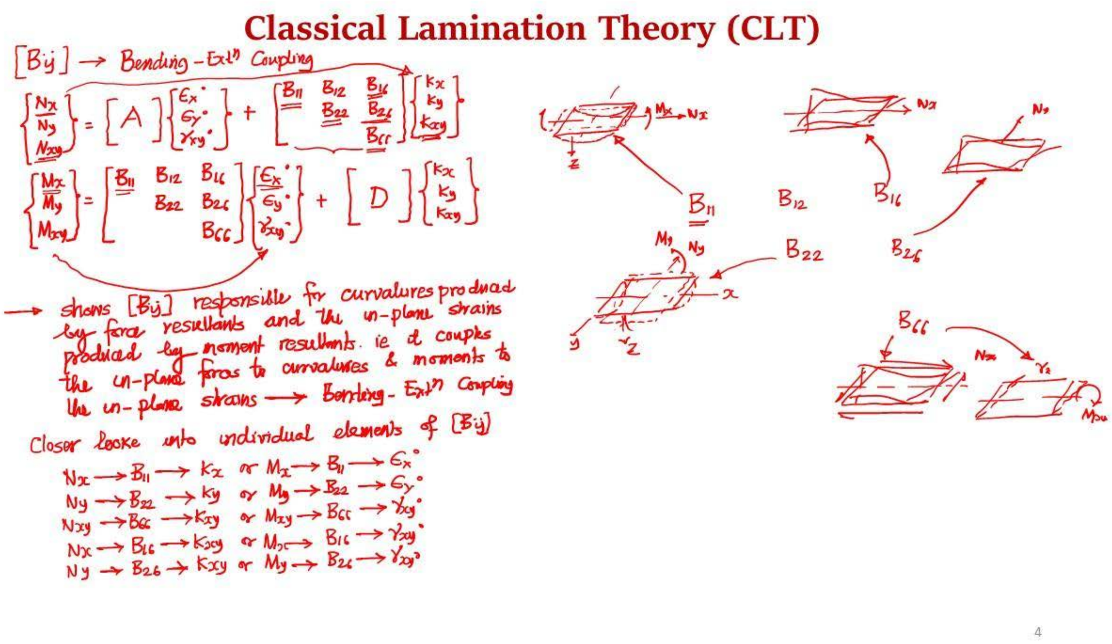

$$
\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ & A_{22} & A_{26} \\ & & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ & B_{22} & B_{26} \\ & & B_{66} \end{bmatrix} \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}
$$

$$
\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ & B_{22} & B_{26} \\ & & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ & D_{22} & D_{26} \\ & & D_{66} \end{bmatrix} \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}
$$

$$
\rightarrow \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} & \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \\ \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} & \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix}
$$

So, [A] is extensional matrix and within [A] matrix there coupling, and note that these are lamina coupling. Then [D] is the bending matrix, but within that [D] matrix there may be coupling. [B] is the coupling matrix that couples the in-plane forces to the curvatures and the moments to the implant strains.

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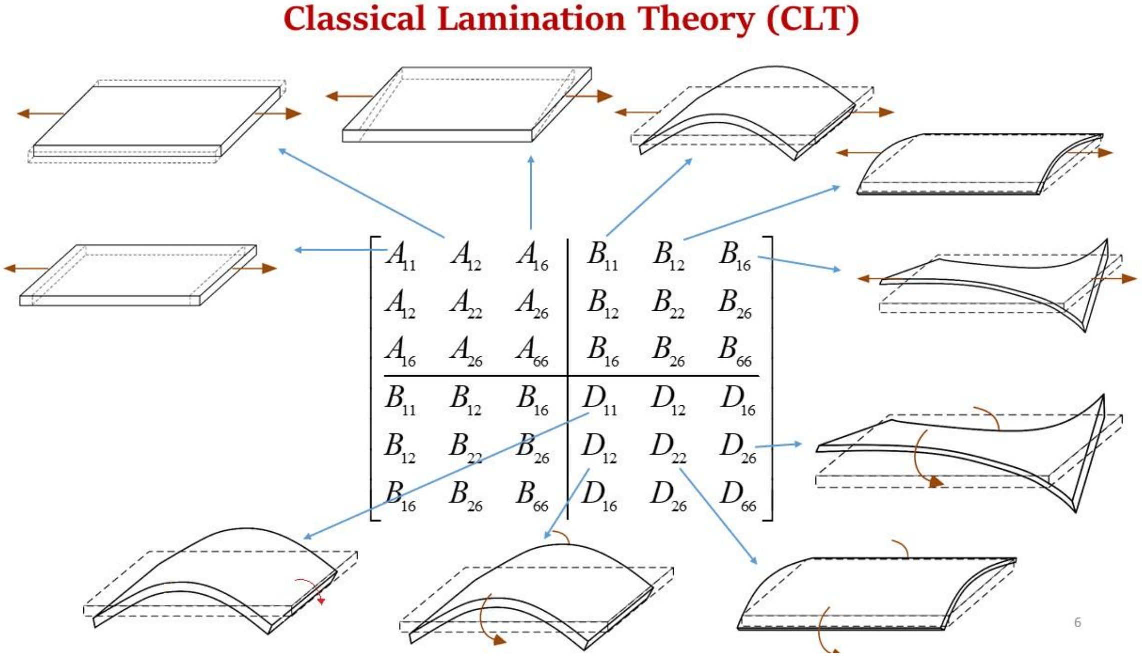

In general, the response of a laminate could be summarized by means of this $\left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right]$ matrix and the role of each term could be explained as has already been discussed. The role of each term in the deformed shape is summarized in the figure.

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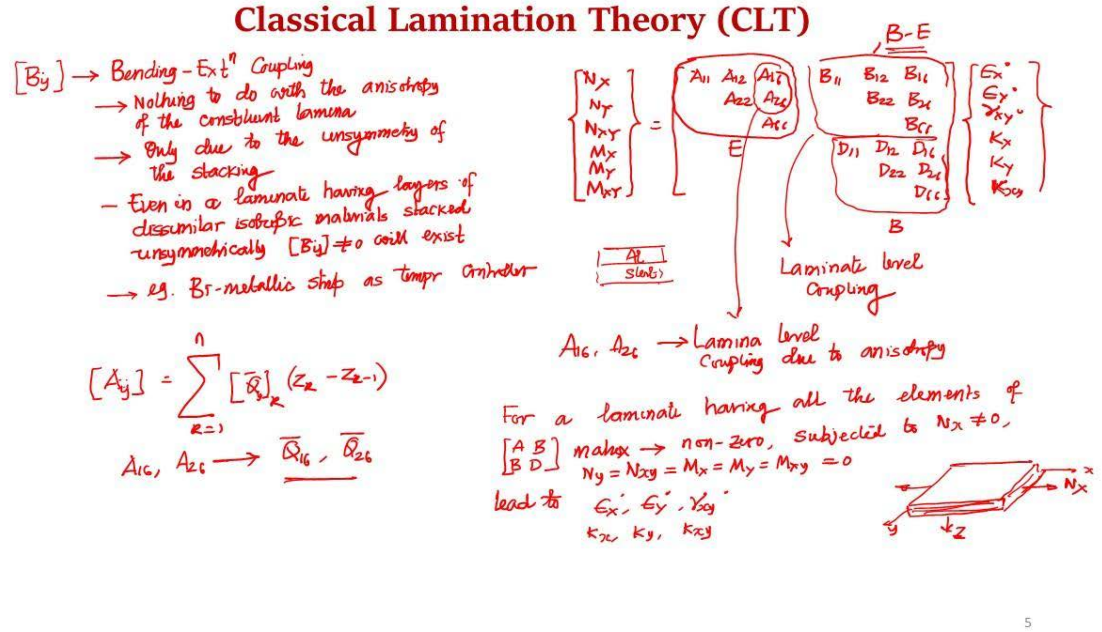

So, this [B] matrix which is the bending extension coupling has nothing to do with the anisotropy of the constituent lamina. It is only due to the un-symmetry of the stacking. Even in a laminate having layers of dissimilar isotropic material stacked unsymmetrically, this coupling stiffness matrix will exist. So, this bending-extension coupling has nothing to do with anisotropy of the laminae. A laminate may be made of number of layers which are all isotropic, but if they are stacked unsymmetrically then they may also have [B] matrix. This is actually laminate level coupling due to unsymmetric stacking and on the other hand this A_{16} and A_{26} are actually lamina level coupling due to anisotropy. This will not be there for isotropic laminae as A_{16} and A_{26} are actually decided by Q_{16} andQ_{26}.

The overall behavior of a laminate depends on what are the terms of this [A], [B] and [D]. Suppose if we have a laminate where all the terms are actually non-zero then it might deform in all possible modes. If a laminate is having all the elements of ABBD matrix as non-zero that means it is fully populated.

In one of the previous lectures, the disadvantages in mechanical characterization of anisotropic material was discussed as could be seen now that in a general laminate it is very difficult to characterize because of this couplings. That means it will have all kinds of in-plane strains as well as all kinds of curvatures subjected to a simple uniaxial load.

Before we actually go more into this ABBD matrix, let us see using CLT laminates are analyzed.

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Classical Lamination Theory (CLT)

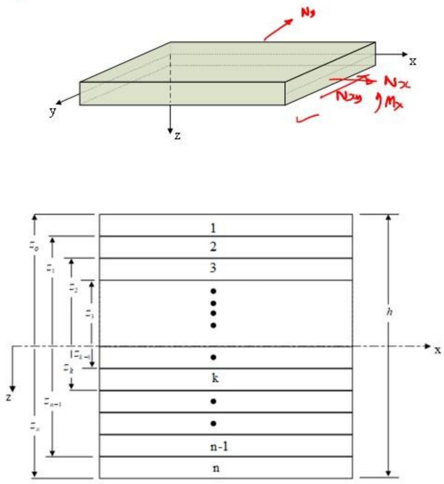
Analyzing a Laminate using CLT :

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{13} & A_{23} & A_{66} \end{bmatrix} & \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{13} & B_{23} & B_{66} \end{bmatrix} \\ \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{13} & B_{23} & B_{66} \end{bmatrix} & \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{13} & D_{23} & D_{66} \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix}$$

Load → (N_x, N_y, N_{xy}, M_x, M_y, M_{xy}) ← Known

Laminate details — no. of laminae (n), material, dimensions } ← Known

stresses, strains → to be determined



In general, the applied load on the laminate is known and the laminate details like the number of layers, material properties of each layer, fiber orientation of each layer, thickness of each layer, their locations, that is the stacking sequence are known. The stresses and strains need to be determined using CLT. This is as follows.

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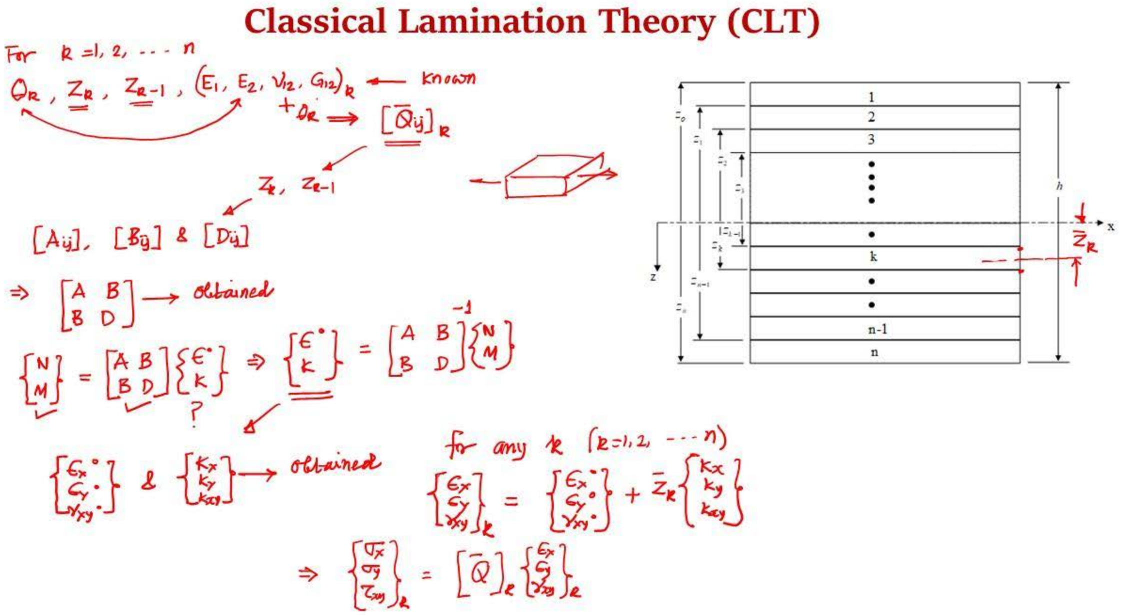

From the properties of each lamina we can determine the reduced stiffness matrix for each lamina as

$$\text{From } E_1, E_2, \nu_{12}, G_{12} \text{ using}$$

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}};\ Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}; Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}; Q_{66} = G_{12},$$

And from the reduced stiffness matrix, reduced transformed stiffness matrix could be obtained knowing the fiber orientation angle for each lamina as

$$\left\{\begin{array}{l}
\bar{Q}_{11} = Q_{11}c^4 + 2\left(Q_{12} + 2Q_{66}\right)s^2c^2 + Q_{22}s^4 \\
\bar{Q}_{12} = \left(Q_{11} + Q_{22} - 4Q_{66}\right)c^2s^2 + Q_{12}\left(s^4 + c^4\right) \\
\bar{Q}_{22} = Q_{11}s^4 + 2\left(Q_{12} + 2Q_{66}\right)c^2s^2 + Q_{22}c^4 \\
\bar{Q}_{16} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)s\ c^3 - \left(Q_{22} - Q_{12} - 2Q_{66}\right)s^3c \\
\bar{Q}_{26} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)s^3c\ - \left(Q_{22} - Q_{12} - 2Q_{66}\right)s\ c^3 \\
\bar{Q}_{66} = \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)c^2s^2 + Q_{66}\left(s^4 + c^4\right)
\end{array}\right\}$$

Therefore, for an 'n' layer laminate, we could evaluate the $\left[\bar{Q}\right]_k$ matrix for any kth lamina (k=1,2,…,n-1,n) and knowing this we could compute [A], [B] and [D] matrix as

$$[A] = \sum_{k=1}^{n}\left[\bar{Q}\right]_k (z_k - z_{k-1})$$

$$[B] = \frac{1}{2}\sum_{k=1}^{n}\left[\bar{Q}\right]_k (z^2_{\ k} - z^2_{\ k-1})$$

$$[D] = \frac{1}{3}\sum_{k=1}^{n}\left[\bar{Q}\right]_k (z^3_{\ k} - z^3_{\ k-1})$$

Now knowing the $\left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right]$ and the applied load $\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix}$, we can determine the mid surface strains and curvatures as

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix}$$

Knowing the mid surface strains and curvatures, strains at any k[th] lamina could be determined as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k = \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + z \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}; \rightarrow \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \left[\bar{Q}\right]_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k$$

$$k = 1, 2, 3, ...n$$

Knowing the global axes strains, material axes strains in any k[th] lamina could be determined as

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \dfrac{\gamma_{12}}{2} \end{Bmatrix}_k = [T]_{\theta=\theta_k} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \dfrac{\gamma_{xy}}{2} \end{Bmatrix}_k$$

Form the material axes strains, material axes stresses could be determined in k[th] lamina as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}_k = [Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}_k$$

Knowing the material axes stresses in each layer we could actually assess whether these layers are safe or fail by using appropriate failure criteria and the corresponding strengths of each lamina.