Mechanics of Fiber Reinforced Polymer Composite Structures
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Lecture-12
Evaluation of Longitudinal Strength

Hello, welcome to the third lecture of the module, micro mechanics of lamina we have been discussing the micro mechanics approach for determination of stiffness and strength of lamina. Now in the last lecture we have discussed in details the mechanics of material approach. Among different approaches of micro mechanics, the most, simple one has been the mechanics of material approach. In the last lecture we have discussed how to determine the engineering constants of a lamina like longitudinal Young's modulus, transverse Young's modulus, in plane shear modulus and Poisson's ratios. We have also discussed in details the limitations of estimating these constants by mechanics of material approach and we understood that the mechanics of material based predictions while agree well in case of longitudinal Young's modulus E_1, however do not agree with the experimental results for the transverse Young's modulus, in-plane shear modulus because those are mostly matrix dominated properties. The reasons for the transverse stiffnesses not agreeing well with the experimental results are mainly because some of the assumptions which actually do not go well with the matrix dominated properties like the transverse Young's modulus and in plane shear modulus.

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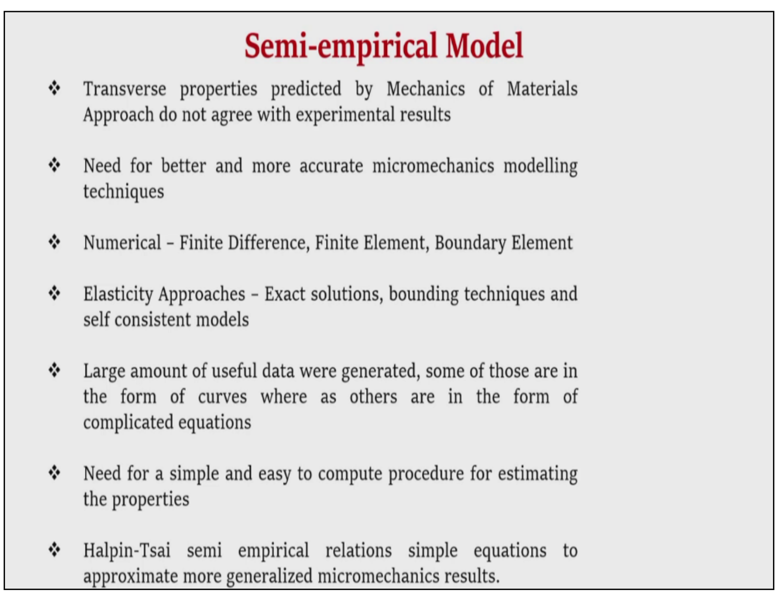

Therefore there was a need for development of better and more accurate procedures to estimate the stiffnesses of the lamina which agree well with the experimental results. Efforts were made with different approaches like numerical approach, elasticity approach, variational approach. In elasticity approach, exact elasticity solutions are developed using the equations of equilibrium, compatibility conditions and boundary conditions. Variational approach uses energy principles and provides upper and lower bounds on the stiffnesses or the properties of the lamina. Numerical analysis like finite element method could actually provide better or and more accurate predictions, but does not provide close form solution and it is case specific. It does not provide a generalized solution, while the elasticity approaches actually provide generalized solutions but leads to a large number of complicated equations and do not cater to a wide range of process variables like volume fraction, different types of fibers and composites, different types of geometries. Therefore there were inherent difficulties in using those by the designers. Therefore there was a need to develop easy and some simple relations which are which are easy for the designer to be used for estimating the properties of for estimating the stiffness of the lamina.

Semi empirical methods actually came handy in that especially the Halpin-Tsai inside method which actually provides a relatively simple equation for estimation of laminar properties in terms of the properties of the fiber matrix and the relative proportions. Even though it is actually approximate it but it covers a wide range that is it is more generalized and therefore it becomes handy for the designers to use those for estimating the properties of the lamina.

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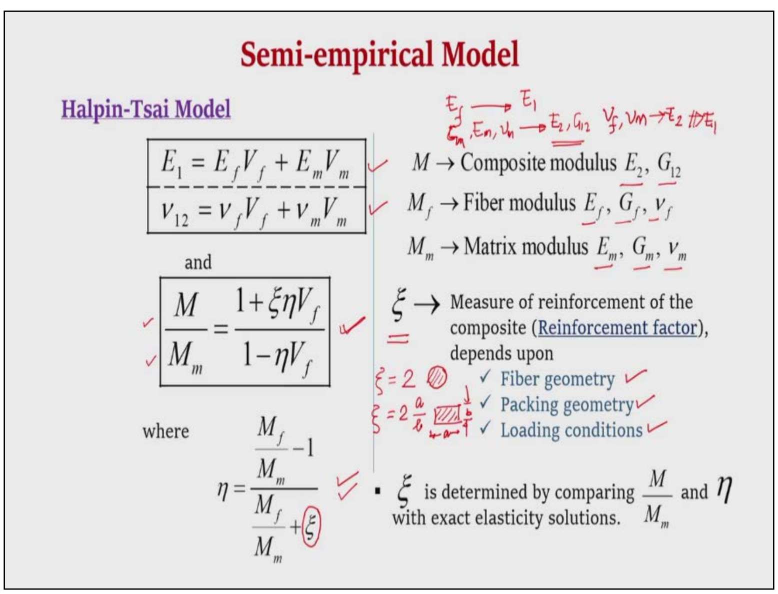

In Halpin-Tsai model, the longitudinal stiffness is given by

$$E_1 = E_f V_f + E_m V_m$$

which is exactly the same what was obtained in mechanics of material approach. Similarly the Poisson's ratio ν_{12} is given by

$$\nu_{12} = \nu_f V_f + \nu_m V_m$$

which is also similar to what was obtained in mechanics of material approach.

However for transverse properties Halpin-Tsai semi empirical model actually interpolated some of the elastic elasticity solutions and provided a much simpler estimate for the transverse stiffness of the lamina. This is given by

$$\frac{M}{M_m} = \frac{1+\xi\eta V_f}{1-\eta V_f}$$

where M actually stands for the transverse Young's modulus E_2 or in plane shear modulus G_{12}. Similarly M_f actually stands for Young's modulus of the fiber or the shear modulus of the fiber and M_m is the matrix modulus that means the Young's modulus of the matrix or shear modulus of the matrix. The term η is defined as

$$\eta = \frac{\dfrac{M_f}{M_m} - 1}{\dfrac{M_f}{M_m} + \xi}$$

This ξ is actually the measure of reinforcement of the composite which is also sometimes called the reinforcement factor and the value of ξ is decided by fiber geometry, packing geometry loading conditions and this ξ is determined by comparing this ratio of the composite transverse modulus to the matrix modulus with the value of η and an exact elasticity solution. This Halpin-Tsai model actually observed to have correlated well with a wide range of composites.

Though it is approximate but it is simple it gives a much simpler estimate and it covers a wide range of process variables unlike many elasticity solutions which are actually restricted to a very narrow band of the design regime. But the difficulty with the Halpin-Tsai relation is determination of this reinforcement factor ξ. By trying with many cases, the value of ξ is proposed as $\xi = 2$ for circular cross section of the fiber and $\xi=2(a/b)$ for rectangular cross section fibers where this is *a* (width) and *b* (height) are the cross sectional dimension of the rectangular fiber (refer slide 5:09).

So, in general we could see that the fiber modulus E_f significantly affects E_1. Therefore E_1 is a fiber dominated property. Then matrix modulus E_m and significantly affects E_2 and G_{12} and it has the fiber modulus has very little effect on E_2 and G_{12} and the Poisson ratios ν_f and ν_m has very little effect on E_2 and no effect on E_1.

So, having understood the mechanics of material approach and the semi empirical approach for determination of the stiffnesses, we will discuss another important topic that is determination of the strength properties of a unidirectional lamina using micromechanical approaches.

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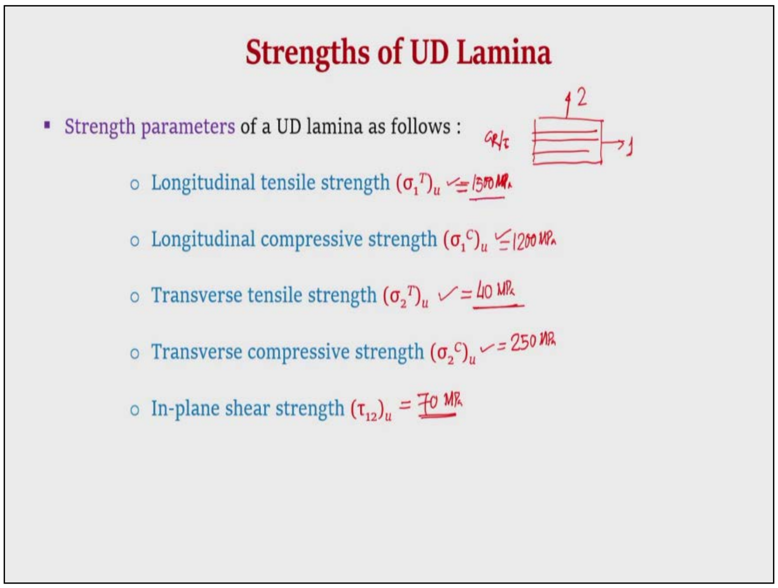

We have already seen in our discussions on macromechanical analysis of an orthotropic lamina that an unidirectional lamina has 5 strength parameters (with reference to the material axes 1-2) viz.

- the longitudinal tensile strength, $(\sigma_1^T)_u$
- the longitudinal compression strength, $(\sigma_1^C)_u$
- the transverse tensile strength, $(\sigma_2^T)_u$
- the transverse compression strength, $(\sigma_2^C)_u$ and
- the in-plane shear strength, $(\tau_{12})_u$
- Here, the material axis 1 is the longitudinal direction and 2 is the transverse direction 1-2 is the principle material direction. Strengths are direction dependent for an orthotropic lamina. Therefore longitudinal tension and longitudinal compression strengths are different, transverse tensile strength in direction and transverse compression strength are

different and we have in plane shear strength. For example, for a typical graphite epoxy lamina $(\sigma_1^T)_u$ =1500 MPa, $(\sigma_1^C)_u$= 1200 MPa, $(\sigma_2^T)_u$ = 40 MPa and $(\sigma_2^C)_u$ = 250 MPa and $(\tau_{12})_u$= 70 MPa.

This was discussed in details in the macro mechanical failure strength failure theories and we saw that the transverse tensile strength being a matrix dominated property is far less compared to the longitudinal tensile strength which is actually controlled by the fiber.

These strengths of a lamina will be decided or influenced by the corresponding strengths of the fiber and the matrix and their relative proportions and in the micro mechanical approach the objective is to relate the strengths of the fiber and the matrix and the relative proportion to the strengths of the lamina ie the five strength parameters of a lamina.

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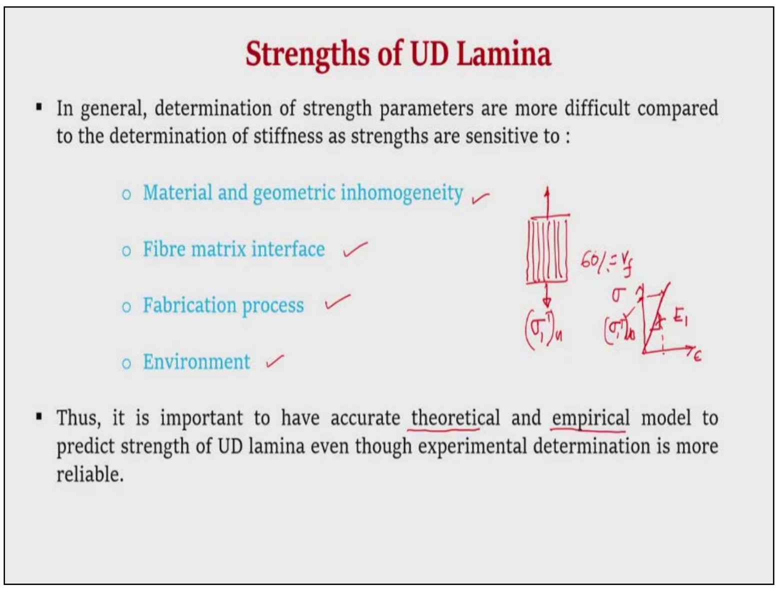

In general the determination of strength parameters are more difficult compared to the determination of stiffness parameters because the strengths are more sensitive to some of the important factors like materials and geometric inhomogeneity, fiber matrix interface, fabrication process, environment.The strengths get degraded sometimes due to change in environment which was addressed during the discussion on hygrothermal stresses in lamina.

Therefore it is important to have accurate theoretical and empirical model.

Suppose a lamina has say 60% fiber volume fraction and 40% matrix and we are really interested to find out what is its longitudinal tensile strength. We load this lamina in tension along direction one in a tensile testing machine and plot the graph between stress and strain both are of course along longitudinal direction and the point at which it fails is nothing but $(\sigma_1^T)_u$ and the and the slope of this curve is nothing but E_1. Now suppose we change the fiber volume fraction from 60% to 50% again we have to do another test, if the material is changed from say glass epoxy to graphite epoxy, we need to it again. It is a time consuming process and it is therefore important that we have accurate theoretical and empirical prediction methodologies where we could predict the strengths of an unidirectional lamina in terms of the corresponding strengths of the constituent fiber and the matrix and the relative volume fractions. But having said so experimental determination is always more reliable. Therefore whatever micro mechanics models are developed need to be carefully validated with the experimental results before those could actually be used with confidence.

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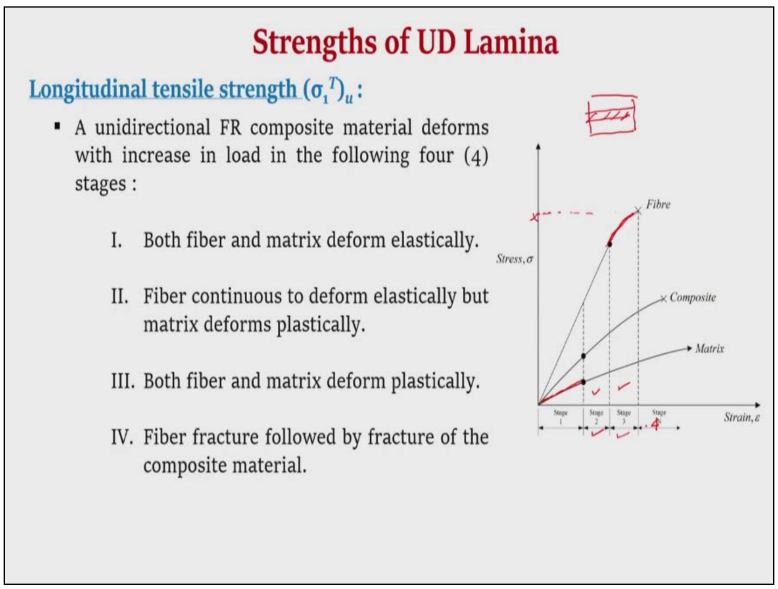

To start with let us first see the longitudinal tensile strength of a lamina. So, in general a lamina which consists of a fiber and the matrix, the fiber is actually comparatively more brittle compared to the matrix and therefore in general when it is loaded both fiber and matrix may deform elastically that means obeying Hooke's law. Then the fiber still continues to deform elastically but the matrix might deform plastically. Then both fiber and the matrix fiber experience plastic till more time and then after that the fiber fracture followed by the fracture of the composite material.

The main objective of adding fibers to the matrix is that we make it stronger and stiffer. Therefore when the fiber fails that may be considered as a failure of the composite.

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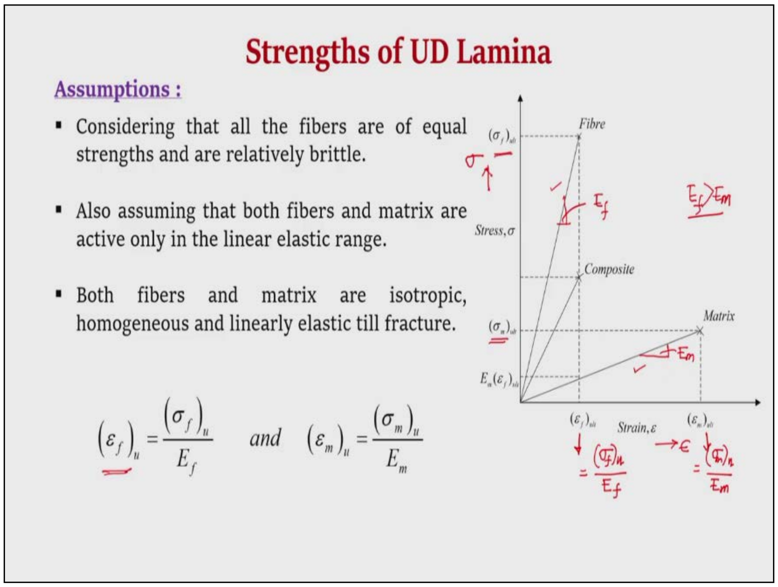

In order to determine the strength of a composite in terms of the strength of the fiber and the matrix it is assumed that the all the fibers are of equal strength and are relatively brittle.

In a in a lamina there may be very large number of fibers and the strengths are actually statistically distributed it is unlikely that all the fibers will have same strengths. But for the simplification of this analysis it is assumed that all the fibers are of same strength. The fibers and matrix are active only in the linear elastic range that is the plastic deformation of the fibers or the matrix is not considered.

Both fibers and the matrix are isotropic homogeneous and linearly elastic till. Now the stress strain curve shows that the fibers are far stiffer compared to the matrix. Therefore the Young's modulus of the fiber (E_f) is much more than the matrix Young's modulus (E_m).

So, knowing the ultimate strength of the fiber ultimate tensile strength of the fiber $\left(\sigma_f\right)_u$ we could obtain the ultimate tensile strain $\left(\varepsilon_f\right)_u$ at which the fiber fails because we have considered the fiber to be linearly elastic till its failure obeying Hooke's law as

$$\left(\varepsilon_f\right)_u = \frac{\left(\sigma_f\right)_u}{E_f}$$

Similarly, the ultimate strain of the matrix at failure again considering that till failure it is linearly elastic and obeying Hooke's law could be obtained as

$$(\varepsilon_m)_u = \frac{(\sigma_m)_u}{E_m}$$

As the stress strain curve shows, the stress strain relationship for the composite will be somewhere in between those for the fiber and the matrix and the slope of this curve is E_1 the longitudinal Young's modulus of the lamina.

Therefore for the lamina, the slope of the stress strain curve of the composite will be decided by the relative proportion of the fiber and the matrix or the fiber volume fraction. Higher is the fiber volume fraction the more this curve will be towards the fiber lower is the volume fraction more this curve will be towards the matrix. In the limit if V_f=1, that means all fiber meaning the fiber testing curve and if V_m=1, that means no fiber then naturally this is the matrix testing curve and in between the slope of the composite curve will be decided by the volume fraction of the fiber as we know that

$$E_1 = E_f V_f + E_m V_m.$$

So, for a given E_f and E_m it is the value of V_f which decides what will be the value of E_1.

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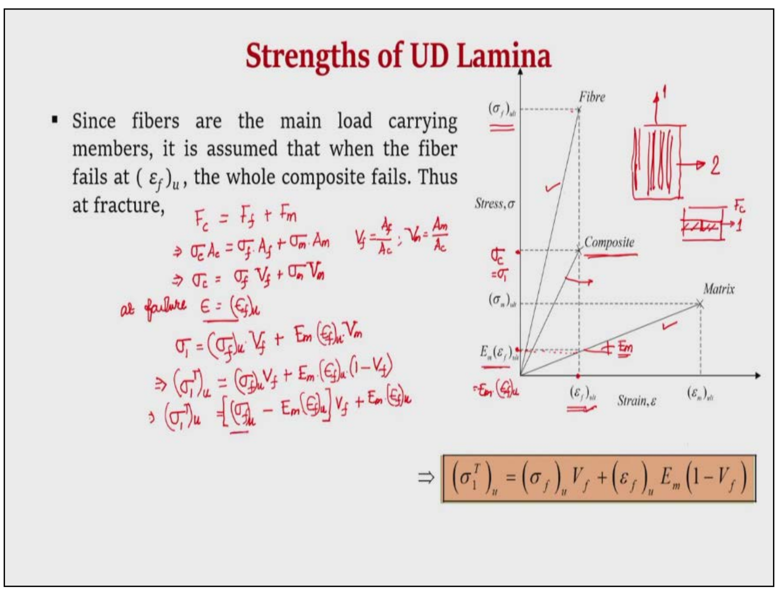

To approach the ultimate tensile stress or the longitudinal tensile strength of a lamina the basic idea is that when we add fibers in matrix the objective is that the strength of the matrix will definitely be improved and the stiffness of the matrix will also be improved that means addition of fiber to the matrix will lead to higher stiffness and higher strength.

And the fibers being the main road carrying member if the fibers fail we consider that the composite fails and it cannot carry any load further. Now if that be so, as could be seen fromm the stress strain curves for a given volume fraction, when this composite or this lamina is actually loaded if we keep on increasing the load, when the strain reaches the ultimate tensile strain of the fiber the fibers fail and if the fibers breaks then the whole composite fails.

Suppose F_c is the load on the composite, then the total load F_c is actually shared by the fiber and the matrix. Now this could be written as

$$F_c = F_f + F_m$$
$$\Rightarrow \sigma_c A_c = \sigma_f \cdot A_f + \sigma_m \cdot A_m \quad V_f = \frac{A_f}{A_c}; V_m = \frac{A_m}{A_c}$$
$$\Rightarrow \sigma_c = \sigma_f V_f + \sigma_m V_m$$

σ_c, σ_f and σ_m are the stresses in composites, fiber and the matrix respectively. A_c, A_f and A_m are the area of cross section of the composites, fiber and the matrix respectively. V_f and V_m are the volume fractions of the fiber and the matrix respectively.

So, from this stress strain curve, the failure condition is that when the fiber strain = $(\varepsilon_f)_u$ that means at failure the stress in the composite along direction, 1 is σ_1 which is nothing but $\left(\sigma_1^T\right)_u$

$$\sigma_1 = \left(\sigma_f\right)_u V_f + E_m \left(\varepsilon_f\right)_u V_m$$
$$\left(\sigma_1^T\right)_u = \left(\sigma_f\right)_u V_f + E_m \left(\varepsilon_f\right)_u \left(1 - V_f\right)$$
$$\left(\sigma_1^T\right)_u = \left[\left(\sigma_f\right)_u - E_m \left(\varepsilon_f\right)_u\right] V_f + E_m \left(\varepsilon_f\right)_u$$

This is the maximum stress the lamina could withstand. The condition is that as the fiber fails the lamina is assumed to have failed. So, this is the expression for longitudinal tensile strength of the lamina.

$$\left(\sigma_1^T\right)_u = \left[\left(\sigma_f\right)_u - E_m \left(\varepsilon_f\right)_u\right] V_f + E_m \left(\varepsilon_f\right)_u$$

Now this is the longitudinal tensile strength in terms of the corresponding strengths of the fiber and the matrix and the relative proportions.

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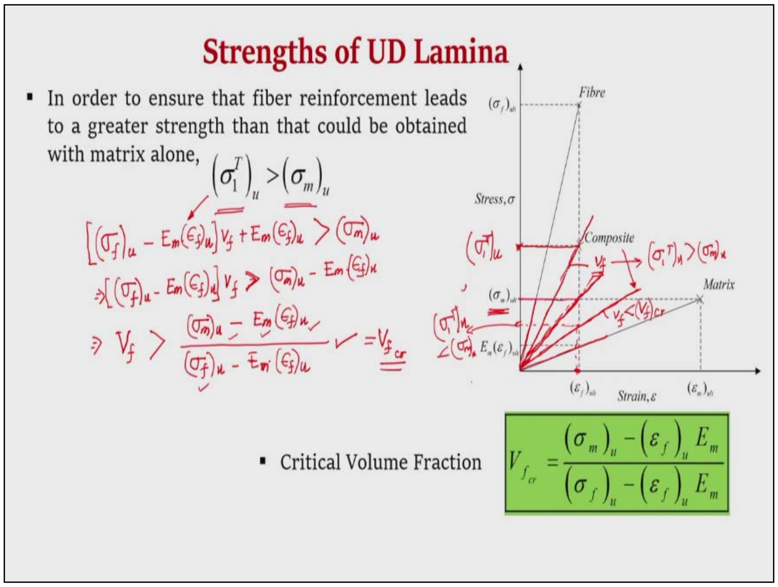

Now the objective of adding fibers is that the fiber volume fraction is such that the strength of the lamina is more than that of the matrix ie $\left(\sigma_1^T\right)_u > \left(\sigma_m\right)_u$. Now how much more that is of course decided by the volume fraction of the fiber as we keep on increasing the volume fraction of the fiber the strength increases.

Therefore we should add fiber such that the strength of the composite that means the longitudinal tensile strength of the lamina must be greater than the strength of the matrix.

$$(\sigma_1^T)_u > (\sigma_m)_u$$

$$\Rightarrow \left[\left(\sigma_f\right)_u - E_m\left(\varepsilon_f\right)_u\right]V_f + E_m\left(\varepsilon_f\right)_u > \left(\sigma_m\right)_u$$

$$\Rightarrow \left[\left(\sigma_f\right)_u - E_m\left(\varepsilon_f\right)_u\right]V_f > \left(\sigma_m\right)_u - E_m\left(\varepsilon_f\right)_u$$

$$\Rightarrow \boxed{V_{f_{cr}} \rightarrow V_f > \frac{\left(\sigma_m\right)_u - E_m\left(\varepsilon_f\right)_u}{\left(\sigma_f\right)_u - E_m\left(\varepsilon_f\right)_u}}$$

So, the fiber volume fraction should be such that the longitudinal science strength of the lamina is more than the tensile strength of the matrix.

So, if the volume fraction ($V_f > V_{f_{cr}}$) means it ensures that the longitudinal Young's modulus of the lamina that means $\left(\sigma_1^T\right)_u$ will be more than the ultimate tensile strength of the matrix, $\left(\sigma_m\right)_u$.

If $V_f < V_{f_{cr}}$ then addition of fiber will not lead to the strength of the laminar more than the strength of the matrix.

Therefore this volume fraction is called the $V_{f_{cr}}$. Now where the stress strain curve of the composite will be relative that for the fiber and the matrix is decided by the volume fraction.

It could be clearly seen that in this case the $\left(\sigma_1^T\right)_u$ is actually more than $\left(\sigma_m\right)_u$ when $V_f > V_{f_{cr}}$.

Now it is clearly visible from this curve that $\left(\sigma_1^T\right)_u < \left(\sigma_m\right)_u$ for $V_f < V_{f_{cr}}$ which is not desirable;

Now what will be the value of $V_{f_{cr}}$ is of course decided by several parameters like $\left(\sigma_m\right)_u$, $\left(\sigma_f\right)_u$, $\left(\varepsilon_f\right)_u$ and E_m and based on these the critical volume fraction will be different for different values of this. Therefore for glass epoxy it will be different and for graphite epoxy it will be different.

So, above $V_{f_{cr}}$ the longitudinal tensile strength of the lamina is more than the ultimate tensile strength of the matrix; if $V_f < V_{f_{cr}}$ then the addition of fiber will not strengthen the composite. So, this is important that we will always try to keep the volume fraction more than the critical volume fraction ($V_f > V_{f_{cr}}$) to ensure that addition of fiber enhances the strength of the matrix.

To summarize to ensure that $\left(\sigma_1^T\right)_u > \left(\sigma_m\right)_u$, the $V_f > V_{f_{cr}}$.

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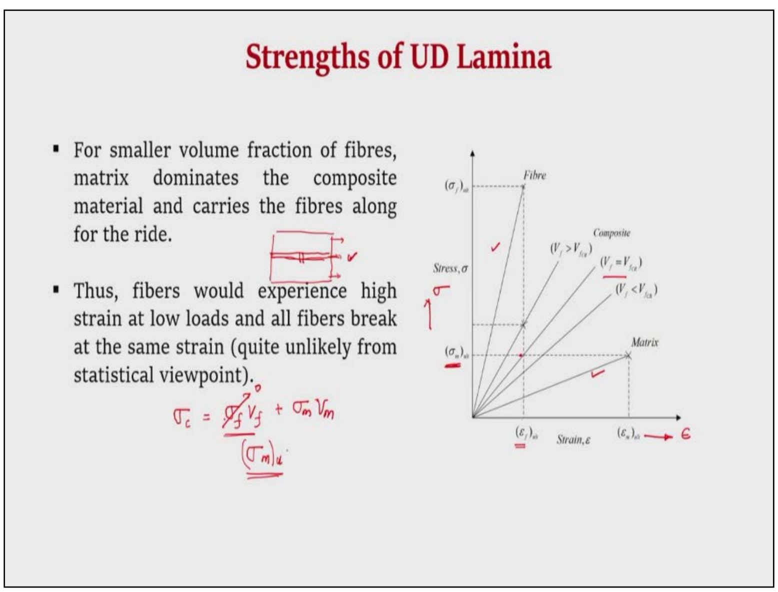

Having understood the significance of critical volume fraction what happens suppose we want for a given composite we add fibers say 1% fiber volume fraction

Now if $V_f = V_{f_{cr}}$ then the $\left(\sigma_1^T\right)_u = \left(\sigma_m\right)_u$ and only when $V_f > V_{f_{cr}}$ the strength will be more than that of the matrix ($\left(\sigma_1^T\right)_u > \left(\sigma_m\right)_u$).

Now suppose the fiber volume fraction is very small volume fraction and when it is loaded the fibers are actually less the matrix actually dominates. So, when it is stretched the matrix will be stressed and the fibers will be carried away along with the matrix. Therefore the fiber will fail first and now because the volume of matrix is much more compared to that of the fiber volume fraction, the matrix can still take the load even if the fiber is not contributing.

$$\sigma_c = \sigma_f V_f + \sigma_m V_m$$

The matrix can take the load till the stress reaches the matrix tensile stress, ultimate tensile strength of the matrix that means the stress reaches this ultimate tensile strength of the matrix and in absence of fibers this is $\sigma_m V_m$

$$\sigma_c = \sigma_f V_f + \sigma_m V_m;$$
$$\sigma_f \to 0 \Rightarrow \sigma_c = \left(\sigma_m\right)_u V_m$$

For example suppose the fiber volume fraction is say 1% that means the 99% of the matrix that means V_m is 0.99 into the matrix tensile strength is equal to the stress in the composite and this is the matrix alone is taking the load this is the maximum stress the composite could withstand.

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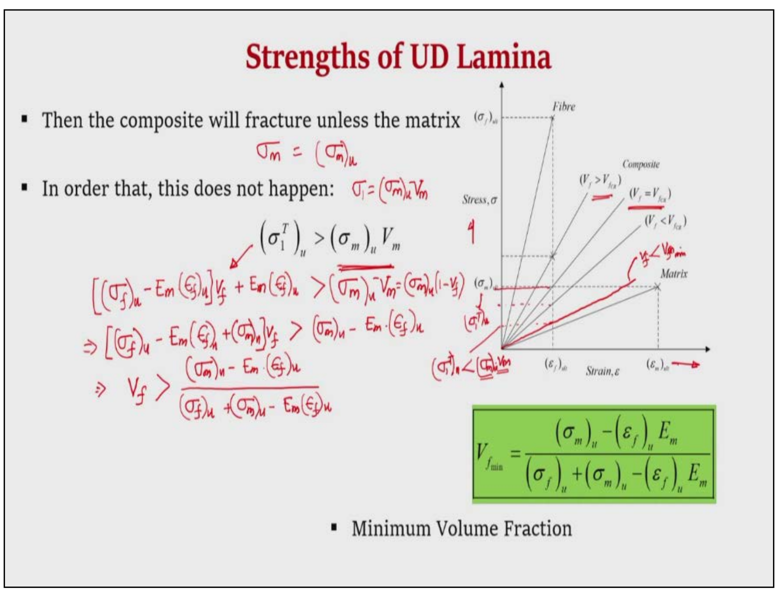

Therefore

$$\left[\left(\sigma_f\right)_u - E_m\left(\varepsilon_f\right)_u\right]V_f + E_m\left(E_f\right)_u > \left(\sigma_m\right)_u V_m = \left(\sigma_m\right)_u\left(1-V_f\right)$$

$$\Rightarrow\left[\left(\sigma_f\right)_u - E_m\left(E_f\right)_u + \left(\sigma_m\right)_n\right]V_f > \left(\sigma_m\right)_u - E_m\left(\varepsilon_f\right)_u$$

$$V_{f_{\min}} = \frac{\left(\sigma_m\right)_u - \left(\varepsilon_f\right)_u E_m}{\left(\sigma_f\right)_u + \left(\sigma_m\right)_u - \left(\varepsilon_f\right)_u E_m}$$

So, this is the minimum volume fraction. The difference between the minimum volume fraction and the critical volume fraction is that the critical volume fraction ensures us that a fiber failure is the failure of the lamina. Minimum volume fraction ensures that the longitudinal tensile strength of the lamina is more than the ultimate strength of the remaining matrix after fiber failure and below minmum volume fraction, increasing fiber volume fraction leads to reduction in strength of the composite.

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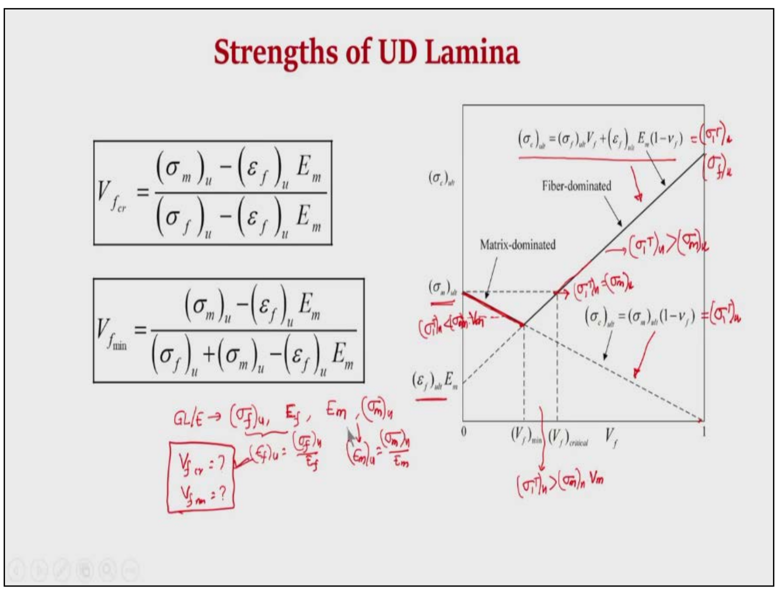

Figure clearly shows the strength of the composite with the fibre volume fraction and it could be seen that as the fiber volume fraction is increased above minimum volume fraction, the strength increases with the increase in volume fraction. Beyond critical volume fraction, the strength of the composite is ensured to be more than that of the matrix.