Mechanics of Fiber Reinforced Polymer Composite Structures
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Module-05
Macromechanics of Lamina-I
Lecture-11
Evaluation of Elastic Moduli

Hello welcome to the second lecture of the module micro mechanics of lamina. In the last lecture the importance of studying micromechanics of lamina and its objectives have been discussed along with different approaches in micromechanics of lamina and different important terminologies used in micromechanics of lamina. Determination of lamina engineering constants using the mechanics of material approach was discussed where the longitudinal Young's modulus and Transverse Young's modulus of a lamina in terms of the Young's moduli of the fiber and the matrix and the volume faction was derived.

In continuation to that today we shall discuss the determination of other engineering constants for a lamina using mechanics of material approach.

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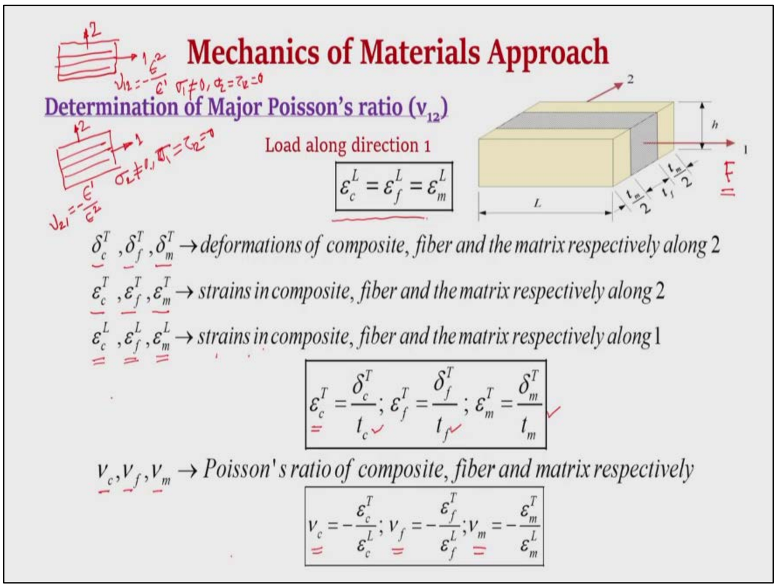

Determination of major Poisson's ratio.

Referring to the figure, with reference to the material axes, there could be two Poisson's ratios viz. ν_{12} (sometimes called major Poisson's ratio) and ν_{21} (sometimes called minor Poisson's ratio) which were defined while discussing macromechanics of lamina and they are related as.

$$\frac{\nu_{12}}{E_1}=\frac{\nu_{21}}{E_2}$$

Now in determination of this major Poisson ratio using mechanics of material approach the same RVE which was used for determination of longitudinal Young's modulus is also considered as shown in the Fig. Here also load is applied along material direction 1 but the deformation along 2 due to the Poisson's effect is observed. Suppose due to the load along 1,

$\delta_c^T\ ,\delta_f^T\ ,\delta_m^T \rightarrow deformations\ of\ composite,\ fiber\ and\ the\ matrix\ respectively\ along\ 2$

$\varepsilon_c^T\ ,\varepsilon_f^T\ ,\varepsilon_m^T \rightarrow strains\ in\ composite,\ fiber\ and\ the\ matrix\ respectively\ along\ 2$

$\varepsilon_c^L\ ,\varepsilon_f^L\ ,\varepsilon_m^L \rightarrow strains\ in\ composite,\ fiber\ and\ the\ matrix\ respectively\ along\ 1$

By definition, the strains along the transverse direction (along 2) are

$$\boxed{\varepsilon_c^T=\frac{\delta_c^T}{t_c};\ \varepsilon_f^T=\frac{\delta_f^T}{t_f}\ ;\ \varepsilon_m^T=\frac{\delta_m^T}{t_m}}$$

Now suppose the Poisson's ratios of the composite fiber and the matrix are

$\nu_c,\nu_f,\nu_m \rightarrow Poisson's\ ratio\ of\ composite,\ fiber\ and\ matrix\ respectively$

By definition of Poisson's ratio,

$$\boxed{\nu_c=-\frac{\varepsilon_c^T}{\varepsilon_c^L};\ \nu_f=-\frac{\varepsilon_f^T}{\varepsilon_f^L};\nu_m=-\frac{\varepsilon_m^T}{\varepsilon_m^L}}$$

Now from the condition of perfect bonding for this RVE the strains in the longitudinal direction (along 1) in the composite, fiber and the matrix are same and hence $\varepsilon_c^L=\varepsilon_f^L=\varepsilon_m^L$.

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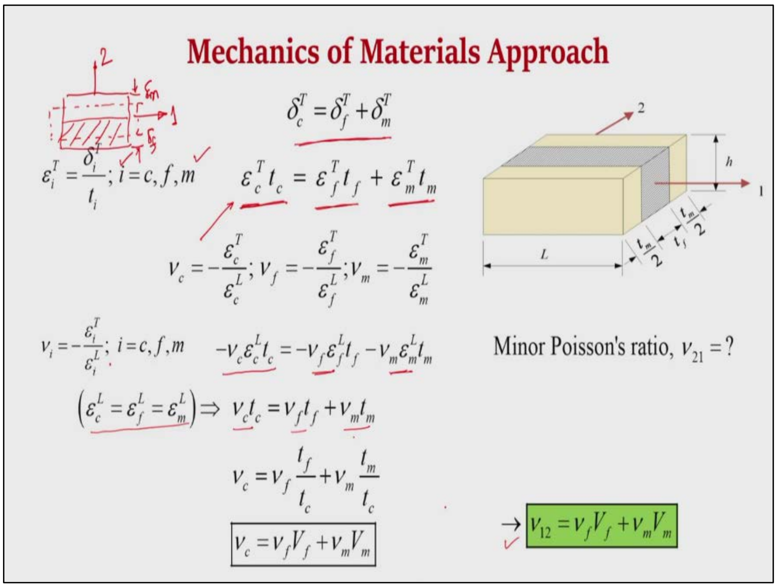

The deformed shape of the RVE showing the strains along 1 and 2 due to load along 1 only is shown in the Fig.

Therefore, the total change in length of the composite δ_c in the transverse direction is the sum of that in the fiber and the matrix i.e.

$$\delta_c^T = \delta_f^T + \delta_m^T$$

Now using the relation

$$\varepsilon_i^T = \frac{\delta_i^T}{t_i}; \; i = c, f, m$$

$$\varepsilon_c^T t_c = \varepsilon_f^T t_f + \varepsilon_m^T t_m$$

Now, writing the transverse strain in terms of longitudinal strain and the Poisson's ratio as

$$\nu_i = -\frac{\varepsilon_i^T}{\varepsilon_i^L}; \; i = c, f, m$$

we get

$$-\nu_c \varepsilon_c^L t_c = -\nu_f \varepsilon_f^L t_f - \nu_m \varepsilon_m^L t_m$$

Now, using $\left(\varepsilon_c^L = \varepsilon_f^L = \varepsilon_m^L\right)$

we get

$$\nu_c t_c = \nu_f t_f + \nu_m t_m \rightarrow \nu_c = \nu_f \frac{t_f}{t_c} + \nu_m \frac{t_m}{t_c}$$

Now taking note of the fact that for the RVE t_f/t_c actually represents the volume fraction V_f and t_m/t_c also represents the volume fraction V_m. Therefore, at the Poisson's ratio could be written as

$$\boxed{\nu_c = \nu_f V_f + \nu_m V_m}$$

Now what is ν_c is the Poisson's ratio of the lamina when the stress is applied along 1 and all other stresses are zero and because of that there is a transverse deformation along 2 therefore this $\nu_c = \nu_{12}$.

$$\boxed{\nu_{12} = \nu_f V_f + \nu_m V_m}$$

So, we get the Poisson's ratios of the lamina in terms of the Poisson's ratios of the fiober and the matrix and the volume fraction. This is similar to what was obtained for the longitudinal Young's modulus of a lamina $E_1 = E_f V_f + E_m V_m$.

Now once ν_{12} is known using the reciprocal relation $\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} \rightarrow \nu_{21} = \nu_{12}\frac{E_2}{E_1}$.

So, the Poisson's ratio ν_{12} is influenced by the Poisson ration of the matrix and Poisson station of the fiber as well as their volume fraction.

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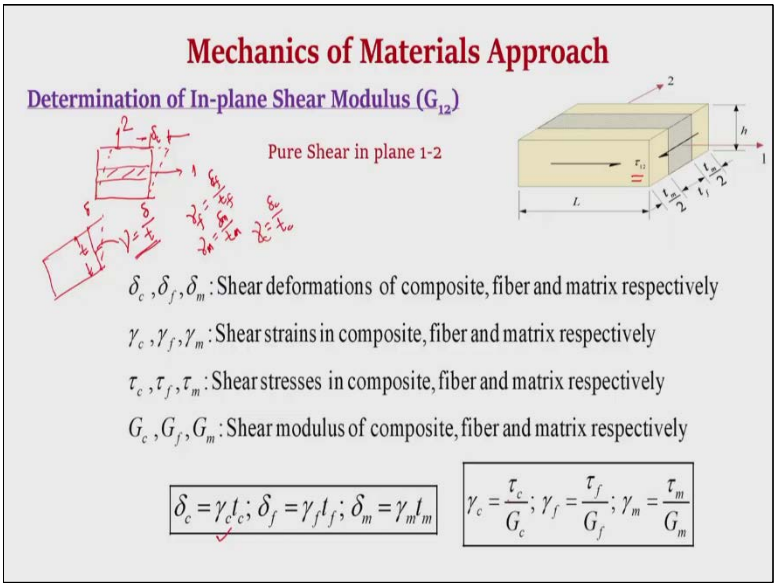

Next, we will discuss the determination of in plane shear modulus of a lamina. In this case again the same RVE is considered but it is now subjected to pure in plane shear τ_{12} and no other stresses. So, suppose because of this in plane shear suppose,

$\delta_c, \delta_f, \delta_m$: Shear deformations of composite, fiber and matrix respectively

$\gamma_c, \gamma_f, \gamma_m$: Shear strains in composite, fiber and matrix respectively

τ_c, τ_f, τ_m : Shear stresses in composite, fiber and matrix respectively

G_c, G_f, G_m : Shear modulus of composite, fiber and matrix respectively

Since the lamina is subjected to in-plane shear stress τ_{12} , the composite, the fiber and the matrix all experience the same shear stress and hence

$$\boxed{\tau_c = \tau_f = \tau_m}$$

Subjected to the in-plane shear, the deformed shape of the RVE is as shown in the Fig. Now using the definition of shear strain (small strain)

$$\boxed{\delta_c = \gamma_c t_c;\ \delta_f = \gamma_f t_f;\ \delta_m = \gamma_m t_m}$$

Also, because it is assumed that the fiber matrix and the composites obey the Hooke's law that is linearly elastic,

$$\boxed{\gamma_c = \frac{\tau_c}{G_c};\ \gamma_f = \frac{\tau_f}{G_f};\ \gamma_m = \frac{\tau_m}{G_m}}$$

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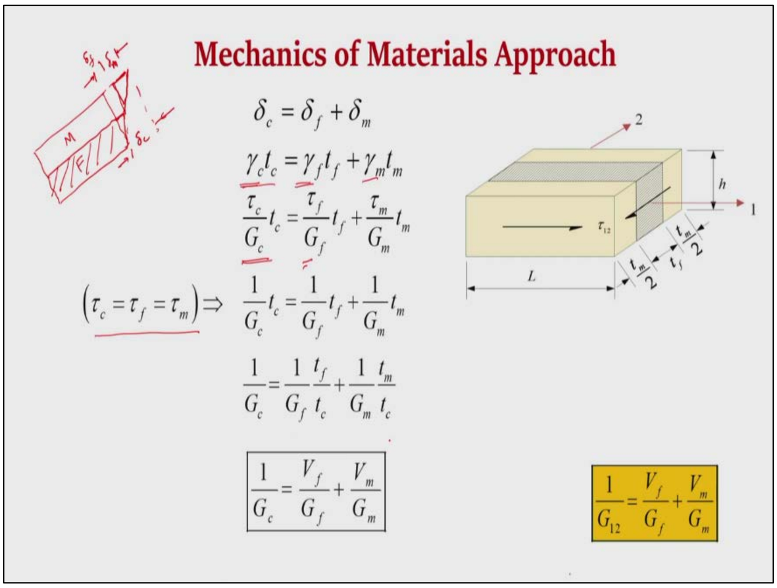

Now referring to the Fig., total shear deformation is nothing but the sum of the shear deformation of the fiber and the shear deformation of the matrix and

$$\delta_c = \delta_f + \delta_m$$

$$\gamma_c t_c = \gamma_f t_f + \gamma_m t_m$$

$$\frac{\tau_c}{G_c} t_c = \frac{\tau_f}{G_f} t_f + \frac{\tau_m}{G_m} t_m$$

Now using

$$\left(\tau_c = \tau_f = \tau_m\right)$$

$$\frac{1}{G_c} t_c = \frac{1}{G_f} t_f + \frac{1}{G_m} t_m \Rightarrow \frac{1}{G_c} = \frac{1}{G_f}\frac{t_f}{t_c} + \frac{1}{G_m}\frac{t_m}{t_c}$$

Now taking note of the fact that for the RVE t_f/t_c actually represents the volume fraction V_f and t_m/t_c also represents the volume fraction V_m

$$\boxed{\frac{1}{G_c} = \frac{V_f}{G_f} + \frac{V_m}{G_m}}$$

Now G_c in this case is nothing but the in plane shear modulus G_{12} therefore

$$\boxed{\frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{V_m}{G_m}}$$

So the in-plane shear modulus of the lamina in terms of that of the fiber and the matrx and the volume fraction is derived and note that it is similar to transverse Young's modulus of the lamina $1/E_2 = V_f/E_f + V_m/E_m$. So, here also ah it is quite clear that the shear modulus of the lamina is actually decided by the corresponding shear modulus of the fiber and the matrix and that the relative proportion.

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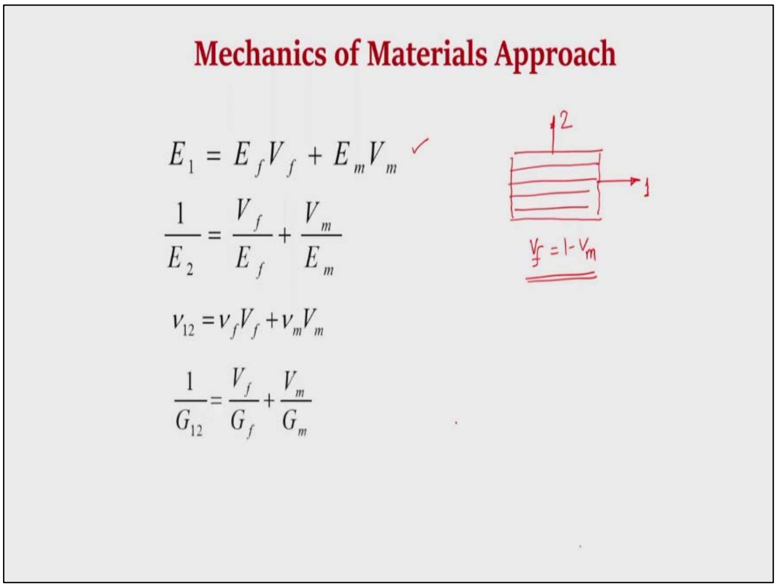

So, using mechanics of material approach the relationship between the composite modulus in terms of the modulus of the fiber and the matrix and the relative proportions are established as

$$E_1 = E_f V_f + E_m V_m$$

$$\nu_{12} = \nu_f V_f + \nu_m V_m$$

$$\frac{1}{E_2} = \frac{V_f}{E_f} + \frac{V_m}{E_m}$$

$$\frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{V_m}{G_m}$$

Now having understood this it, the assumptions that were made to develop these relationships were clearly stated as

- Fiber matrix and composites are all obey Hooke's
- Fibers are of uniform strength, uniform dimensions, equally spaced
- There is no void
- Perfect bonding.

Now it is important that we understand how accurately these expressions developed using mechanics of material approach could actually predict the composite modulus in terms of the modulus of the fiber and the matrix.

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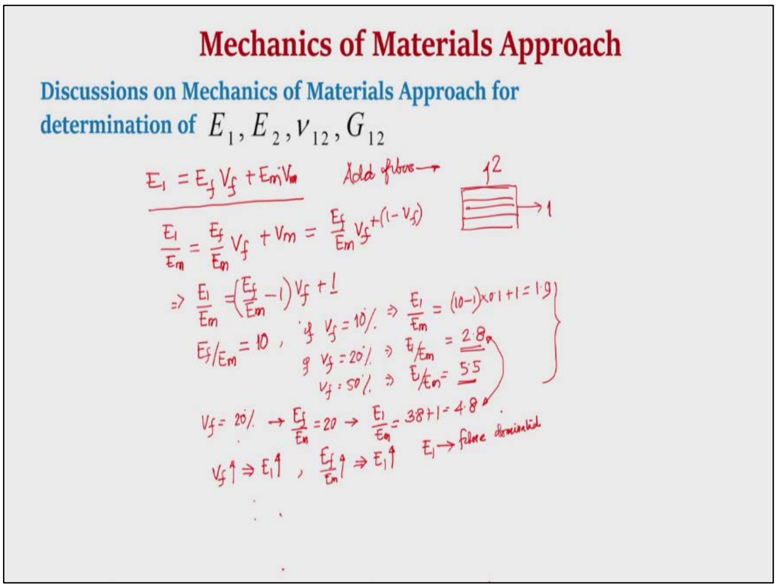

So, let us start with the longitudinal Young's modulus E_1. So, we have obtained that

$$E_1 = E_f V_f + E_m V_m$$

So, why do we add fibers. Suppose in a matrix if we add fibers naturally, we would like to see that the stiffness increases, because fiber stiffness is far higher compared to that of the matrix. Therefore with the addition of fiber in the matrix, the stiffness increases. Let us see how the fiber properties actually influence E_1.

Say we can write this as

$$E_1 = E_f V_f + E_m V_m$$
$$\Rightarrow E_1 / E_m = \left(E_f / E_m\right) V_f + V_m$$
$$\Rightarrow E_1 / E_m = \left(E_f / E_m\right) V_f + \left(1 - V_f\right)$$
$$\Rightarrow E_1 / E_m = \left(E_f / E_m - 1\right) V_f + 1$$

E_1 is the longitudinal Young's modulus of the lamina, E_f and E_m are the Young's moduli of the fiber and the matrix respectively. Generally, the fiber modulus is far higher compared to that of the matrix.

Suppose $E_f/E_m = 10$, then in such a case if suppose $V_f = 10\%$ (10% fiber and 90 % matrix) that leads to

$$E_1/E_m = (10-1) \times 0.1 + 1 = 1.9$$

This means that adding 10% fiber leads to 1.9 times increase in the modulus of the composite compared to the matrix modulus. So, adding only 10% fiber leads to 90% increase in the Young's modulus.

Similarly, for $V_f = 20\%$ (20% fiber and 90 % matrix) leads to

$$E_1/E_m = (10-1) \times 0.2 + 1 = 2.8.$$

So, increasing the fiber volume fraction by 2-fold E_1/E_m is also increased from 1.9 to 2.8. Similarly, for $V_f = 50\%$ then $E_1/E_m = 5.5$. So, we could see that as we increase the volume fraction of the fiber for a given E_f/E_m the composite longitudinal Young's modulus also increases almost by the same order.

Now suppose for a given V_f, say $V_f = 20\%$, if we increase E_f/E_m , say $E_f/E_m = 20$ this leads to

$$E_1/E_m = (20-1) \times 0.2 + 1 = 4.8.$$

So, for a given volume fraction if E_f/E_m is doubled E_1 increases from 2.8 to 4.8. So, what we observe is that as V_f increases it has significant effect on the increase longitudinal Young's modulus and for a given V_f as E_f/E_m increases there is a significant increase in the longitudinal Young's modulus.

So, the fibers both in terms of its modulus as well as in terms of its volume fraction actually significantly influence E_1 and therefore this E_1 is actually a fiber dominant property. So, the addition of fibers actually we understood now that addition of fibers actually influences the longitudinal Young's modulus.

Next we try to understand what happens when we add the fiber in terms of the load carrying capacity of the composite and in terms of stress strain relationship of a composite.

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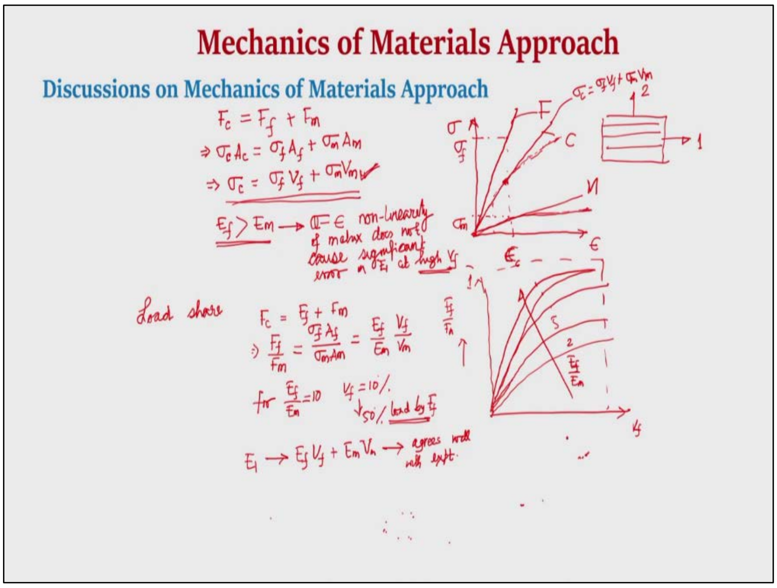

Suppose the load F_c applied along direction 1 is shared by the fiber (F_f) and the matrix (F_m). So,

$$
\begin{aligned}
&F_c = F_f + F_m \\
&\Rightarrow \sigma_c A_c = \sigma_f A_f + \sigma_m A_m \\
&\Rightarrow \sigma_c = \sigma_f (A_f / A_c) + \sigma_m (A_m / A_c) \\
&\Rightarrow \sigma_c = \sigma_f V_f + \sigma_m V_m
\end{aligned}
$$

where A_c, A_f and A_m are the cross-sectional areas of the composite, fiber and the matrix respectively and σ_c, σ_f and σ_m are the stress in the composite, fiber and the matrix respectively. For the considered RVE (ref Fig.), $A_f/A_c = V_f$.

Suppose we have the stress strain relationship for fiber and the stress strain relationship for the matrix. Now for a given volume fraction and at a given strain suppose ε_c we know what is σ_f. Similarly for a given ε_c we know what is σ_m, the matrix stress. Therefore, knowing the volume fraction, V_f we also know what is σ_c using $\sigma_c = \sigma_f V_f + \sigma_m V_m$.

That means the stress in the composite corresponding to a given strain could be obtained and thus we could obtain the stress strain curve for the composite for a given volume fraction.

Now in this in the micro mechanics approach while determining E_1 it was assumed that the matrix and the fiber behaves linearly elastic and obeys Hooke's law. Now suppose the matrix does not obey Hooke's law i.e., the matrix is non-linear beyond certain strain value as shown in the Fig. Now in deriving $\sigma_c = \sigma_f V_f + \sigma_m V_m$ we still did not put the assumptions of linearity. Only when we write the stresses in terms of strains and Young's modulus we put the assumptions of linearity.

Now in this in this expression the assumption of linearity is still not use and therefore we can still use this to obtain the composite stress strain curve using the matrix stress strain curve which is non-linear. Only thing is the curve for the composite stress strain will also be non-linear. But knowing that E_f is far greater than E_m, the influence of the non-linearity of the polymer matrix will not be significant in determination of the stress strain curve of the composite. Also, at higher volume fraction it is dominated by the fiber therefore E_m or stress strain non-linearity of matrix does not cause significant error in E_1 especially at high volume fraction. Therefore, many times it is not considered, but fiber behaves like linear till its failure. But matrix may be having nonlinearly but it does not have much influence on the determination of E_1 using mechanics of material approach.

The next thing is the load share.

$$
\begin{aligned}
& F_c = F_f + F_m \\
& \Rightarrow F_f / F_m = (\sigma_f A_f)/(\sigma_m A_m) \\
& \Rightarrow F_f / F_m = (\varepsilon_f E_f A_f)/(\varepsilon_m E_m A_m) \\
& \Rightarrow F_f / F_m = (E_f / E_m)(V_f / V_m) \quad [\text{since}\, \varepsilon_f = \varepsilon_m]
\end{aligned}
$$

E_f is generally much higher compared to that of the matrix. Therefore, say for $E_f/E_m = 10$ and say $V_f = 10\%$, this leads to $F_f/F_m = 1$. This means that addition of 10% of fiber actually leads to 50% of the load shared by fiber. Therefore, it is significant.

If we plot this, the volume fraction versus F_f/F_m for different E_f/E_m as shown in the Fig., it could be seen that as volume fraction increases, the load share of the fiber increases for a given volume fraction. As E_f/E_m increases the load share also increases. Therefore, E_1 is actually a fiber dominated property.

And it is observed that the predicted E_1 by agrees well with experimental observations.

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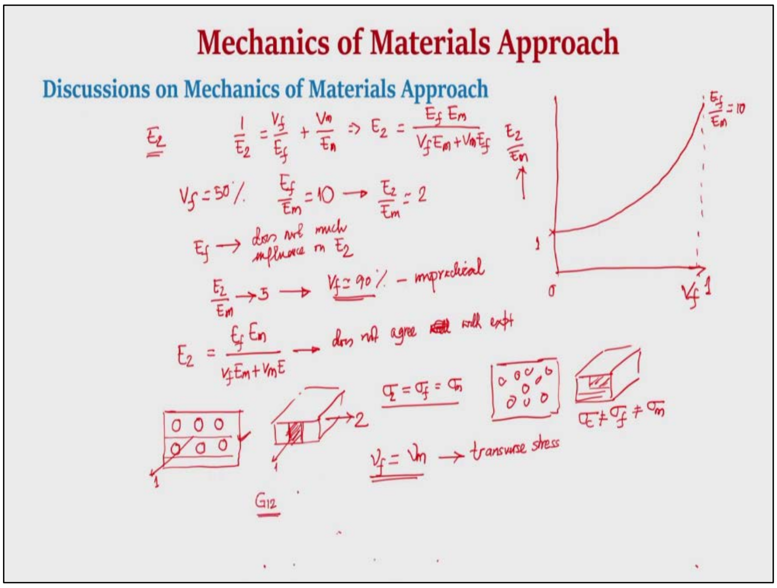

Next let us see then influence on E_2. We know

$$1/E_2 = V_f/E_f + V_m/E_m$$
$$\Rightarrow E_2 = E_f E_m / (V_f E_m + V_m E_f)$$
$$\Rightarrow E_2/E_m = E_f / (V_f E_m + V_m E_f)$$

Now if we plot this if we plot this E_2/E_m as a function of V_f as shown in the Fig.

So, for V_f =1 this becomes $E_2/E_m = E_f/E_m$. Suppose for a given V_f =50% for a given E_f/E_m = 10, E_2/E_m =1.88. That means adding 50% fiber leads to an increase in the matrix modulus only by 2 times. Now if you compare this with longitudinal modulus, adding only 10% of fiber leads to almost 90% increase in the matrix Young's modulus. So, E_f does not have much influence on E_2. To achieve a E_2/E_m =5, we need almost 90% fiber volume fraction which is impractical. 90% fiber volume fraction is not achievable as in that case there will be not much space left for waiting of the fibers and the fibers will touch each leading to imperfect adhesion of the fiber in the matrix. Therefore, transverse Young's modulus E_2 is not influenced by the modulus of the fiber as well as the volume fraction of the fiber. It is actually a matrix dominated property.

It is observed that the expression for E_2 from the mechanics of material approach does not agree well with experimental results. Now the reason for this is that in the RVE considered, it was assumed that the fibers in the thickness directions are regularly spaced as shown in the Fig. and the RVE is simplified.

But in actual case the fibers will be randomly spaced and it is not stacked as it is shown here. Therefore, actually the stress in direction 2 may not be uniform but it is shared ie. the stress in the composite is not equal to the stress in the matrix and stress in the fiber but is actually shared. That is the reason for the discrepancy.

And the second reason is the mismatch in Poisson's ratio of the fiber and the matrix. So, because of this there will be stresses induced in the fiber and the matrix. So, that is another reason which is not taken into account in this simplified RVE therefore the transverse Young's modulus which is obtained from mechanics of materials approach does not actually predict results accurately and it does not match with the experimental observations. On the other hand, the E_1 agrees well with the experimental observations.

Same is the reason that even G_{12} which is a matrix dominated property and also does not agree well the experimental observations. The mechanics of material approach actually gives us simplified simple relations where the modulus of the composite could be obtained from the corresponding modulus of the fiber and the matrix and the relative proportions.

However, except the longitudinal Young's modulus E_1 which actually agrees well with the experimental observations, the matrix dominated properties like transverse Young's modulus E_2 and the shear modulus G_{12} actually do not agree with the experimental results.

So, in summarizing, the mechanics of material approach could provide us a simplified relationship for E_1, E_2, ν_{12} and G_{12}. However, the predictions of E_2 and G_{12} by mechanics of material approach do not agree with the experimental observations.