Mechanics of Fiber Reinforced Polymer Composite Structures
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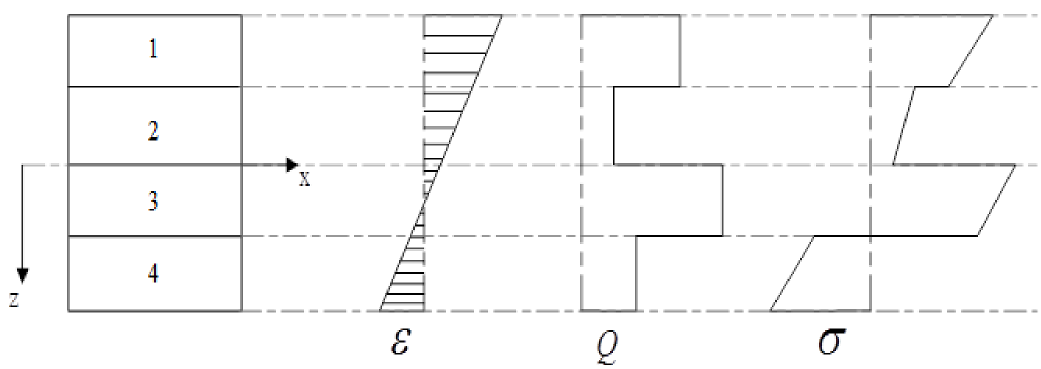
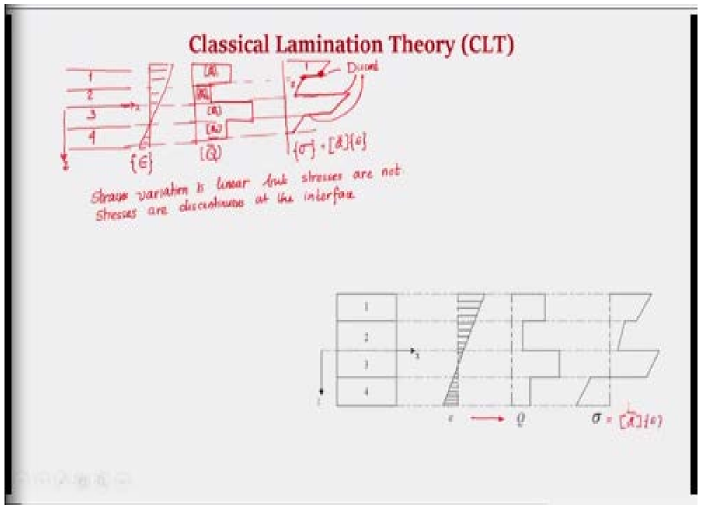
Module-07
Elastic Behaviour of Laminates-I
Lecture-19
Classical Lamination Theory-Part II

Hello and welcome to the 3rd lecture of this module and we have been discussing a classical lamination theory for analyzing of FRP laminated plate. In the last two lectures assumptions made in classical lamination theory in terms of the displacement field, in terms of the strain displacement relationship and in terms of the stress-strain relationship have been discussed. Then based on those assumptions, strains at any layer in the lamina could be expressed in terms of the mid surface strains and curvatures. However, it was understood that even though the stains vary linearly across the thickness, the stresses do not since the stiffnesses are different in different layers shown in the Fig.



It is therefore not straightforward to relate the stresses in the laminate to the strains because in an n-layer laminate the stresses could be different in each layer and at the interface, the stresses are discontinuous. **(Refer Slide Time: 01:27)**



Since the stresses are discontinuous across the thickness and in each layer the stresses are different, it is convenient to actually represent the stresses in terms of force and moment resultants.

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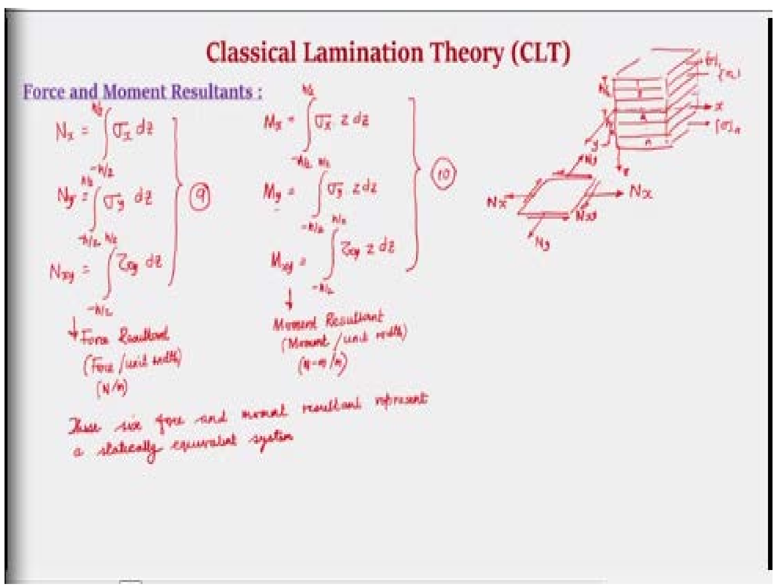

Here, the total effect of stresses in each layer of the laminate is represented by means of a corresponding forces assumed to act at the mid surface of the laminate known as force resultants. For example, the effect of σ_x in all the layers of the laminate is represented by a single force as

$N_x = \int_{-h/2}^{h/2} \sigma_x dz$ acting at the mid surface of the laminate, where, *h* is the thickness of the laminate.

This is with reference to the coordinate axes x-y-z of the laminate fixed at the mid surface.

Similarly, we can write $N_y = \int_{-h/2}^{h/2} \sigma_y dz$ and $N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} dz$, as the force resultants for σ_y and τ_{xy} acting at the mid surface of the laminate.

Since the forces acting on each layer are different, therefore they will have a net moment and that is represented by the corresponding moment resultants. For example at the moment resultant due to σ_x is $M_x = \int_{-h/2}^{h/2} \sigma_x z dz$, due to σ_y and τ_{xy} are $M_y = \int_{-h/2}^{h/2} \sigma_y z dz$ and $M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z dz$ respectively. Thus the stresses in each layer are represented by means of equivalent forces and an equivalent moments which are actually acting at the mid plane. Note that in this case the stresses are only multiplied by the thickness and not by the area and hence these force resultants are actually force per unit width or force per unit length, therefore the unit is Nm^{-1}. Similarly, the moment resultants are actually moment per unit width or length, therefore it is unit is $N\text{-}m\ m^{-1}$.

Therefore, together these six force and moment resultants (11) and (12)

$$\left.\begin{aligned} N_x &= \int_{-h/2}^{h/2} \sigma_x dz \\ N_y &= \int_{-h/2}^{h/2} \sigma_y dz \\ N_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} dz \end{aligned}\right\} \quad (11) \qquad \text{and} \qquad \left.\begin{aligned} M_x &= \int_{-h/2}^{h/2} \sigma_x z dz \\ M_y &= \int_{-h/2}^{h/2} \sigma_y z dz \\ M_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} z dz \end{aligned}\right\} \quad (12)$$

represent a statically equivalent system of the stresses in the laminate.

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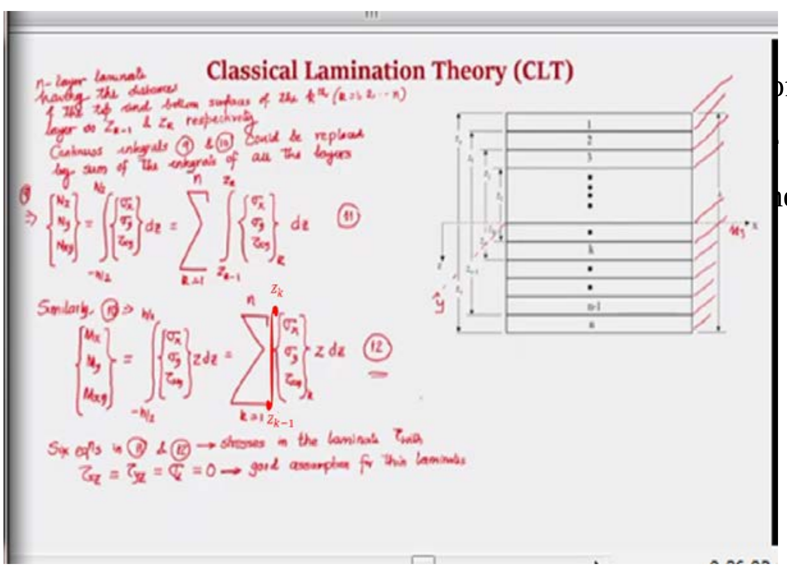
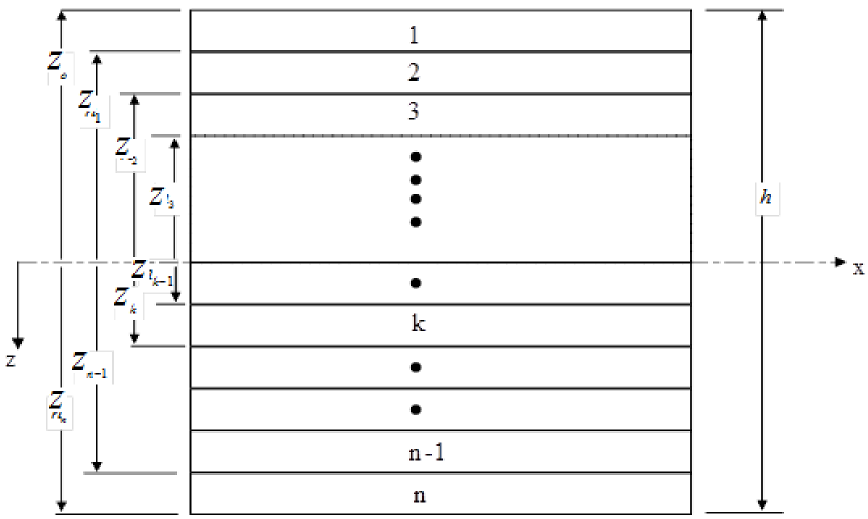


of the top and bottom surfaces of the mid surface as shown in the figure. e y-z plane.



Now the stresses are not continuous across the thickness but within one layer the stresses are thickness and hence the continuous integrals in (10) and (11) could actually be replaced by summing the integrals in each thickness over all the layers like

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k dz \quad (13)$$

and

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} zdz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k zdz \quad (14)$$

Now that the stresses are actually represented by the force and moment resultant let us apply the stress strain relationship.

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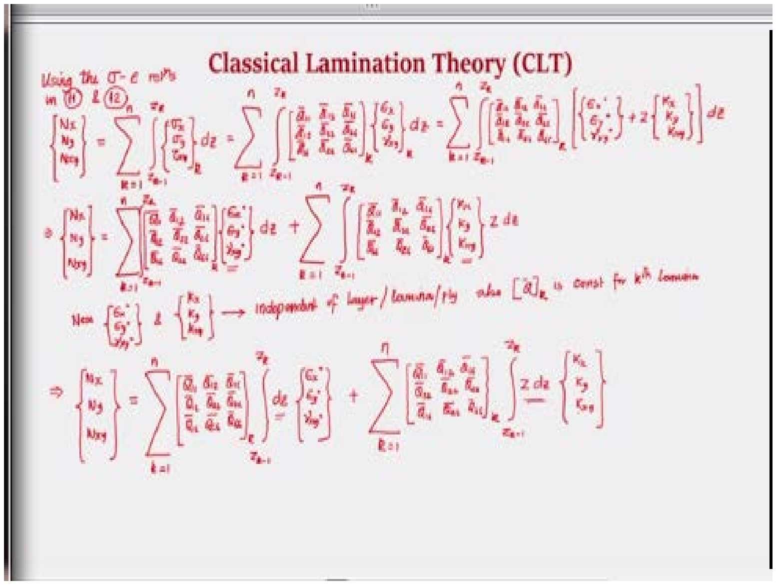

Using the stress strain relations for the kth lamina as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k$$

And also using the strain in the kth lamina in terms of the mid surface strains and curvatures as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k = \begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix}_k \left[\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} \right]$$

in equation (13)

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix}_k \left[\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} \right] dz$$

$$\Rightarrow \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} dz + \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix}_k \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} z dz \quad (15)$$

The in-plane force resultants N_x, N_y and N_{xy} could be expressed in terms of the mid surface strains and the curvature.

In this integration, the reduced transformed stiffness for the kth layer $\begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k$ is constant across that particular kth lamina thickness and hence could be taken out of the integration. In addition the mid surface strains $\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix}$ and the curvatures $\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$ are constant for the whole laminate and hence could actually be taken out of the summation sign

Therefore, after integration, (15) could be written as

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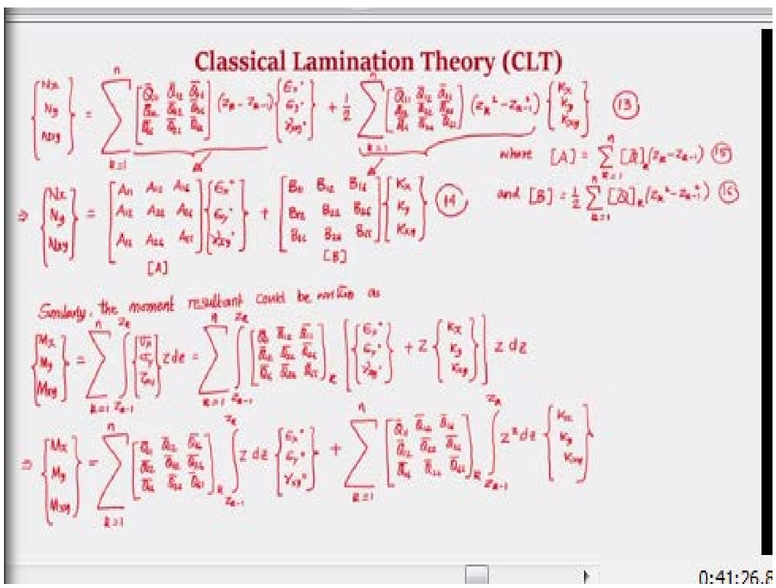

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k (z_k - z_{k-1}) \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \frac{1}{2} \sum_{k=1}^{n} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k (z_k^2 - z_{k-1}^2) \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} \quad (16)$$

which could be simplified as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} \quad (17)$$

where $$[A] = \sum_{k=1}^{n} \left[\overline{Q_k} \right] (z_k - z_{k-1}) \quad (a) \qquad [B] = \frac{1}{2} \sum_{k=1}^{n} \left[\overline{Q_k} \right] (z_k^2 - z_{k-1}^2) \quad (b)$$

[A] and [B] both are 3 × 3 matrix.

Similarly, using the stress strain relations for the kth lamina as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k$$

And also using the strain in the kth lamina in terms of the mid surface strains and curvatures as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k = \begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix}_k \left[\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} \right]$$

in equation (14) the relations between the moment resulatants and the mid surface strains and curvatures could be obtained as

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k zdz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix}_k \left[\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} \right] zdz \quad (18)$$

In this integration, the reduced transformed stiffness for the kth layer $\begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix}_k$ is constant across that particular kth lamina thickness and hence could be taken out of the integration. In addition the mid surface strains $\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix}$ and the curvatures $\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$ are constant for the whole laminate and hence could actually be taken out of the summation sign

Therefore, after integration, (18) could be written as

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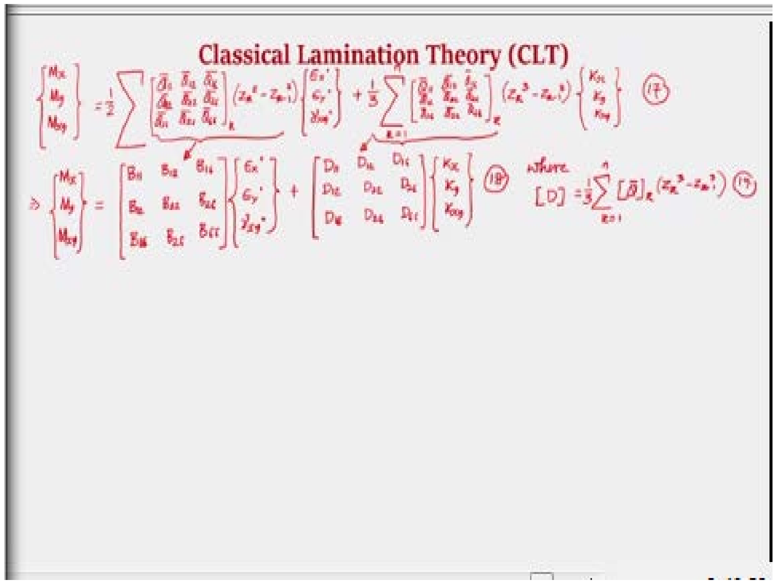

Therefore, we obtain this and then we can do this simple integration and we get

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \frac{1}{2}\sum_{k=1}^{n}\begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix}_k (z_k^2 - z_{k-1}^2)\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \frac{1}{3}\sum_{k=1}^{n}\begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix}_k (z_k^3 - z_{k-1}^3)\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} \quad (19)$$

where [A] and {b} are already defined and [D] is defined as

$$[A] = \sum_{k=1}^{n}\left[\overline{Q_k}\right](z_k - z_{k-1}) \ (a) \quad [B] = \frac{1}{2}\sum_{k=1}^{n}\left[\overline{Q_k}\right](z_k^2 - z_{k-1}^2) \ (b) \quad [D] = \frac{1}{3}\sum_{k=1}^{n}\left[\overline{Q_k}\right](z_k^3 - z_{k-1}^3) \ (c)$$

Combining (17) and (19), we get the relationship between the force and moment resultants and the mid surface strains and curvatures as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix} \quad (20)$$

That means for a given laminate, given the forces (force and moments), the mid-surface strains and curvatures could be determined using the ABBD matrix. In a laminate if the applied forces and moments are known, then N_x, N_y, N_{xy}, M_x, M_y, M_{xy} could be obtained by diving those by the width/length. Therefore this (20) is the constitutive relation for the laminate obtained using classical lamination theory.

Note that these [A], [B] and [D] matrices are the functions of the properties of each layer ($\overline{[Q]}$ obtained from E_1, E_2, G_{12}, ν_{12} and θ), fiber orientation of each layer, thickness of each layer and the location of each layer with reference to the mid surface. So knowing the material of the laminate and the stacking sequence , these A,B and D matrices could be determined and knowing these ABD, the strains and curvatures in a laminate corresponding to the applied forces could be determined.