Evolutionary Computation for Single and Multi-Objective Optimization

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Lecture - 20
Classical Multi-Objective Optimization Methods

Welcome to the session on Classical Multi-Objective Optimization Method. As of now, we have discussed about the introduction of multi-objective optimization in which we discuss the concept of dominance, the Pareto-optimality. We also discussed approaches to multi-objective optimization. In this particular session, we will be focusing on some of the methods that can be used to solve multi-objective optimization problems.

Now, here we refer these method as classical methods, it is only because we want to differentiate these methods with EC techniques. Also, these methods are relatively older or we can say EC techniques are relatively newer and therefore, we are calling this method as a classical optimization method.

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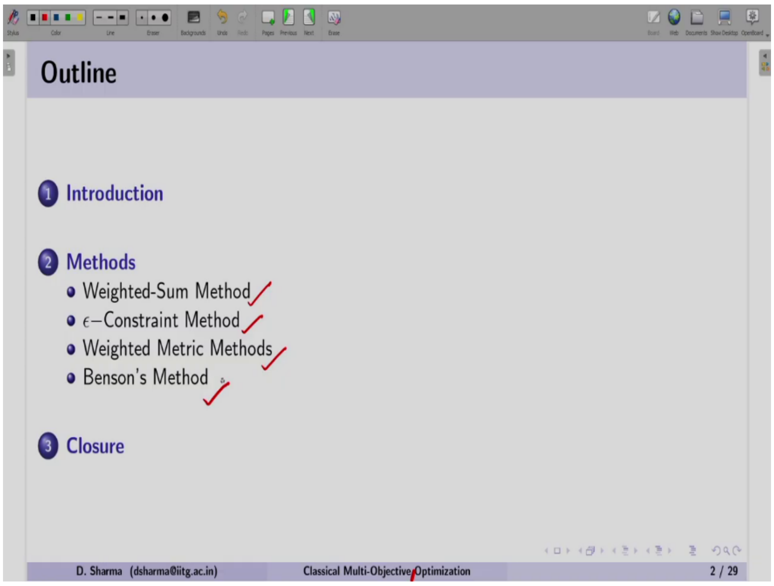

In this particular session, we will be going through the introduction and thereafter we will discuss few methods. The method includes weighted-sum method, epsilon constraint method, weighted metric method, and Benson method. Although, there is a large class of methods that can be considered as a classical methods, we will be focusing on these 4 methods. Thereafter, we will close this session.

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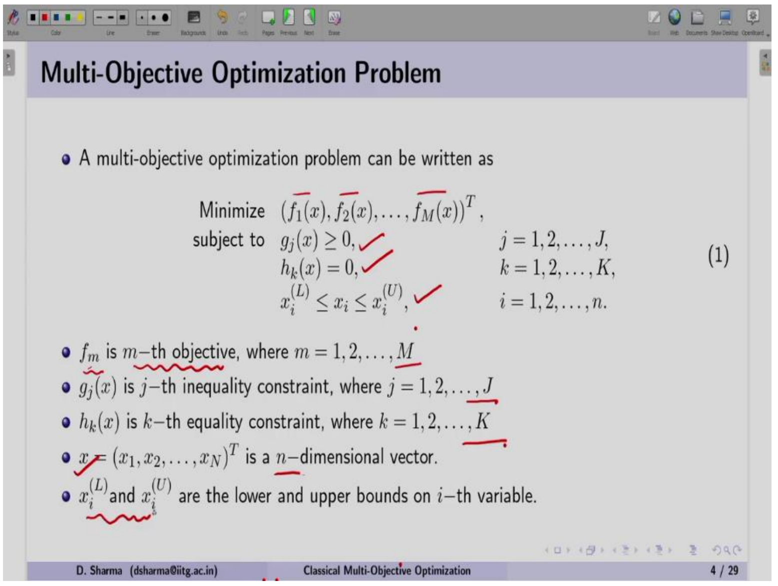

$$Minimize \quad (f_1(x), f_2(x), \ldots, f_M(x))^T$$

$$subject\ to \quad g_j(x) \geq 0, \qquad j = 1,2, \ldots, J,$$

$$h_k(x) = 0, \qquad k = 1,2, \ldots, K$$

$$x_i^{(L)} \leq x_i \leq x_i^{(U)}, \qquad i = 1,2, \ldots, n$$

Let us begin with introduction. As we remember a multi-objective optimization problem can be written as, as we can look into the equation number 1, we want to minimize the objective function. So, we have multiple functions here. Now, in this particular case, we have written all the functions as a vector.

So, we can consider we have a vector of objective function. This particular function is subjected to inequality constraint, equality constraint, and we have a variable bounds. Now, here when we are referring any f m, it means that we are referring to the m-th objective in the objective vector and the size of the objective vector is capital M.

Similarly, we can have inequality constraint that is up to J, and we can have equality constraint up to capital K. The variable vector x, it is a column vector having a dimension n. Similarly, we have bounds that is lower and upper bound on each decision variable.

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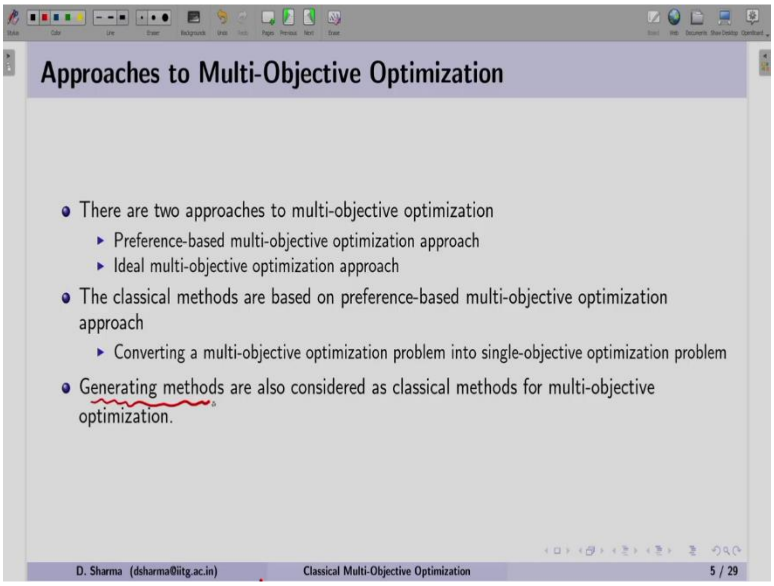

We know that there are two approaches that can be used to solve multi-objective optimization problem. One is called preference-based approach; another is called ideal multi-objective optimization approach. As we can see here the classical method are based on preference-based multi-objective optimization method in which we generally convert a multi-objective optimization problem into single-objective optimization problem.

There are other methods as well, such as generating methods. These methods can also be considered as classical methods for multi-objective optimization. Since, the classical methods are based on the preference-based approach. So, let us have a recap of what we mean by preference-based approach.

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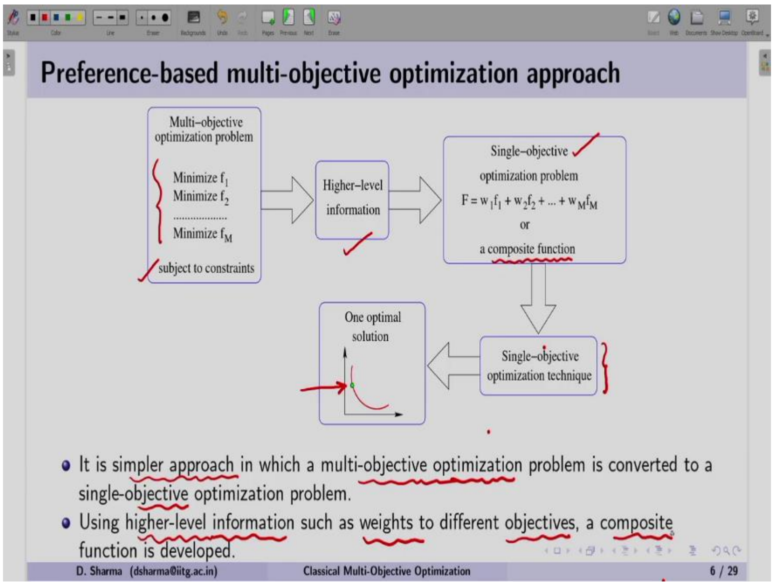

So, if we look at here, suppose we have M number of objectives all of them are conflicting in nature and these objectives are subjected to constraints. Thereafter, we use some higher level information that information will help us to convert this multi-objective optimization problem into single objective optimization or we can make some other composite functions that can be solved.

Now, since the problem is single objective optimization, we can use any single objective optimization technique to solve them. And finally, we are going to get one optimum solution as we can see here and this optimum solution will be one of the Pareto-optimal solution. This approach is found to be simple and why because we are converting all the objectives into a single objective.

Since, we have to convert into a single objective we need higher level information as we discussed, and one of the way is called weights that when that can be used with the different objectives and we can make a composite function. With this introduction let us understand these classical methods for multi-objective optimization. We will start our discussion with weighted-sum approach.

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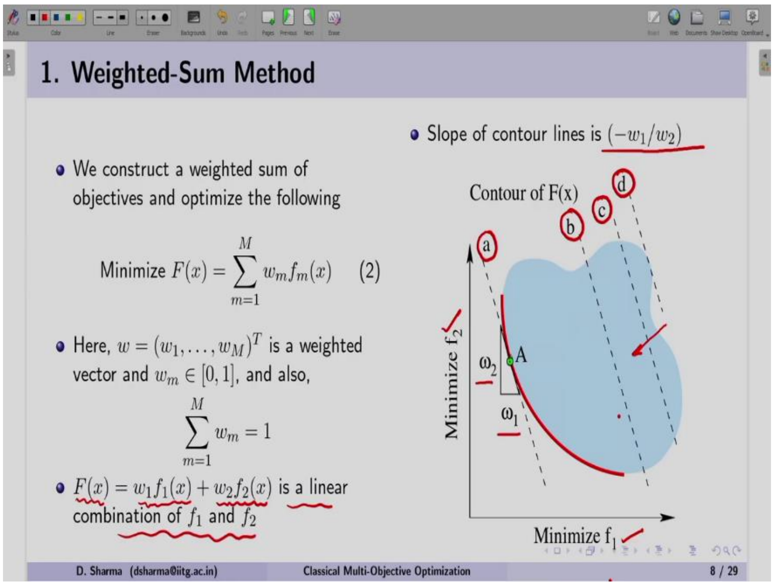

As it can be seen here the weighted sum approach in which we will be making a composite function by adding the weight into the objective function and then we take a summation over all the objective. Here these w's for every objective will constitute a weight vector which is generally written as omega.

$$Minimize\ \ F(x) = \sum_{m=1}^{M} w\_m\, f_m(x)$$

This omega is a weight vector and the important point is that all the weight vectors that is for each objective that should take the value between 0 to 1. Moreover, the summation of the weights should be equals to 1. So, in this case then we will be using the weighted sum method we will be multiplying the weight into the objective function, we will be taking this summation.

However, we have to be careful or we have to taken into an account that the omega values for each objective should lie between 0 to 1 and the summation of all omegas should be 1. Let us take a case of two objective problem now, in this case our composite function F x can be written as omega 1 f 1, similarly omega 2 f 2. Looking at this particular equation we can see it is a linear combination of f 1 and f 2.

If we look into the figure on the right-hand side here we want to minimize the f 1, minimize f 2, in this case since these are the linear combination. So, the contours which are represented by a, b, c, and d, so, these are the contours of the composite function f. Now, here we can see that a specific value of say omega 1 and omega 2 will decide the slope of the contour as we can see on the top.

So, this particular slope will be telling us that for example, we want to minimize the F x, so these contours will be moving as we can see here and finally, the optimal solution for the corresponding omega that will be A. And from this particular figure what we can understand that if we change the value of omega 1 and omega 2, in this case, we are going to get a different slope that different slope will help us to find out the other solution on the Pareto-optimal front.

So, in any case, we are, we want to generate multiple solutions on the Pareto-optimal front then we need to take different sets of omega 1 and omega 2 that will help us to find those points. Now, since our problem can have multiple constraints as well, so the modified multi-objective optimization problem can be written into the single objective form.

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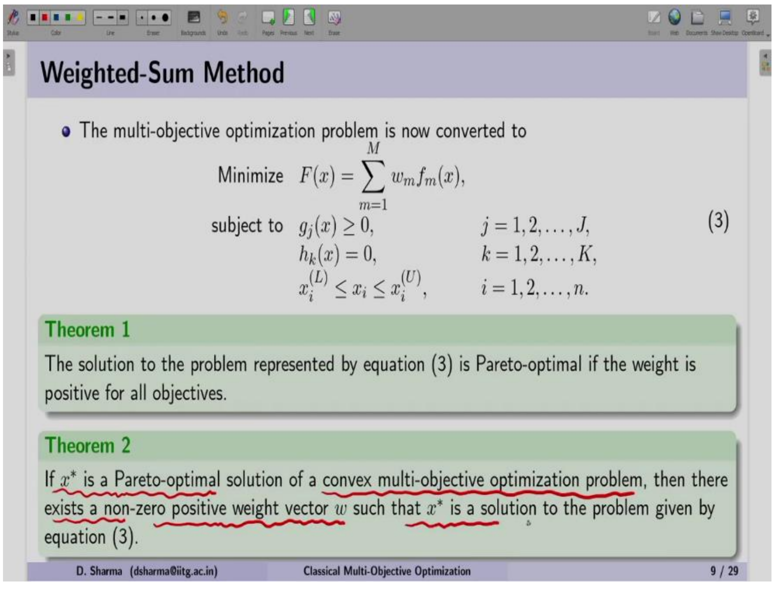

$$Minimize\ \ F(x) = \sum_{m=1}^{M} w\_m\ f_m(x)$$

$$subject\ to \quad g_j(x) \geq\ 0, \quad j = 1,2, \ldots\ , J,$$

$$h_k(x)\ =\ 0, \quad k = 1,2, \ldots, K$$

$$x_i^{(L)} \leq x_i \leq x_i^{(U)}, \qquad i = 1,2, \dots, n$$

We can see in equation number 3, that we want to minimize the composite function which is given as capital F x and while minimizing this we also have to take care of the constraints as well as the variable bound. Let us discuss theorem number 1, in this particular theorem, it says that the solution to the problem represented by equation 3 is a Pareto-optimal, if the weight is positive for all objective.

So, as we can see in the equation number 3, this is the modified single objective optimization problem. If we are going to have all the weights positive, then it is going to give us a one optimum solution. We also have theorem 2, in which if x star is the Pareto-optimal solution of a convex multi-objective optimization problem, then there exist a non-zero positive weight vector w such that x star is the solution to the problem given by equation number 3.

So, what we can understand from the theorem number 1, that if we are going to take positive value of omega or we have this omega vector, if we are going to have the positive value that will be corresponding to one optimal solution. However, in the theorem 2, it says that if the problem is a convex problem, then for every Pareto-optimal solution there is a non-zero omega vector that can help us to find that particular solution.

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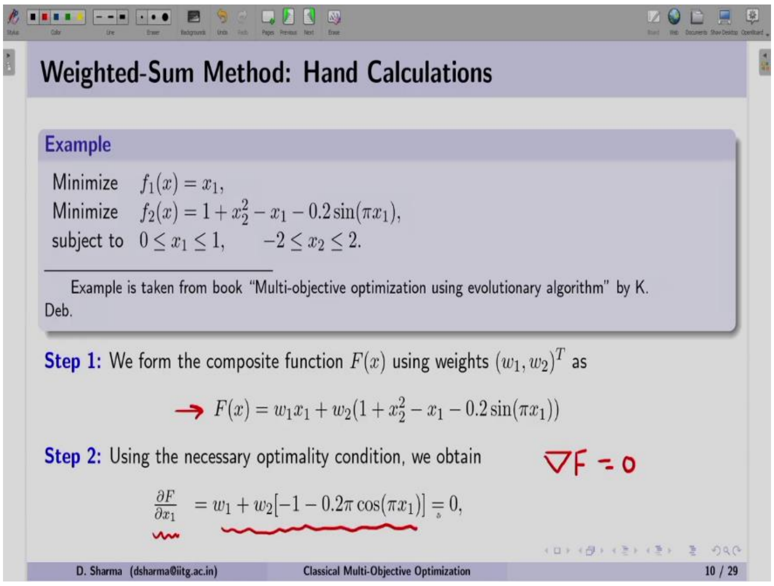

$$Minimize \quad f_1(x) = x_1$$

$$Minimize \quad f_2(x) = 1 + x_1 + x_2^2 - 0.2 \sin(\pi x_1)$$

$$bounds \qquad 0 \leq x_1 \leq 1 \quad and -2 \leq x_2 \leq 2$$

$$F(x) = w_1 x_1 + w_2\left(1 + x_2^2 - x_1 - 0.2 \; sin(\pi \, x_1)\right)$$

$$\frac{\partial F}{\partial x_1} = w_1 + w_2 \left[-1 - 0.2 \, \pi \; cos(\pi \, x_1)\right] = 0,$$

Now, having understanding about the weighted sum method let us perform some hand calculation. For this, we have taken a simple case in which we have just two objectives. As we can see that we want to minimize f 1, we want to minimize f 2 and both the variables which are x 1 and x 2 they are lying between 0 to 1 and minus 2 to plus 2.

This particular example, we have taken from the book Multi-objective optimization using evolutionary algorithm by Professor Deb. So, what we can do here is that the step 1 suggest we have to form the composite function F x using weights which is w 1 and w 2.

So, since we know the form, we can directly write the capital F x is equals to omega 1 x 1 omega 2 f of 2. So, since the equation of f 2 is given on the top, we construct this objective function. Now, as we know that the current problem which we have converted into the single objective optimization, it is a unconstrained problem. So, for this particular problem which is unconstrained and single objective, we can use our optimality conditions.

The first optimality condition is the necessary optimality condition that we can find it by making the gradient of the function equals to 0. In this case, the first component will become dou F by dou x 1. So, basically, we are differentiating this capital F with respect to x 1 only. And this will give us the equation as we can see here and we have to put it equals to 0.

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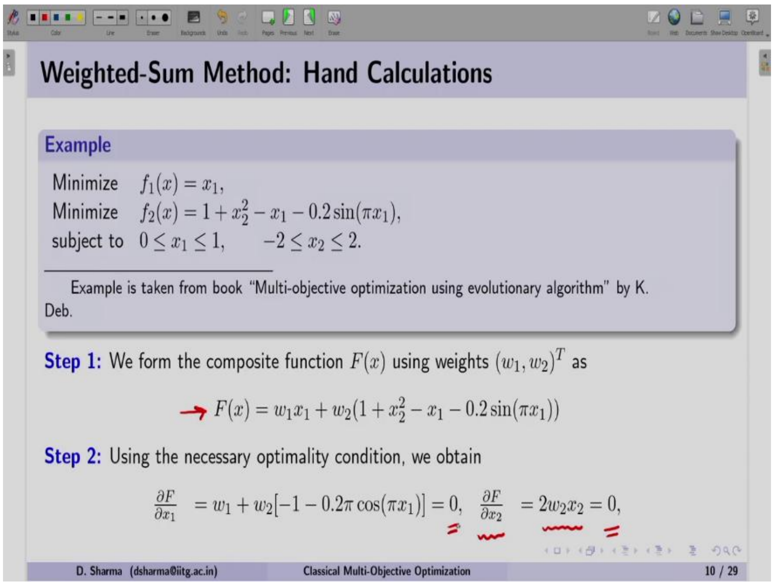

The 2nd component we will get it when we will again differentiate the capital F with respect to x 2 and this coming out to be 2 omega 2 x 2 that is also equals to 0. So, both the components we are making equal to 0, so that we can find the value of x 1 and x 2.

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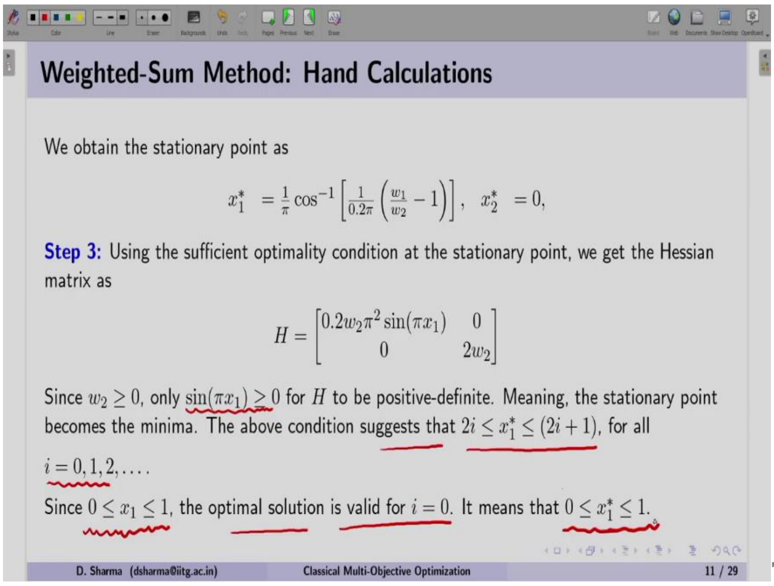

Type equation here.

$$x_1^* = \frac{1}{\pi}\ cos^{-1}\left[\frac{1}{0.2\ \pi}\left(\frac{w_1}{w_2} - 1\right)\right]$$

$$x_2^* = 0,$$

When we equate both the component of the gradient of the capital F equals to 0, we will get this stationary point. As we can see here, the stationary point x 1 star is given and x 2 star also given. Now, in this, for this particular problem x 2 star is always 0 and x 1 star has the value that depends on omega 1 and omega 2.

Since, we know that the stationary point can be maxima or minima or it can be an inflection point, so we need to find out the Hessian. So, using this sufficient optimality condition, we find the Hessian of Hessian matrix, since it is a two-variable problem, so the Hessian matrix is 2 by 2. Since, the matrix is simple we will be looking into the principal components only.

Now, here to H to be the positive definite, so we can see that the component which is given as 2 omega 2 that should be greater than and equals to 0. By putting this condition that omega 2 should be greater than and equals to 0, the another principal component should also be greater than 0.

If we take only the function with respect to x 1, we will say that the sin of pi of x 1 should be greater than 0. Now, by looking into the condition and we want to make H should be positive definite, we can say that the point the stationary point which we get at the top which is x 1 star and x 2 star becomes the minimum point.

Now, here if we look at this particular condition when we have sin of pi x 1 greater than equals to 0, this particular condition suggest that the x i should lie between 2 i and 2 i plus 1 for all the values of i starting from 0 to 1, but at the same time we also know that x 1 should lie between 0 to 1, therefore, the optimum solution for the given problem is only valid when we take i equals to 0.

So, in this case we can see that the optimum solution will be lying when we say that x 1 star is lying between 0 to 1. So, for the given problem the range of x 1 is given whenever it is lying between 0 to 1. It is going to give us an Pareto-optimal solution along with that x 2 star should be 0.

In the previous equation, we found that x 1 is calculated with respect to the values of omega 1 and omega 2 and x 2 star was 0. So, in this case, we can see that what value of omega 1 and omega 2 can give us the extreme solution on the Pareto-optimal front. So, when we say extreme solution meaning that the solution which is corresponding to the minimum of f 1, on the Pareto-optimal front, similarly the other solution which is minimum of f 2 on the Pareto-optimal solution.

So, let us identify these two extreme solutions as well as what are the corresponding values of omega 1 and omega 2.

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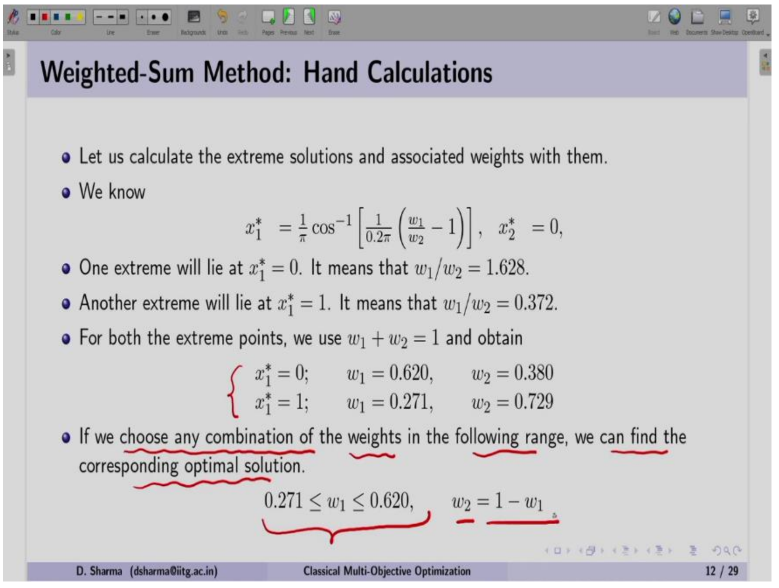

$$x_1^* = \frac{1}{\pi} cos^{-1}\left[\frac{1}{0.2\,\pi}\left(\frac{w_1}{w_2} - 1\right)\right], \qquad x_2^* = 0$$

$$x_1^* = 0; \quad w_1 = 0.620, \quad w_2 = 0.380$$

$$x_1^* = 1; \quad w_1 = 0.271, \quad w_2 = 0.729$$

$$0.271 \leq w_1 \leq 0.620, \qquad w_2 = 1 - w_1$$

So, in the previous slide we understood that x 1 star depends on omega 1 and omega 2 and x 2 star should always be 0. Now, since, one of the extreme will be lying on x 1 is equals to 0, if we equate this particular condition into the equation here, so what we can find is that cos 0 is 1, by equating it we will get this omega 1 by omega 2 is 1.628.

The another extreme solution will be lying at x 1 star equals to 1. When we are putting this condition in the given equation here we can find the value of omega 1 and omega 2 will become 0.372. So, here from these two expression, we can find that what could be the range of omega 1 and omega 2.

So, for both extreme points we can use omega 1 plus omega 2 equals to 1. Meaning that 1 will be 1 will be independent variable and omega 2 can be found in terms of omega 1. So, let us take x 1 star equals to 0. In this case, since we are using omega 1 plus omega 2 equals to 1 and the ratio which is given on the top by using these two equation we can get omega is equals to 0.620 and omega 2 is 0.380.

Similarly, for another extreme solution where we write x 1 star as 1, and using the two equation as omega 1 plus omega 2 0 and omega 1 divided by omega 2 as 0.372, it will give us omega 1 as 0.271 and omega 2 as 0.729. So, if we choose any combination of the weights in the following range, we can find the corresponding optimal solution.

Meaning that from the above condition as we can see x 1 star equal to 0 and x 1 star equals to 1 that will give us this particular range that when we are going to change omega 1 from 0.271 to 0.620. And we can find the value of omega 2 by substituting into the equation. The solution corresponding to these different values of omega within the range can find the Pareto-optimal solution for the given problem.

Now, since this particular method is found to be quite simple and straight forward, and we can always use our concepts for single objective optimization this method offers various advantages. So, let us look at the advantages first and then we will discuss the disadvantages.

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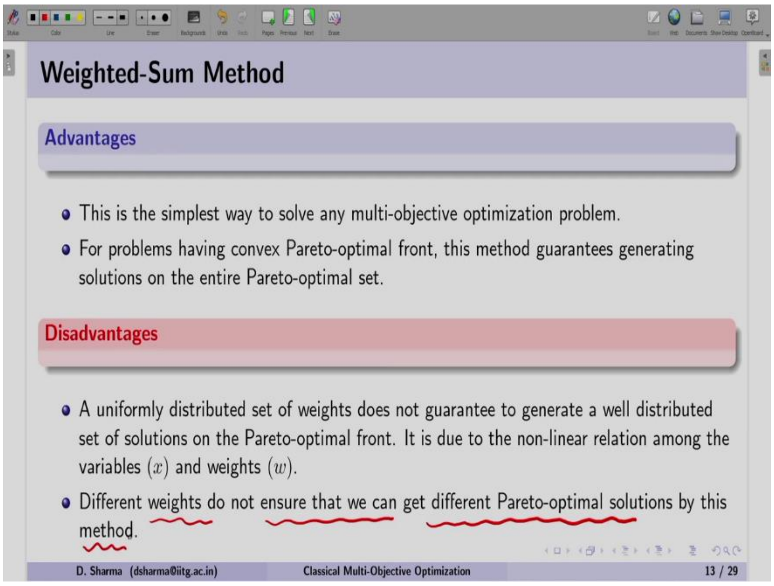

So, the first very first advantage is this is the simplest way to solve any multi-objective optimization problem. The problem, which we have solved that help us to understand that converting multi-objective problem into single objective, it is actually reducing the complexity of the problem and making into a single objective.

Second advantage could be that for the problems having convex Pareto-optimal front. This method guarantees generating solution on the entire Pareto-optimal set. So, we discuss the theorem number 2 in which these if we are solving a convex problem. Convex problems mean the problem having convex Pareto-optimal front.

So, when we are choosing the value of omega 1 and omega 2, it will be corresponding to one of the Pareto-optimal solution on the front. Now, let us discuss what could be the disadvantages with this particular method. So, first disadvantage is a uniformly distributed set of the weights does not guarantee to generate a well distributed set of solution on the Pareto-optimal front. It is due to the non-linear relation among the variables and the weights.

Second is, different weights do not ensure that we get the different Pareto-optimal solution by this method. It means that the omega 1, omega 2 value for a one particular set of solution and another omega 1, omega 2, omega 3 values for another solution, if we run it our optimization algorithm, they may not give you two distinct solution. Sometimes this solution can converge to a one Pareto-optimal solution.

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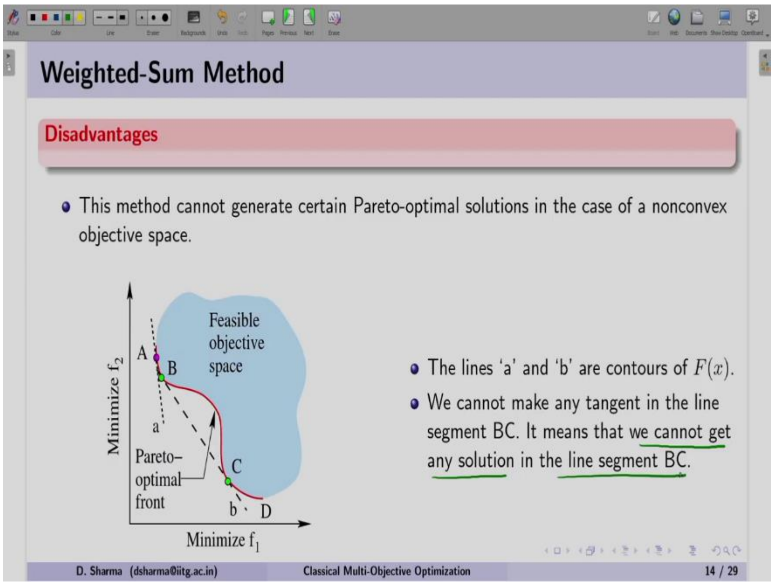

Let us discuss the third which is an important disadvantage for the given method that this method cannot generate certain Pareto-optimal solution in case of non-convex objective space. So, let us understand that. In the figure, we can see that the red line is represented by the Pareto-optimal front. Now, if we look at the point A, we are going to get a contour as written by the small a value.

Similarly, if you find the contour at the point b then we are going to get another contour. Now, the problem is as and when we are trying to find out any contour that is tangent to the feasible objective space, for example, such as at the middle now. Now, at this particular point we cannot find the tangent and therefore, if inside this particular segment BC, we cannot get any solution in the line segment BC.

Now, if we look at this particular Pareto front, in this example only the segment BC is the non-convex and the other portion of the Pareto-optimal front is the convex one. So, the different weights of omega 1 and omega 2, can generate the solution other than the segment BC.

So, therefore, here as and when we are solving a problem which is having a non-convex Pareto-optimal front, this method cannot generate the optimal solution in that particular segment which could be the important limitation, since many problems could have non-convex Pareto-optimal front.

As of now, we have discussed the very simplest method and we found that this particular method we cannot use for non-convex Pareto-optimal front. So, in order to eliminate this particular problem, we have an important method which is called epsilon constraint method.

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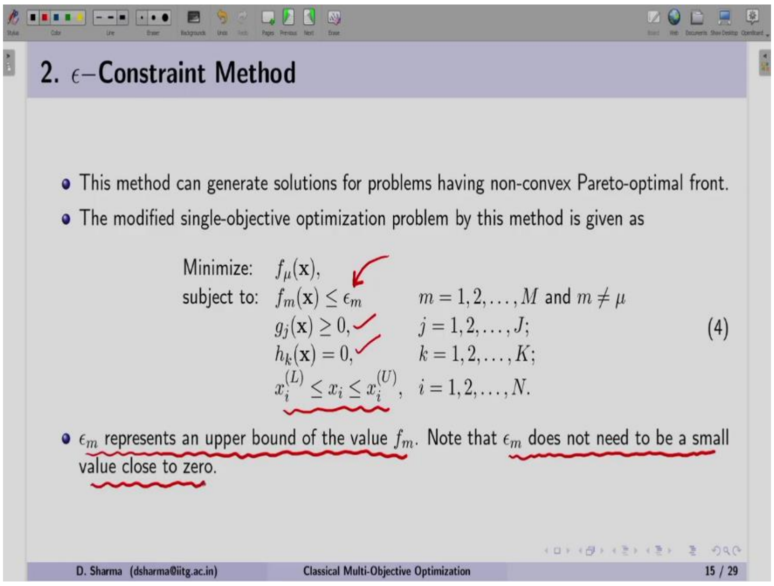

$$Minimize: \quad f_\mu(x)$$

$$subject\ to: \quad f_m(x) \leq \epsilon_m \qquad m = 1,2,\ldots,M \text{ and } m \neq \mu$$

$$g_j(x) \geq 0, \qquad j = 1,2,\ldots,J$$

$$h_k(x) = 0, \qquad k = 1,2,\ldots,K$$

$$x_i^{(L)} \leq x_i \leq x_i^{(U)}, \qquad i = 1,2,\ldots,n.$$

So, what is this? So, in this particular epsilon-constraint method, we can generate solutions for the problems having non-convex Pareto-optimal front. This particular method converts the multi-objective optimization problem by considering only one objective to minimize and rest of the objectives we will be having some constraint.

Now, as can be seen here that the objectives from m equal to 1 to capital M and this small m should not be the same as mu. So, as we can see that we have chosen f mu to minimize, so this particular objective we are removing, but for rest of the objectives we are putting a constraint here. Apart from that we can have the constraints like g j of X these are the original constraint of the problem as well as the equality constraints and the a variable bound.

So, here if we looked at this particular value called epsilon m, so generally we use this epsilon m value as a small value, but in the case of epsilon constraint method we are using or we are representing epsilon m value as an upper limit on the objective function which can be a bigger value. And therefore, we can see that epsilon m represents an upper bound of the value f m. As I mentioned earlier, epsilon m does not need to be a small value close to 0. So, this we have to be careful.

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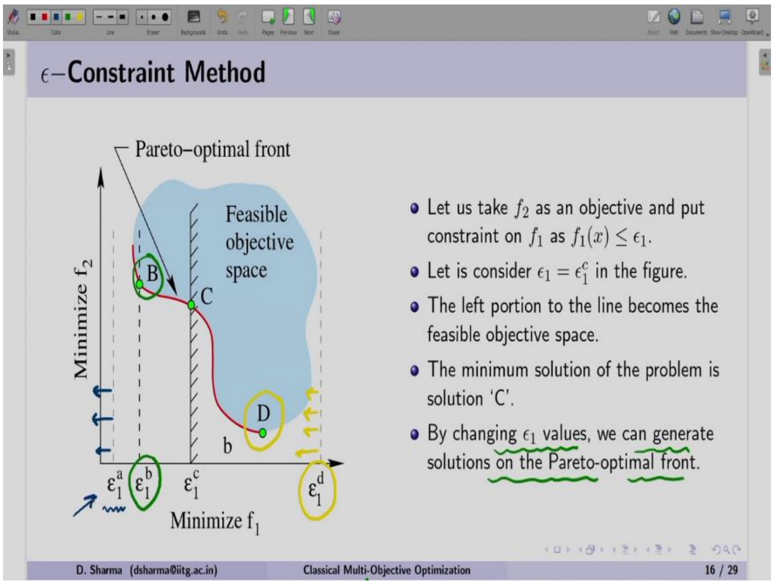

Now, let us understand this method graphically. So, we can see the figure on the left-hand side in which we want to minimize f 1, minimize f 2, and the red line is represented by the Pareto-optimal solution. So, for a given problem let us take f 2 as an objective function and we are going to put a constraint on f 1 as f 1 is equals to or smaller than epsilon 1.

Let us consider that epsilon 1 is epsilon e c. So, here epsilon e 1 c, we can locate on the f 1 axis. Now, looking at this particular constraint we can see that the region which is on the left-hand side, so basically this region is the feasible region and we have to find what is the optimal solution for the given problem?

Now, since we want to minimize f 2 and we know that this particular region is only the feasible region, so here when we will be minimizing f 2, we are going to get a point C, which is the minimum solution for the given problem. So, in order to get different points on the Pareto-optimal front, we can change the value of say epsilon 1 and we can generate solution on the Pareto-optimal front.

So, if we look into the figure, for example, we have chosen certain value called epsilon 1 b, when we will be putting this particular constraint and finding the minimum solution for it we are going to get a solution B. Now, let us consider another epsilon value as epsilon 1 d.

In this case, what we will find that this particular epsilon value is actually out of the feasible objective space, but when we are putting as we know that region on the left-hand side is the feasible region when we are going to minimize the f 2 objective with this constraint the solution which we are going to get is the solution D.

Let us look into the another situation now. Suppose, we have taken epsilon 1 a, now here when we are using it and we remember that the left-hand side of this line is the feasible one; when we are going to solve this problem, we know that we are not going to get any solution because all solutions are become infeasible solution.

So, from this discussion we can see that when we are taking epsilon at say point C and at point B we are going to get a different value of epsilon. In one case, when we take a point D in terms of epsilon on the left-hand side we are going to get one of the extreme solution on the Pareto front.

On the other hand, when we take a point A in terms of epsilon this particular situation cannot generate a solution for us because there is no feasible search space for the algorithm. Meaning that when we are deciding the epsilon value, we have to be careful what value of epsilon should be taken within the range, so that it should generate Pareto-optimal solution for the given problem.

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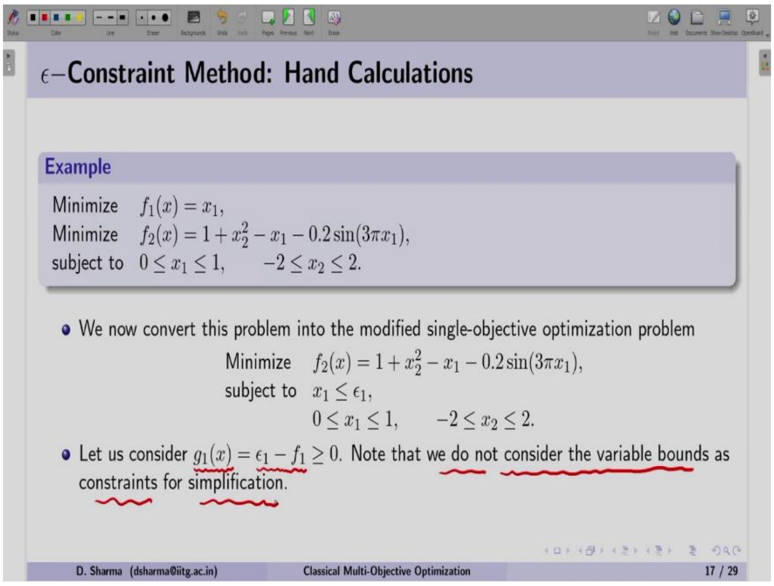

$$Minimize \quad f_1(x) = x_1$$

$$Minimize \quad f_2(x) = 1 + x_2^2 - x_1 - 0.2\, sin(3\pi\, x_1)$$

$$subject\ to \quad 0 \le x_1 \le 1, \quad -2 \le x_2 \le 2.$$

Now, let us solve a problem using the epsilon constraint method. We are considering the same problem which we have solved earlier. We want to minimize f 1, we want to minimize f 2 and this problem has just two variable bounds which are x 1 will be lying between 0 to 1 and x 2 will be lying between minus 2 to plus 2. We now have to convert this multi-objective problem into the modified single objective optimization problem.

So, let us consider that we want to minimize f 2, we are putting a constraint on f 1 which is f 1 should be smaller than epsilon 1 and we have the ranges. So, let us consider that we have just one constraint which we are representing as g 1 and we are writing rewriting this constraint as epsilon 1 minus f 1 should be greater than and equals to 0.

Here, we should note that that we are not considering the variable bounds as constraint only for the simplification. However, if we are solving some other kind of a problem, we should consider the variable bounds as our constraints.

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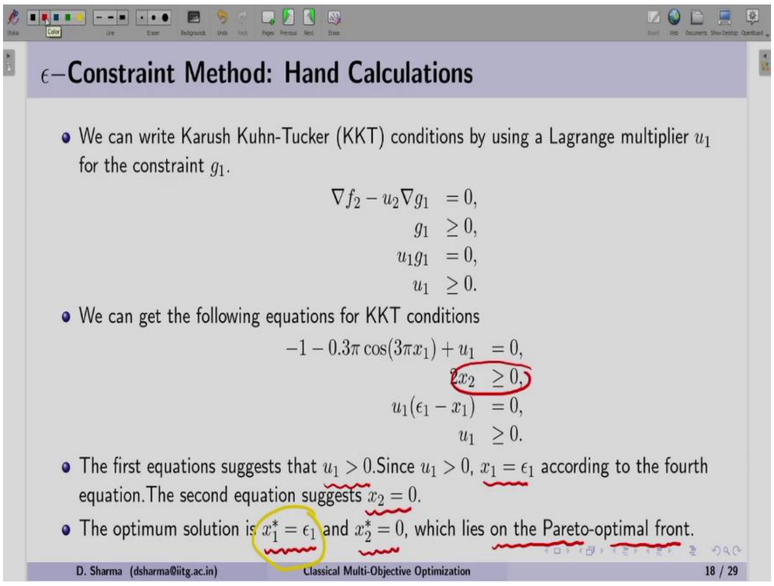

$$\nabla f_2 - u_2 \nabla g_1 = 0,$$

$$g_1 \geq 0,$$

$$u_1 g_1 = 0,$$

$$u_1 \geq 0.$$

Now, we know that when we will be solving a constraint optimization problem the optimality condition can be found using Karush Kuhn-Tucker conditions. So, these conditions are also known as KKT condition. Let us assume that we are taking u 1 as a Lagrangian multiplier for the constraint g 1.

$$-1 - 0.3\,\pi\, cos(3\pi x_1) + u_1 = 0,$$

$$2x_2 \geq 0,$$

$$u_1\,(\epsilon_1 - x_1) = 0,$$

$$u_1 \geq 0.$$

So, using the KKT condition, the first condition is the optimality condition where we will be finding the gradient of the Lagrangian function, then we have this constraint equation, third is the complementary slackness condition, and finally, the Lagrangian multiplier can take u 1 greater than an equals to 0.

By using this KKT conditions we can write the equations. So, as we can see, the first equation is representing the optimality condition, second equation is our variable bound, the third equation is the complementary slackness and the fourth equation is u 1 greater than equals to 0.

Now, if we look at the equation number 1 here. So, as we can see that this particular part will go on to the right-hand side and for any value of x 1, u 1 will always be greater than 0. Since, u 1 is going to be 0, looking at the complementary slackness condition the epsilon 1 minus x 1 should be equals to 0, meaning that x 1 equals to epsilon 1. And the second equation: Now, if you look at this particular equation now, this equation suggests that x 2 should be 0.

So, from this these KKT conditions suggests that u 1 will be greater than 0, x 1 is equals to epsilon 1 and x 2 should be 0. From these condition, we can say that the optimal solution is x 1 star is equal to epsilon 1 and x 2 star is equals to 0. And this particular point will be lying on the Pareto-optimal front. So, what is the interesting point about this particular condition is that if we change the value of epsilon 1, we are going to get different Pareto-optimal solutions.

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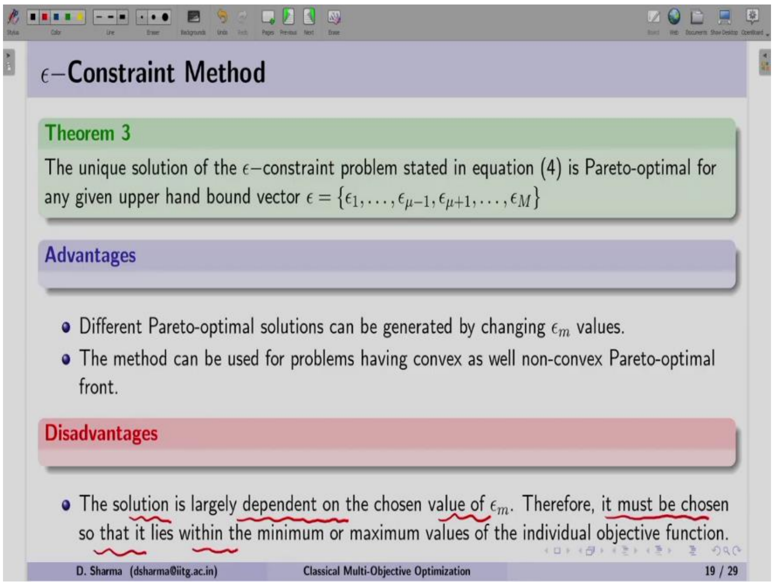

So, we have another theorem for epsilon-constraint method. This theorem says that the unique solution of the epsilon constraint problem stated in equation number 4, that is the modified single objective optimization problem using epsilon constraint method is a Pareto-optimal for any upper hand bound vector that is given as for a given value of epsilon.

So, that theorem is giving an idea that if we change the value of epsilon these epsilon values will be corresponding to different values of the Pareto-optimal solution on the front. Now, since as we have understood that this particular method is good in a way that we can solve the problems which have convex and non-convex problems. So, therefore, this method offers various advantages. So, let us go them one by one.

First is obvious that different Pareto-optimal solutions can be generated by changing the value of epsilon m that can be that can be seen from the theorem 3 as well. Similarly, the method can be used for problems having convex as well as non-convex Pareto front. From our analysis we have understood that this method is good, it can be used for any kind of multi-objective optimization problems, but there are certain disadvantages.

So, the major disadvantage is the solution is largely dependent on the chosen value of epsilon m, therefore it must be chosen, so that it lies within the minimum or the maximum value of the individual objective function value. So, as we have discussed earlier with the help of a figure that we took 4 points, 4 epsilon values corresponding to A, B, C and D. When the values are chosen such as B and C, we are going to get the Pareto-optimal solution and both of them are different.

But when we are taking other values which are out of the bound, so as we can understand from the epsilon value corresponding to the point t that in that case whenever we will be minimizing f 2, we are going to get a same solution which is D. So, in this case, if we are taking many such epsilon values which are out of the bound and in all of the cases, we are going to get the same solution.

On the other hand, if we take epsilon value corresponding to point A, then we are not going to get any solution because there is no feasible objective space. Therefore, we have to be careful when we have to use the epsilon value. So, we should know what is the range of epsilon value so that we can generate different Pareto-optimal solution on the front.

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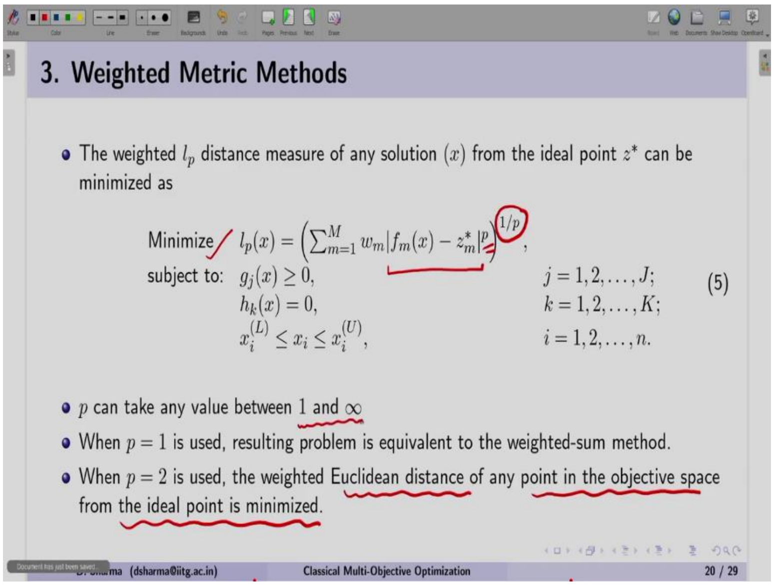

$$Minimize \quad l_p(x) = \left(\sum_{m=1}^{M} w_m \ |f_m((x)) - z_m^*|^p \ \right)^{\frac{1}{p}},$$

$$subject\ to: \ g_j((x)) \ \geq \ 0, \qquad j = 1,2, \ldots, J$$

$$h_k((x)) \ = \ 0, \qquad k \ = 1,2, \ldots, K;$$

$$x_i^{(L)} \ \leq \ x_i \ \leq \ x_i^{(U)}, \qquad i = 1,2, \ldots, n.$$

We will now discuss the weighted metric method. As we can see here the weighted l p distance that measure is that distance measure of any solution x from the ideal point z star and that we want to minimize. Now, looking at this equation, so we are writing this l p x, here we can see that we are finding the difference between the solution x with respect to the ideal point z.

And this difference is multiplied with the weight vector which is omega m and then we are taking a summation and finally, we will have 1 by p. Now, this weighted l p, l p distance this particular objective function, now we can see that we have converted multi-objective optimization problem into a composite function which is l p and this particular problem can also be subjected to the problem constraints and the variable bound.

Now, here looking at the equation number 5, the value of a p can take any value between 1 to infinite. Now, when we are changing the value from 1 to infinite, the function l p will behave differently. So, let us look into it. So, when p equals to 1 is used, the resulting problem is equivalent to weighted-sum method. So, as we can see in equation number 5, when p equals to 1 meaning that we are taking a difference between these two values and then multiplying with the omega m.

So, that is very similar, why because our ideal point will not change in our simulation, only corresponding to the value of x the objective function will change. So, therefore, it is quite similar to the weighted-sum method. So, p equals to 2 become the Euclidean distance of any point in the objective space from the ideal point and this we are going to minimize as per the weighted metric method.

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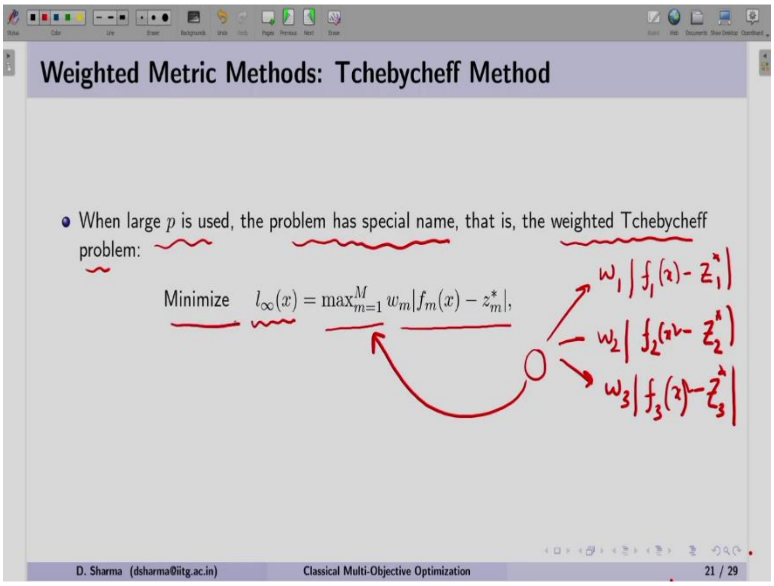

$$Minimize \quad l_\infty(x) \; = \; max_{m=1}^{M} \, w_m \, |f_m(x) - z_m^* \; |,$$

Since, p can take any value from 1 to infinite let us take a large value. So, when we take p large, the problem has a special name that is called the weighted Tchebycheff problem. So, this particular problem is now converted as, so as we can see this is l infinite x and here we have the maximum m from 1 to capital M and we are finding which has the maximum value of w m or omega m and the difference between f m minus z m star.

Meaning that suppose I have 3 objective problem, so, we will be calculating the difference for all the objective function. So, let us write it here and the third one is the third objective. Now, if we get any value, so we will be looking which particular value has the maximum value, so that particular value is taken here and this value the maximum value which we want to minimize here.

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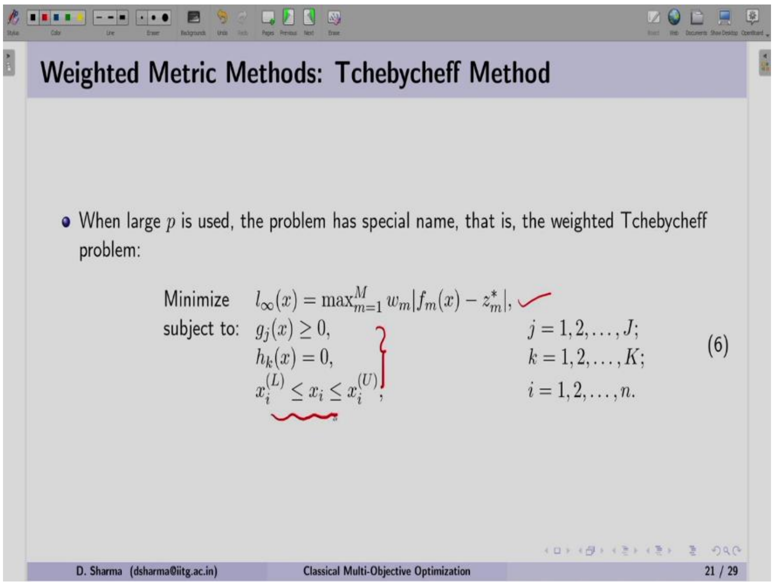

$$Minimize \quad l_\infty(x) = max_{m=1}^{M} w_m |f_m(x) - z_m^* \;|,$$

$$subject\ to: \quad g_j(x) \geq 0, \& \; j = 1,2, \ldots, J;$$

$$h_k(x) = 0, \quad k = 1,2, \ldots, K;$$

$$x_i^L \leq x\_i \leq x_i^U, \& \; i = 1,2, \ldots, n.$$

Now, as we remember that after converting the multi-objective problem into single objective problem, we could have our original constraints as inequality and equality constraint with the variable bounds.

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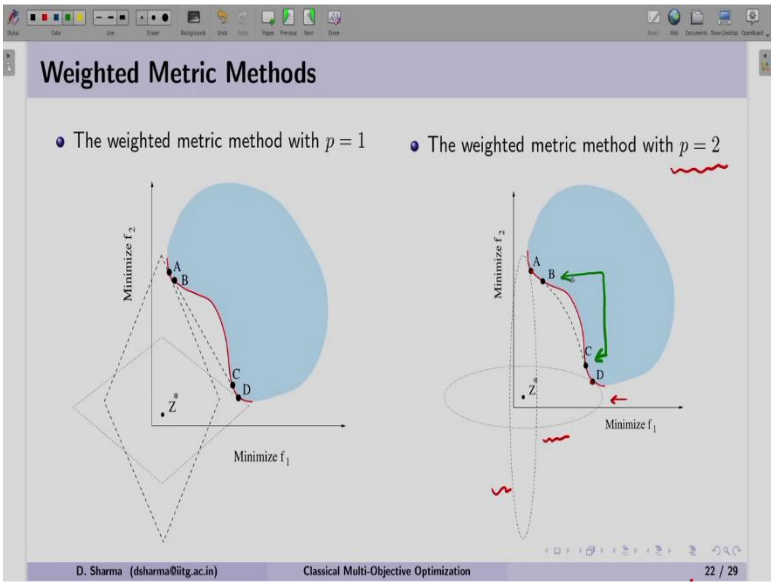

So, let us see the contours of the weighted metric method with the different values of p. Let us consider p equals to 1. Now, when we consider p equals to 1. Now, as we remember it is similar to the weighted metric method. So, therefore, as you can see the contours are like these straight lines. So, as we take a point A, we are going to get a contour as we can see in the picture. Similarly, at a point D also, if we look then we are going to get a different contour.

The important point is these contour cannot be generated between the range of or between the segment B and C. And therefore, when we keep p equals to 1, we may not able to generate any solution between segment B and C, let us take a another case where we have p equals to 2.

Now, when p equals to 2 means we are finding an Euclidean distance, the l, the metric function will become an ellipse; as of now here as we can see the contour at A is one ellipse, contour at point D is another ellipse, that are going to give us the point A and D. However, again in this particular segment B and C, we cannot generate such contour and therefore, we cannot find any point in between B and C.

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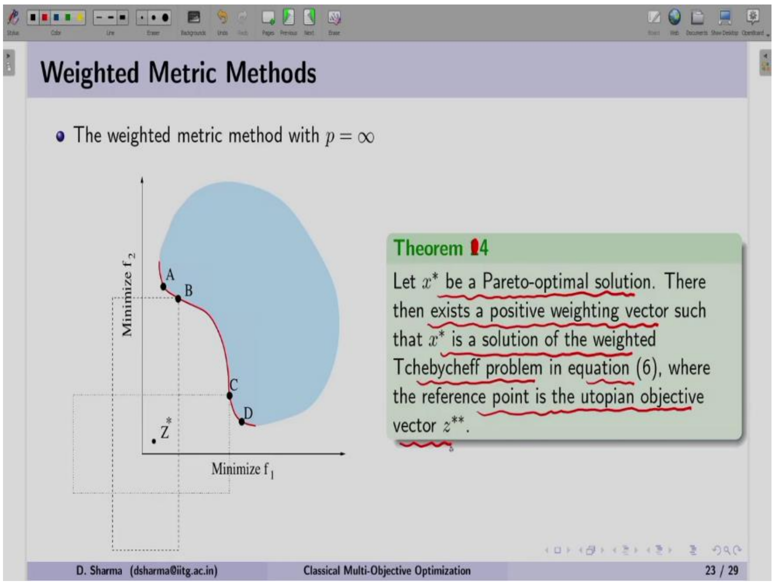

Suppose, we take a infinite value of a p, and as we remember when we consider p equals to infinite the problem or the method is referred as Tchebycheff function, the metric method with p equals to infinite the contours can be seen here.

Now, these contours, looking at these contours we can easily see that even if there is a point inside, we can have such kind of a rectangular box meaning that different values of omega can help us to find out the solution between the segment B and C.

So, from our previous discussion, we understand that we can generate any solution from the one extremes to the point B, similarly another extreme to point C which are easy to find. However, when we talk about any point between in the segment B and C, it cannot be generated. But with the weighted metric method having p equals to infinite, we can even generate a solution on the non-convex Pareto front.

So, based on that we have our fourth theorem right now, it says that let x star be the Pareto-optimal solution then there exist a positive weight vector such that x star is the solution of weighted Tchebycheff problem in equation 6, where the reference point is the utopian objective vector z double star. So, as we remember the difference between the ideal point and the utopian point is that when we minimize the objective function independently, that will give us the ideal point.

When we are subtracting the epsilon values into the ideal point, we will get the utopian point. We have discussed this weighted matrix method as we can understand the different values of a p can generate different value different Pareto-optimal solution, especially when we take p equals to infinite value then we can generate solution on the non-convex Pareto-optimal front. So, as we can understand that there are certain advantages with this method, let us discuss those advantages one by one.

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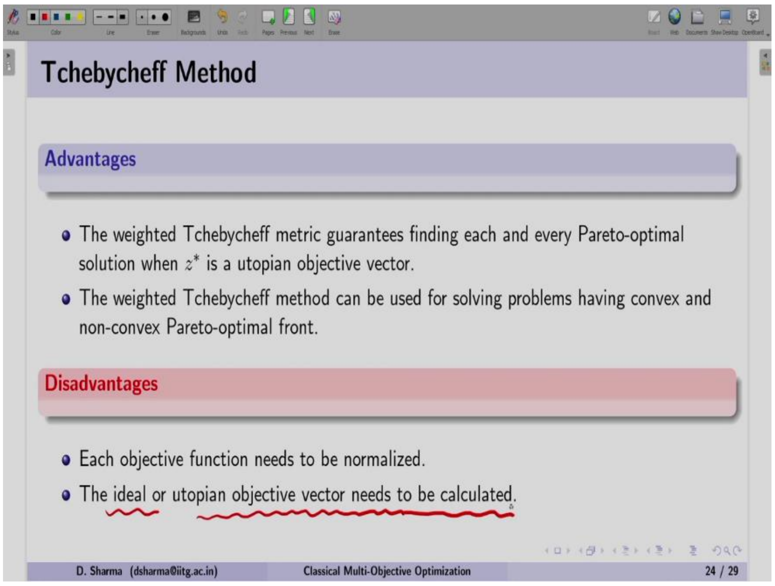

So, first advantage is the weighted Tchebycheff metric guarantees finding each and every Pareto-optimal solution when z star is an utopian point and that is what we discussed in the theorem. So, we keep on changing the weights, we will get the different Pareto-optimal solution. The weighted Tchebycheff method can be used for solving problems having both or convex or a non-convex Pareto-optimal front, which is an important property because the problem can have different nature of Pareto-optimal front.

But there are certain disadvantages with this method. So, the first major disadvantage is the objective function needs to be normalized. Second is the ideal or the utopian objective vector needs to be calculated. Now, since we have mentioned here that the objective function should be normalized this is also valid for the weighted sum method.

Since, this normalization is needed because our search should not biased towards any of the objective function, so we need normalization, meaning that we have to perform extra computation to normalize it. Similarly, the calculations are done with respect to the z star or sometimes z star star which is ideal utopian point. So, these points need to be calculated before we start our optimization.

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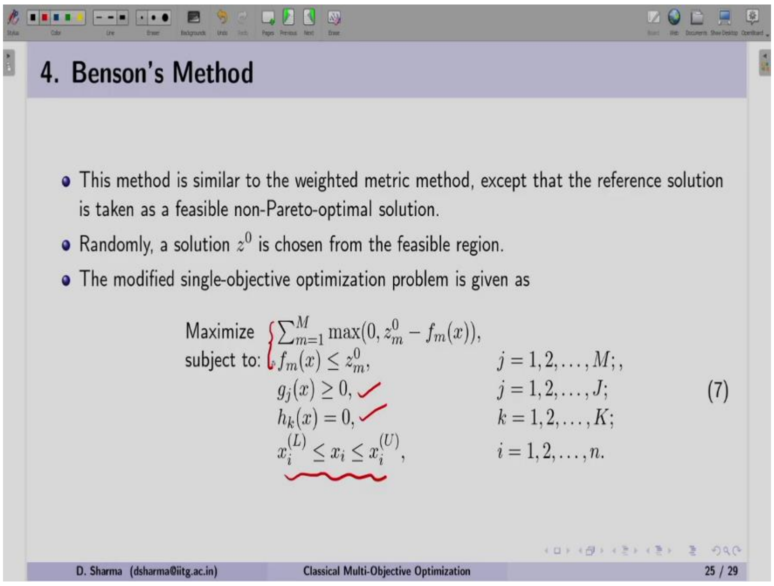

$$\text{Maximize} \quad \sum_{m=1}^{M} max(0, z_m^0 - f_m(x)),$$

$$\text{subject to:} \quad f_m(x) \leq z_m^0, \quad j = 1,2, \ldots, M;$$

$$g_j(x) \geq 0, \quad j = 1,2, \ldots, J;$$

$$h_k(x) = 0, \quad k = 1,2, \ldots, K;$$

$$x_i^{(L)} \leq x_i \leq x_i^{(U)}, \& \; i = 1,2, \ldots, n.$$

As of now we have discussed 3 methods, now let us discuss the last method as a part of this session which is Benson method. This method is similar to weighted metric method except that the reference solution is taken as a feasible non-Pareto-optimal solution, meaning that, we are going to choose a solution z 0 which should which is a feasible, but non-Pareto-optimal solution from the feasible region.

When we use Benson method the modified single objective optimization problem can be written as we can see we want to maximize. So, there is a first change we can see. Second is we want to minimize for the summation m equals to 1 to capital M and we are looking for the maximum value between 0 or the difference between the point which we have chosen minus the objective function value.

Along with that we also have another constraint such as f m of x should be smaller than and equals to z m. So, this extra constraint will also be included into the formulation. Along with that we can have problem constraints and the variable bounds. So, that can be seen here that these two changes are needed into the multi-objective optimization formulation.

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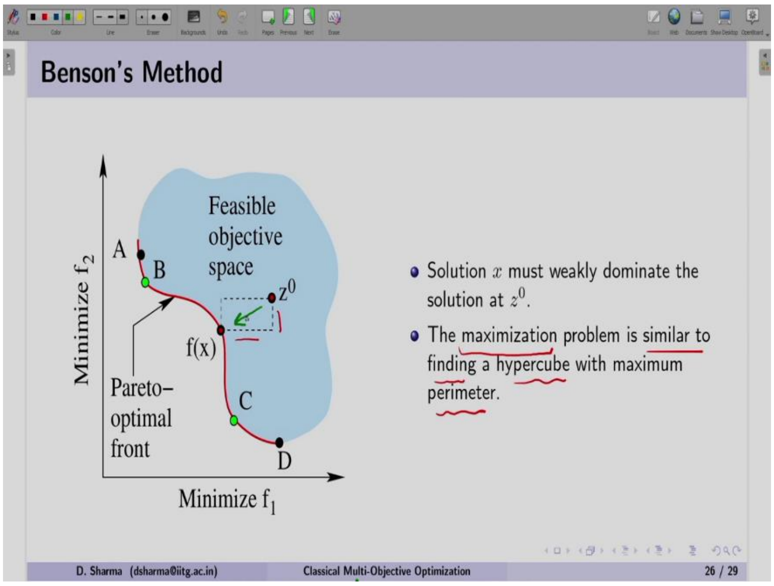

Let us understand with the help of an of a graphical example here. Here we want to minimize f 1 as well as minimize f 2, for this particular problem the feasible objective space is given, Pareto-optimal front is also given. Now, let us have this solution x. Now, generally when we choose solution z 0, so solution x must weakly dominate the solution at z 0.

So, that is the first condition. And second is this particular problem as we have discussed this is a maximization problem. So, this problem is similar to finding hyper cube with a maximum parameter. As can be seen in this figure that we have say z 0 and we have the solution z x and we can see since it is a minimization problem z 0 is dominated by x.

Now, when we are solving a maximization problem or maximizing the parameter as we can see here meaning that we are basically pushing this solution, so that the solution x should hit on the Pareto surface and will become the Pareto-optimal solution for their given problem.

So, one interesting point that we can observe after going through the weighted metric method and Benson method, in weighted matrix method we are actually minimizing the difference with respect to the point z star which is an ideal point. We know that z star cannot be achieved. So, minimizing that difference means we are pulling a point x towards the z star.

So, when there is a, so this particular point will be converging and finally, we will converge to the Pareto-optimal front because it cannot reach to the z star. In the Benson case, we are maximizing the difference of a point x with respect to z 0 which is which should be weakly dominated at least.

So, in this case, we are pushing this particular solution x 0 with respect to z 0, so that after some iteration this point will hit the Pareto-optimal front and will become the Pareto-optimal solution. Now, let us understand what could be the advantages of using the Benson method.

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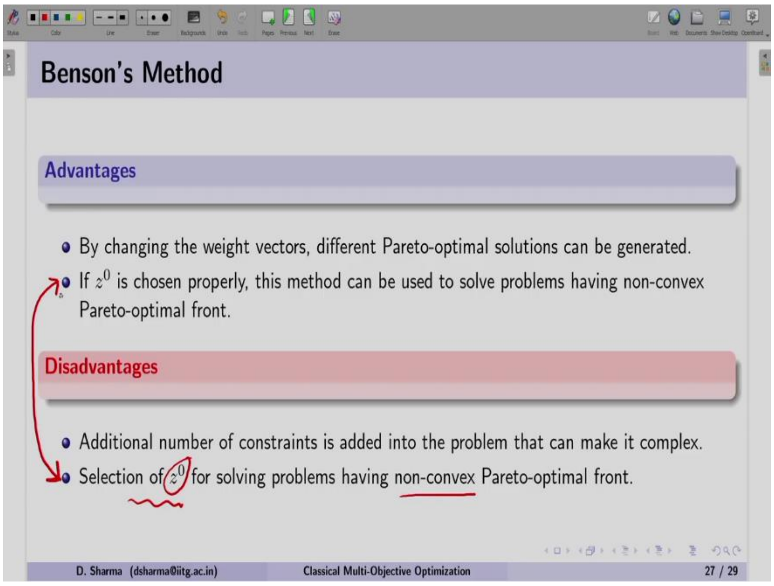

So, the first advantage is by changing the weight vectors the different Pareto-optimal solutions can be generated. Second, if we choose this z 0 properly this method can be used to solve problems having non-convex Pareto-optimal front which is an important property for us because we do not know the problem has a convex or a non-convex Pareto-optimal front, so we have to choose z 0 accordingly.

Since, there are certain advantages, but there are certain disadvantages as well with this method. First is additional number of constraints is added into the problem that can make it complex. So, as we have remember, in our formulation when we are when we are maximizing the Benson method, but the at the same time we put a limit on the objective functions as well. So, we are actually including extra constraint.

As we remember in the epsilon constraint method as well we are including the extra constraints into the formulation and that can also make it complex. Second its advantage is that we have to choose the value of z 0, because if this selection is not proper, we cannot solve the non-convex Pareto-optimal front so problems having non-convex Pareto-optimal front.

So, as we can see at one point of a time when this Benson method is giving us the advantages based on z 0 that could be disadvantages if we do not choose z 0 properly.

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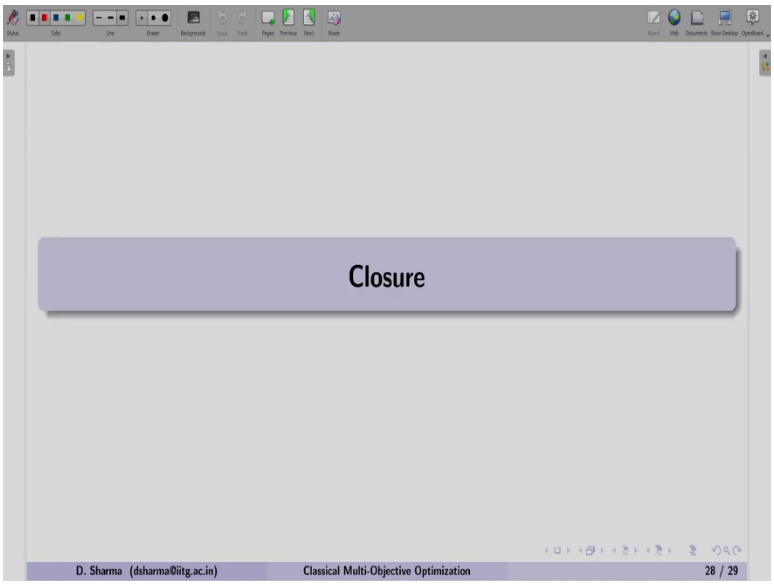

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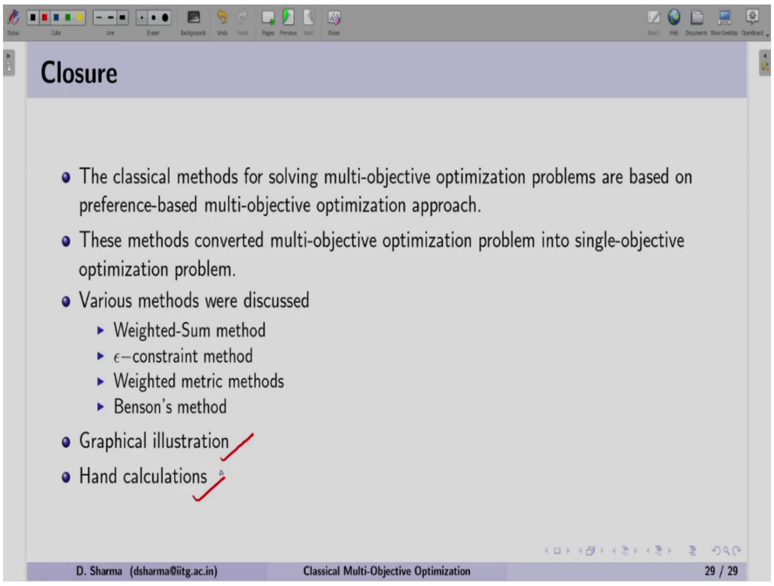

Now, we have come to the closure of this session. What we have seen that the classical methods for solving multi-objective optimization problems are based on multi-objective optimization approach. So, using this preference-based multi-objective optimization approach all these methods converted the multi-objective problem into single objective optimization problem.

While doing so, we have discussed various methods in which the weighted sum method was used where omega m, f m, and the summation over it was used epsilon constraint method keep one objective and rest of the objective will become constraints. Weighted metric method introduces the ideal point and the parameter p, and that was used for generating the solution using different weights.

Similarly, the Benson method introduced a one point called z 0 that is a dominated or weakly dominated point with respect to the current point. And all these methods we have discussed all of them the graphically, so that we can understand their behavior, their methodology, and we perform hand calculation for two of the method such as weighted sum method and epsilon constraint method.

So, in this particular session, what we have understood? That the given multi-objective optimization problem can be solved using preference-based approach and a and the approaches which we have discussed everyone needs certain information either in terms of weights which are omega 1, omega 2, omega 3 etcetera or we need details about epsilon which in which is in the case of epsilon constraint.

Once these higher-level information is available the problem is a single objective problem and we can solve those problem using any of the EC techniques which we have discussed so far. With having understanding on the classical multi-objective methods, now I conclude this session.

Thank you.