Evolutionary Computation for Single and Multi-Objective Optimization
Dr. Deepak Sharma
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Lecture - 19
Multi-Objective Optimization: Ranking and Diversity

Welcome to the session on Multi-Objective Optimization. This particular session, we will be focusing on Ranking and Diversity. So, basically convergence and diversity, these two aspects we will discuss.

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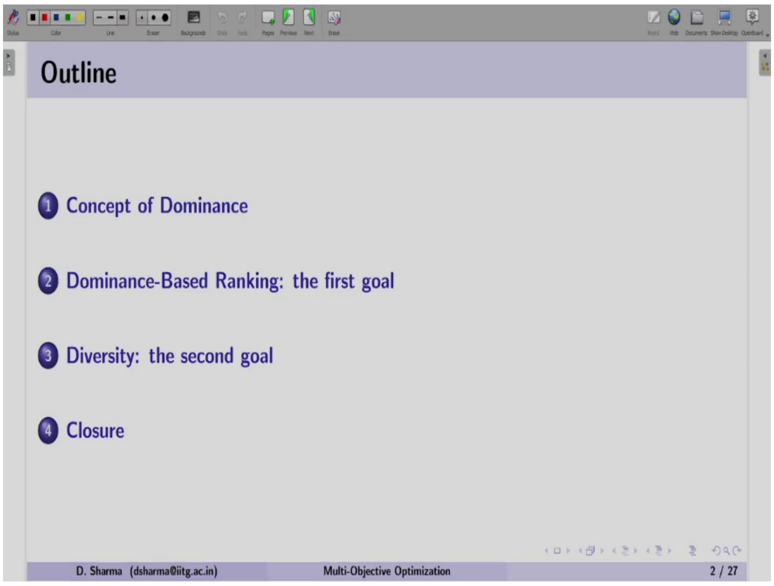

So, the outline of this particular session is we will start with the concept of a dominance as a recap. Thereafter, we will be addressing the two goals or the two issues or the two challenges that we need to tackle when we want to use EC techniques for multi-objective optimization. So, first will be dominance-based ranking followed by the diversity, and then we will conclude this session.

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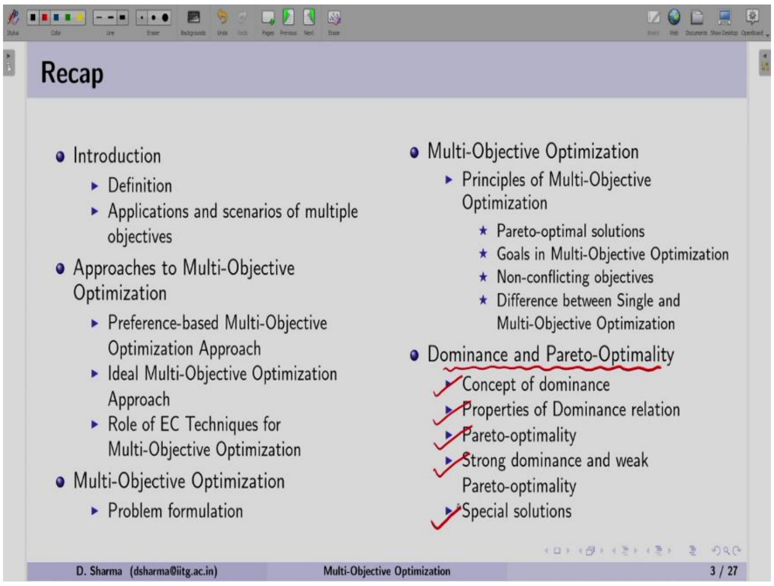

So, before we start let us have a recap of the previous session. In that particular session, we have gone through the introduction of multi-objective optimization. And thereafter, we discuss about the approaches of multi-objective optimization said that includes preference-based multi-objective optimization, and ideal multi-objective optimization approach.

We also talk about the role of EC techniques for multi-objective optimization. Under the multi-objective optimization topic, we discussed the mathematical formulation followed by the principles. Under the principle, we discussed the Pareto-optimality, Pareto-optimal solutions, the goals, non-conflicting objectives as well as the difference between single and multi-objective optimization.

And at the last we discussed about the dominance and Pareto-optimality, where we focused on concept of dominance, properties of dominance relationship, Pareto-optimality, strong dominance and weak Pareto-optimality as well as the special solution.

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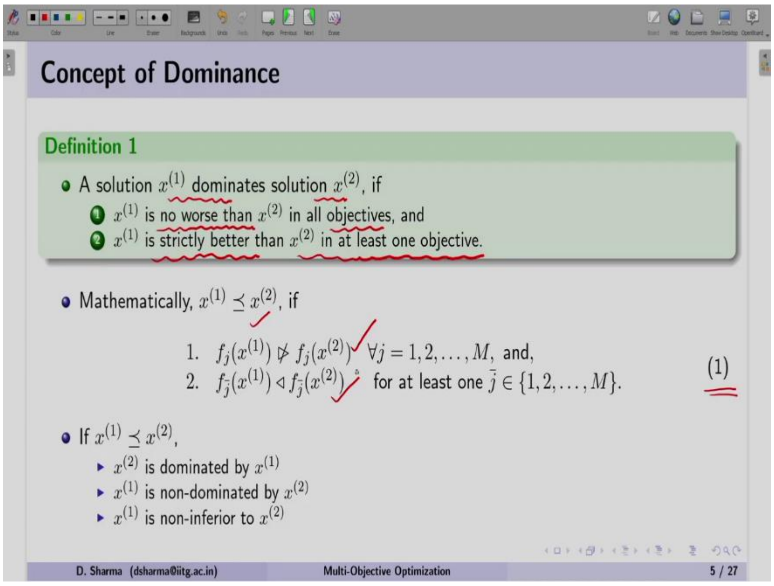

$$1. \quad f_j\left(x^{(1)}\right) \not\triangleright f_j\left(x^{(2)}\right) \quad \forall\, j = 1,2, \ldots, M, \qquad and$$

$$2. \quad f_{\bar{j}}\left(x^{(1)}\right) \triangleleft f_{\bar{j}}\left(x^{(2)}\right) \quad for\ at\ least\ one\ \bar{j} \in \{1,2, \ldots, M\}.$$

Now, let us begin with the concept of a dominance. This particular dominance this concept we have already understood it. So, let us discuss one more time as a recap. Here when we say the solution x 1 dominates solution x 2, it means that x 1 is no worse than x 2 in all objectives.

And x 1 is a strictly better than x 2 in at least one objective. So, by following this particular definition, mathematically we can see in equation number 1 that x 1 is dominating x 2, so that it should not be worse in any of the objective, and it should be; it should be strictly better in at least one objective. So, that way we can say x 1 is dominating x 2.

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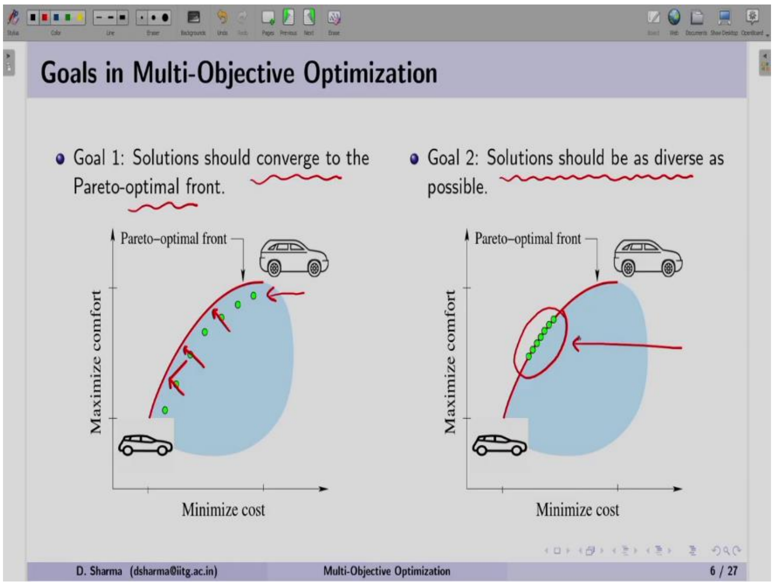

As we have understood that there are two goals for solving a multi-objective optimization. So, the first goal is we should have a convergence; another goal is we should have a diversity. So, if we look into the example here, so the solution should converge to the Pareto-optimal solutions.

In this case, when we are solving, so as we can see the solution, for example, generated by the algorithm, they are diverse, but they have not converge to the Pareto-optimal set. So, our first objective is these solutions should converge to the Pareto-optimal set. The second goal is the solution should be diverse as possible.

Now, if we look it look at into, if we look at the figure on the right hand side, we can see there are set of solutions which are converged, but they are representing a small set of Pareto optimal front. So, in this case, our algorithm should able to generate the solution that must be nicely distributed on the Pareto-optimal solution. And therefore, while solving a multi-objective optimization problem, we have to satisfy or cater two goals that is the convergence and diversity.

In this particular session, for convergence, we will focus our discussion on dominance-based rankings. These dominance-based ranking uses Pareto-optimality or the concept of a dominance for comparing the solution. So, the first goal as we have the first goal, the convergence, we will discuss the dominance-based ranking now.

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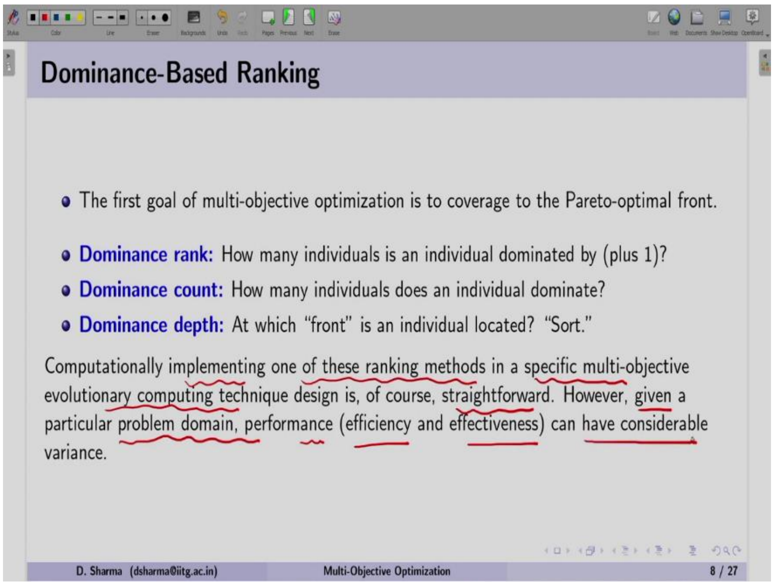

Here the as we discussed earlier the first goal of multi-objective optimization is to converge to the Pareto-optimal set. There are various ways we can prefer this particular convergence. The first way is the dominance-rank. It suggests that how many individuals is an individual dominated by and then we are adding plus 1.

Now, as we have; as we remember we are referring as a we are referring the members of a population as solutions some time points, and sometimes individual. So, here when we are referring to an individual, it means we are talking about a solution. So, what this dominance-rank says that how many solution is an individual or a solution dominated by.

Second is the dominance count it says that how many individuals does an individual dominate. Now, you can make a difference between these two definitions as the first definition says that how many individuals is an individual dominated by, and the second says that how many individual does an individual dominate.

The third method that we can use is called say dominance depth. Now, this says that at which front is an individual located. So, basically kind of a sorting we are going to use there. So, computationally implementing one of these ranking method is an in a specific multi-objective evolutionary computing technique design is, a straightforward.

However, given a particular problem domain, preference performance basically the efficiency and the effectiveness that can have a considerable variance. So, let us discuss these three methods one by one. We will start our discussion with dominance-rank.

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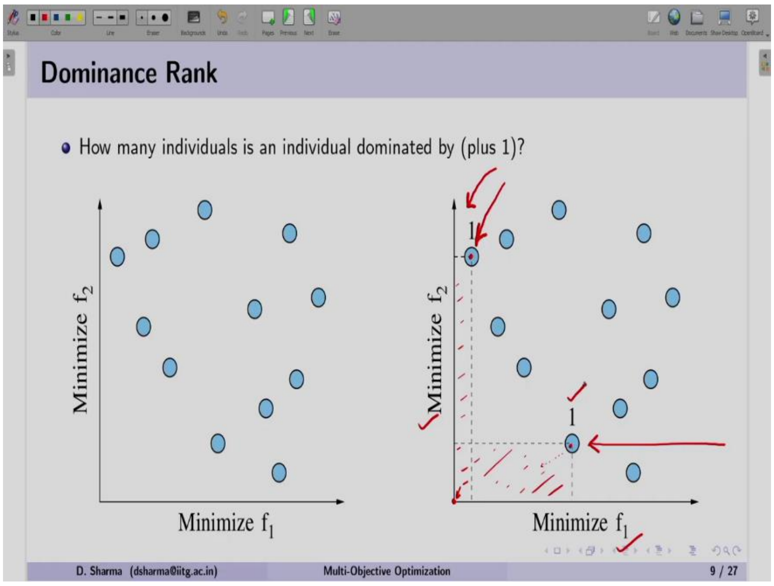

As we have understood dominance-rank says that how many individual is an individual dominated by, and then we are adding plus 1. So, let us take a case in which we want to minimize f 1 and minimize f 2, and we have taken some solution in this objective space. Now, as it says that how many individual is an individual dominated by.

So, let me take this particular solution. So, we have to find whether this solution is dominated by any member. So, what we can use is a, the concept of dominance can be used and we can see. Since it is two objective problem we have f 1 and f 2, and both of them are minimization.

So, the easiest way is to if to find whether any solution is dominating this or a not. So, from this particular solution, if I look at the origin, then we will see is there any solution which is lying in this particular rectangle or not. So, this is only valid for minimization problem and to objective problem. So, as we can see that there is nothing available.

So, this means that this solution is the non-dominated solution. So, no one is there. So, 0 plus 1. So, the dominance-rank of this solution is 1. As a second case let me take the another solution. Now, this particular solution again if I see that from this particular solution if I look towards the origin, now as you can see in this particular rectangle we do not have any solutions. This means this is not dominated by anyone. So, the rank of this solution is 0 plus 1, 1. So, the 1 which I am writing in this particular figure meaning that it has a rank 1.

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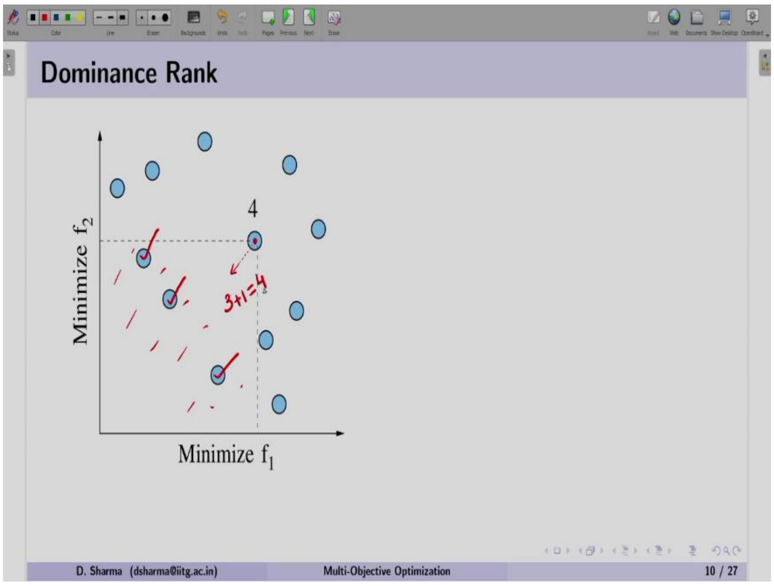

Let me take another case. Now, here we have taken this particular solution. As per our previous discussion since we have both objectives and both are minimization time, if I look at the towards the origin and then we draw this particular a rectangular box. Here in this particular box as we can see there are three solutions, 1, 2, and 3. These three solutions are actually dominating our solution. So, the rank of this solution will become 3 plus 1 equals to 4 that I have mentioned here.

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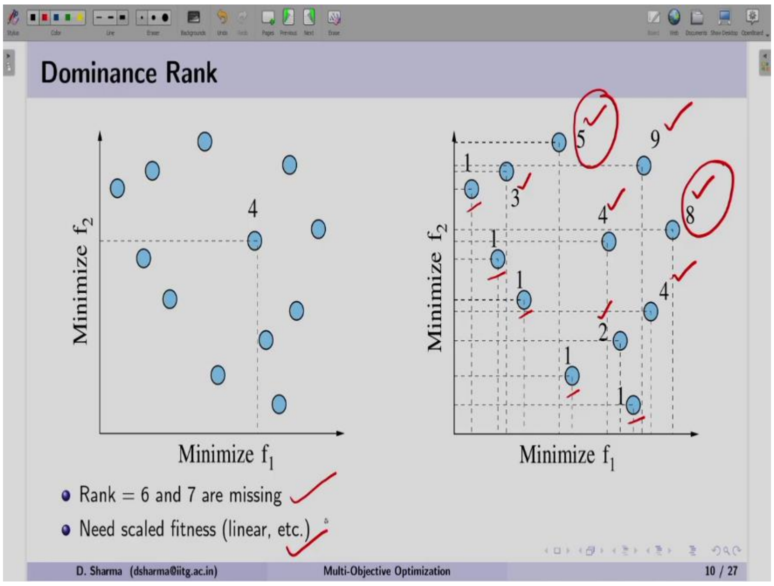

If we are going to follow the same definition, we can find that the solutions where 1 is written these are the non-dominated solutions; because these solutions are not dominated by anyone. Thereafter, we have other solutions where 2, 3, 5, 4, 4, 8 and 9 are written. So, these are the dominance-rank.

Now, what is the important point about the dominance-rank here is, if we look at the rank, so after rank number 5, we have 8. So, there is no rank which 6 and 7. So, the solutions having rank 6 and 7 are missing. So, this can create sometimes issues when we are selecting the solution with the EC techniques.

And therefore, we need some kind of linear scaling that can be used with the dominance-rank method. The best part of this particular method is that the solutions which are having rank 1, all of them are non-dominated solution.

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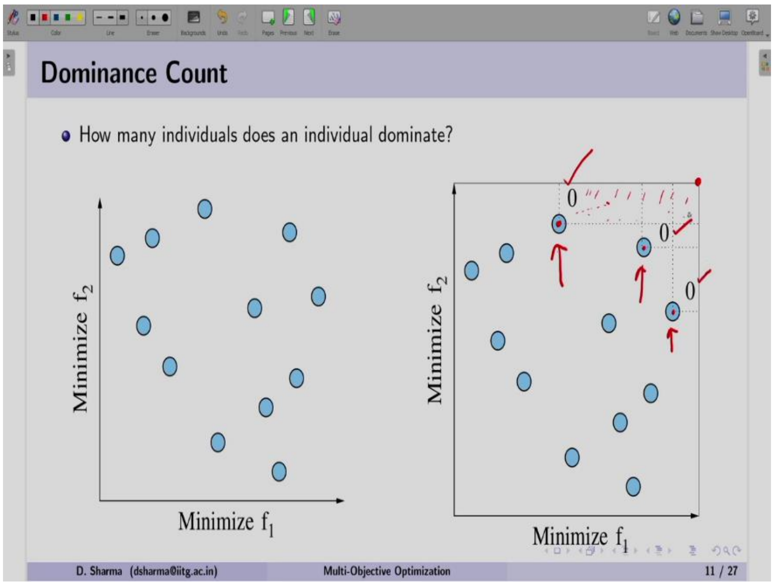

Now, let us discuss the second method which is called dominance count. As it says how many individual does an individual dominate. So, we will be taking the same set of solutions as f 1 and f 2 we want to minimize, and we have the same set of solutions now. Now, let me take a solution which is given here on the top.

Now, in this particular solution, in our previous discussion, we are going to use concept of a dominance to find out whether this particular solution is dominating anyone. So, in this case, what we will be doing is, we will be looking at the diagonally opposite corner of this origin. And if we look into this particular direction and that is going to make a rectangular box here.

And inside this particular box, if there is no solution meaning that this solution is not dominating anyone. And therefore, we have written 0. Similarly, if I take the another solution, for this also, there is nothing, so it is not dominating anyone. And similarly the third solution. So, therefore, their rank are 0.

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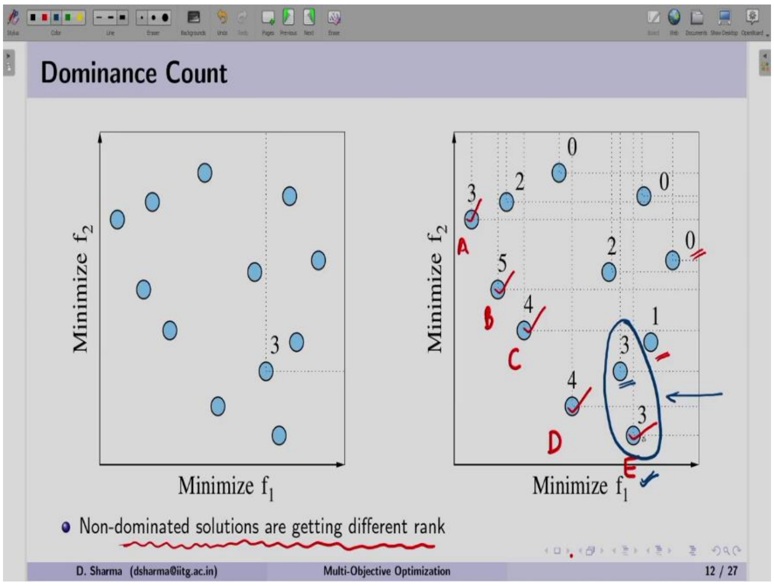

Let me take the another solution now. In this particular solution and the extreme diagonally opposite corner of the from the diagonal, now if we look into it, we are going to get this particular rectangle. Inside this rectangle, I can see there are three solutions 1, 2 and 3.

Since this particular solution is dominating 3 solution, so the dominance count for this solution is 3. Similarly, if we are going to find the dominance count for every solution, what we can find is we have the solution with a 0 count; means not dominating any one, solution count with a 1 means dominating only 1 solution and so on.

Here the interesting observation is as we have understood from the dominance-rank method, the solution the following solutions these solutions are the non-dominated solution. Let me mark them as A, B, C, D, and E, all these solutions are the non-dominated solutions as we have understood using dominance-rank.

Now, if we look at their rank, now all these solutions can have a different rank. So, the dominance count method will give me different rank for the non-dominated solution, so that is why at the bottom we have mentioned non-dominated solutions are getting different rank.

Another observation is if we look at say for example, these two solutions, now both the solutions have dominance count 3. But we know solution E is the non-dominated solution;

however, solution this particular solution is the dominated solution. Since both have the same rank, we cannot differentiate which solution is the dominating solution.

So, therefore, with the dominance count method, we have to devise a way so that we can differentiate what solutions are the non-dominated solutions as compared to the other solutions. Now, we will move to the third method that is called dominance depth.

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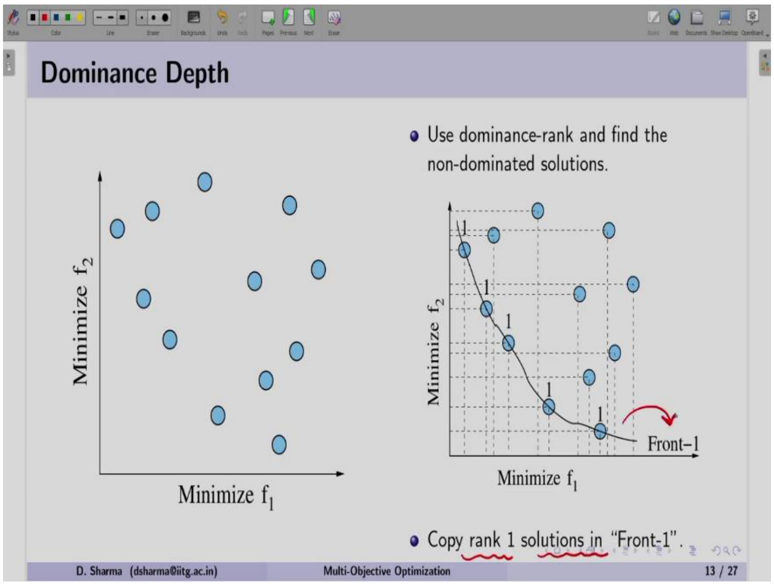

In this particular dominance depth method, we are taking the same set of solution for minimizing f 1 and f 2. So, in this one, let us use the dominance-rank method first, and find the non-dominated solution. Since this particular process we did it earlier. So, we know that the solutions which are marked with 1 all are having rank 1, and they are become the non-dominated solution. So, what we will do is we will copy all the rank 1 solution in front-1.

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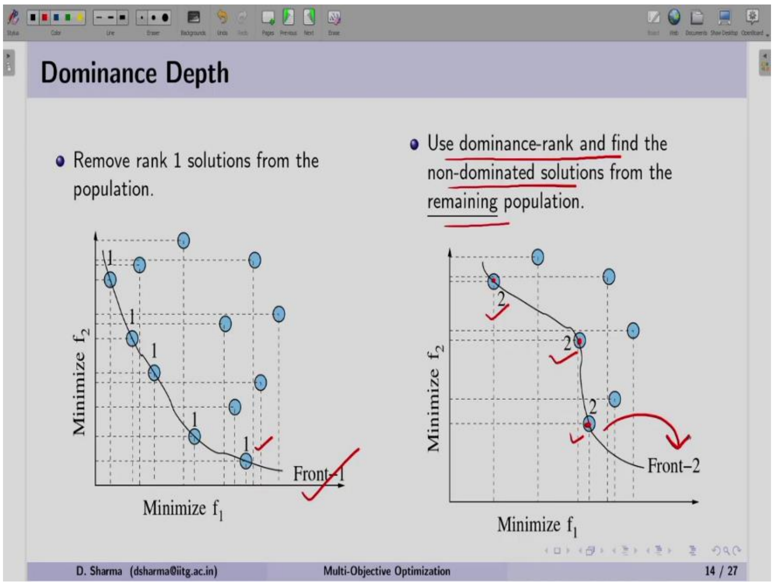

So, we are copying into the front-1. Now, once it is copied we will remove these rank 1 solution from the population meaning that since these solutions are copied, so we are going to remove them from the population. Now, if we remove them, we are left with the solution as we can see on the right hand side.

Now, on the remaining solution, we will again perform the dominance-rank, and find the non-dominated solution from the remaining population. If we do so, we will get these solutions as our non-dominated set. Since, we know that we have already saved rank 1 solution. So, we will save the solution as rank 2 solution, and we will be copying them into front-2.

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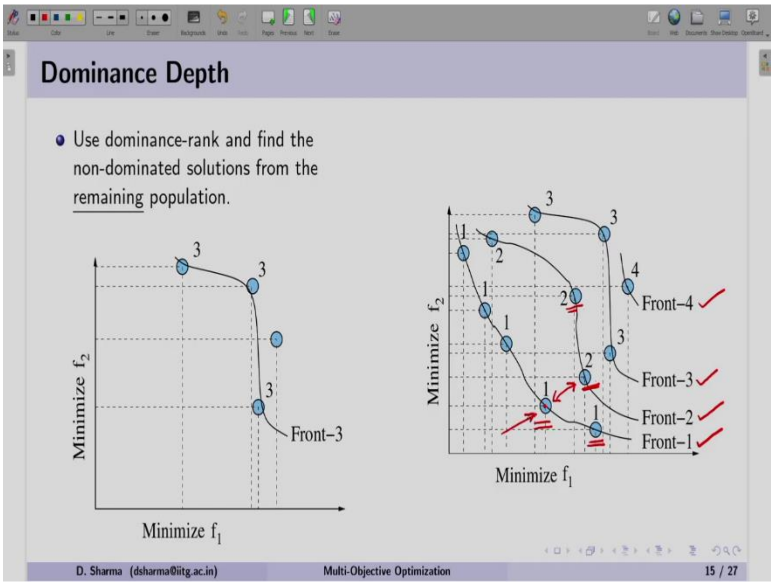

Since we now in the next step, we will be removing them those solutions we are left with these solutions. Again we will use say dominance-rank and find the non-dominated solution in the remaining population. As we can see these are the solutions which are non-dominated as of now. And rank 1 and rank 2 are already assigned. So, we assigned rank 3 and all these solution will be copied to front-3.

If we follow this particular procedure, we can see on the right hand side that the solutions are copied into front-1, front-2, front-3, and front-4, or we can say the solutions are sorted into the different fronts. The solutions which are lying in say front-1 all of them have rank 1; similarly the solutions which are lying in rank 2, so they are they have a rank 2; and similarly rank 3 and rank 4 solution.

So, here when we are going to compare these two solution, so we know we know that solution having a rank one is better than 2. So, we will be emphasizing the rank 1 solution. So, with these three methods, we can use the concept of a dominance in the different ways, we can identify the good solutions or the non-dominated in solution over the other solutions.

These three methods we can use for the better convergence in order to satisfy the first goal of multi-objective optimization that is the convergence to the Pareto-optimal set. Now, we are moving towards the second goal. And the second goal is we should have the diversity among the solutions. And to achieve that, diversity we have different methods.

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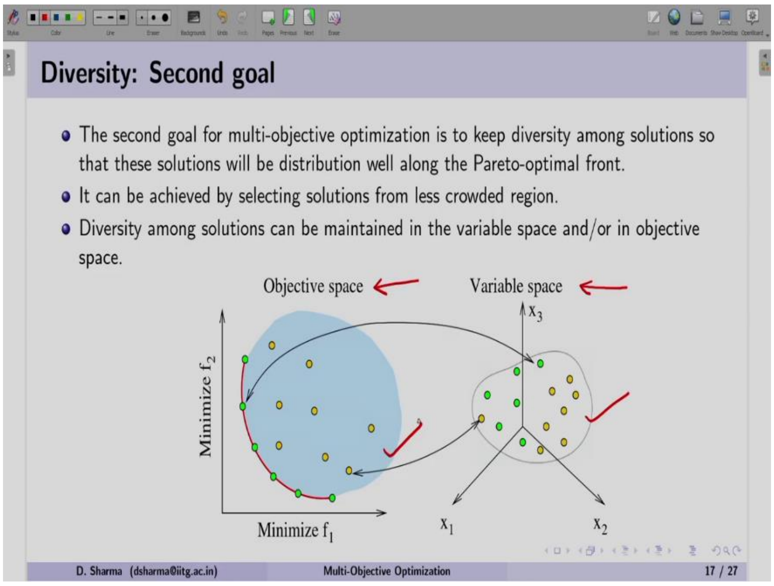

So, in this, now we are going to discuss the diversity. The second goal as we have mentioned for multi-objective optimization is to keep the diversity among the solution, so that these solutions will be distributed well along the Pareto-optimal front and that is the whole objective or the second objective for multi-objective optimization. This can be achieved by selecting the solutions from the less crowded region.

Now, since we have to maintain the diversity, and also we know that we have two spaces one is called variable space and other is called the objective space. So, we can maintain the diversity in the variable space or in the objective space. So, let us understand that how we can maintain the diversity among the solution. So, the very first method is called weighted-vector method.

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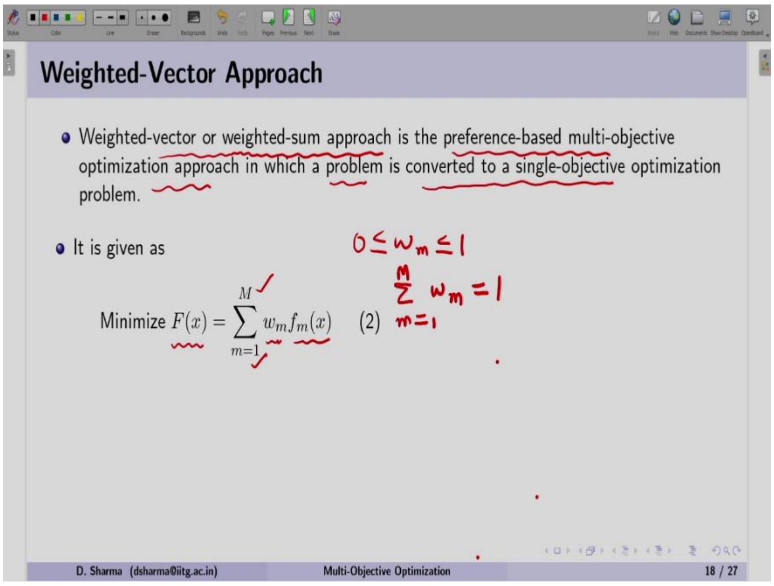

$$Minimize\ \ F(x) = \sum_{m=1}^{M} w\_m\, f_m(x)$$

The weighted-vector or a weighted-sum approach is the preference based multi-objective optimization approach in which a problem is converted to a single objective optimization. As we can see in equation number 2, we are minimizing say capital F of x that is nothing but the summation of the all the objectives from 1 to M, and these objective functions are multiplied by the weights called w m.

The condition is that all the weights should lie between 0 to 1, and also the summation of the weights should be equals to 1. So, with these two conditions, we can convert our multi-objective problem into single objective optimization problem using equation number 2.

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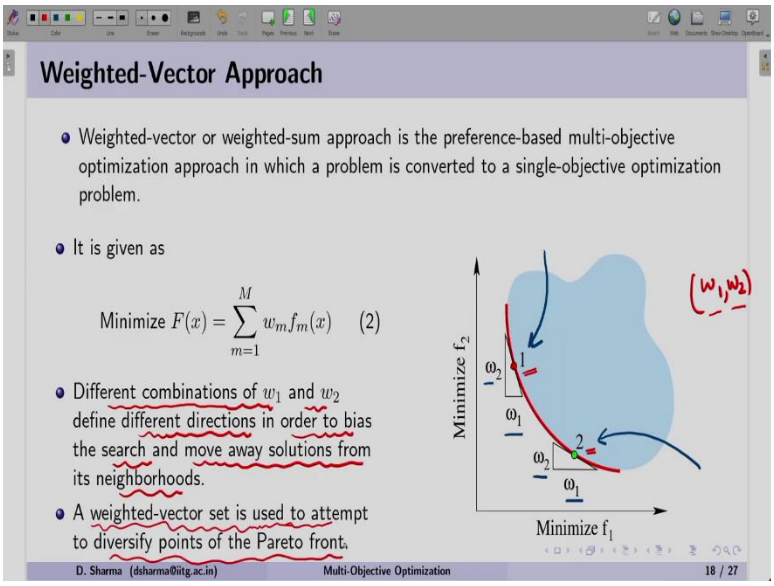

Now, if we look into the figure on the right hand side, we have taken two points. As we can see here that the first point is the point 1, another point is point 2. Now, at this particular point, if we are assigning say omega 1 and omega 2, and different set of omega 1 and omega 2, you can see that these two different sets of omega 1 and omega 2 will allow these two solutions 1 and a 2 to converge to the different set of points on the Pareto-optimal set.

Therefore, the different combination of omega 1 and omega 2 defined define different directions in order to bias the search and move away solution from the neighbourhood; meaning that, if I am going to; if I take different values of omega 1 and omega 2 for all the points, when we are using different values of omega 1 and omega 2, we are biasing the search for different solutions. So, for example, different value of omega 1 and omega 2 will bias the search for solution 1, so that it will converge to a given point as shown in the figure.

Similarly, if we take different values of omega 1 and omega 2 for solution 2, it will be biasing the search where we can see the solution number 2. Moreover, these two different values of omega 1 and omega 2 will move away these solutions in its neighborhood. So, that way we can have the diversity among the solution.

A weighted-vector set as we have understood is used to attempt to diversify solution of the Pareto front. So, we keep on changing the value of a w 1 and w 2 for different solutions, and we can expect that these solutions will be converge to the different points on the Pareto-optimal set.

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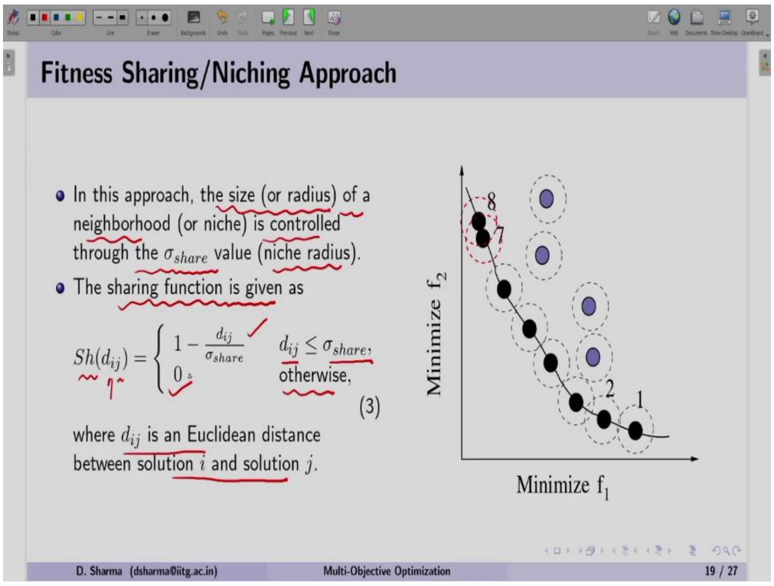

$$Sh(d_{ij}) = \begin{cases} 1 - \dfrac{d_{ij}}{\sigma_{share}} & d_{ij} \le \sigma_{share} \\ 0 & otherwise \end{cases}$$

The another method by which we can emphasis the diversity among the solution is called fitness sharing or niche approach. In this particular approach, the size of the size or sometimes we call it as a radius of a neighborhood is controlled through a one parameter called sigma share. So, we are going to find this is called a niche radius. In this way, we will be calculating the sharing function as given as the sharing function for a solution i with respect to j.

So, this says that d ij. So, d ij is a Euclidean distance between solution i and solution d j. If the distance between them is a smaller than and equals to the sigma share say the radius, then we will be calculating the sharing function as given here. If it is already far means the solutions are far from each other, then this sharing function has a 0 value.

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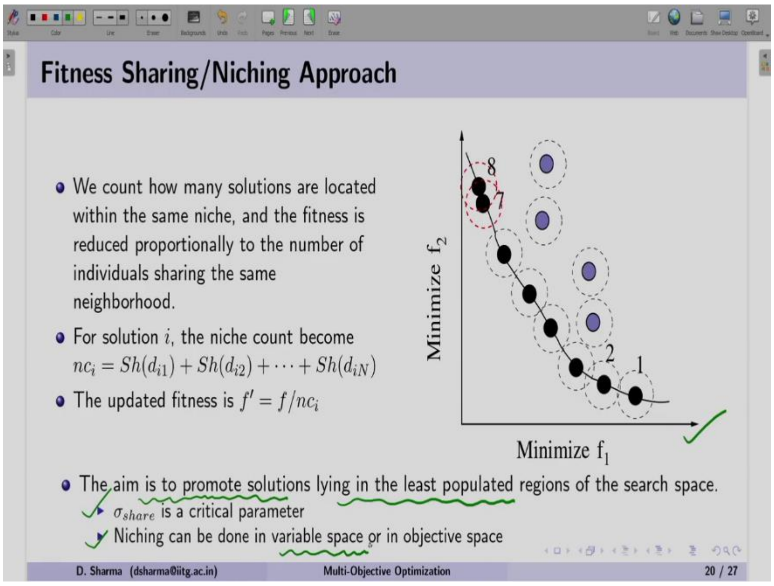

$$nc_i = Sh(d_{i1}) + Sh(d_{i2}) + \cdots + Sh(d_{iN})$$

Now, when we calculate this sharing function? Now, we can count. So, basically we can count how many solutions are located within the same niche, and the fitness is reduce proportional to the number of individual sharing the same neighborhood. Now, here since we have calculated the sharing function, so as you can see here we for a solution i, if we count the niche count as given as nc i, this is equals to sharing function value i with respect to 1, then with respect to 2, and similarly for all the solution in the population.

And the updated fitness of the solution will become f divided by the niche count. So, let us understand this particular method with the help of a figure here. Now, as we can see the solution 1, 2 and other solution they are quite far. Since they are quite far, so they are niche count is 1.

$$f' = f/nc_i$$

But if we look at the solution number 7 and solution number 8 here, these solutions are quite close to each other. Now, using these radius, the red the circles which are drawn using red lines, now here you can see that these two solutions having they are closed and therefore, they are sharing fitness which we have referred as Sh d ij these values will not be 0.

Once we are doing this, so the niche count for these two solutions have a value more than 1. So, at the end, the fitness of the solution number 7 and a solution number 8 will be reduced, why, because these two solutions are close to each other. So, as we can understood from this particular method that when the solutions are club together or they are clustered in a same region, these solutions will be having the reduced fitness value.

Since it is going to be reduced fitness value, then the selection of these solution will be less as compare if we take a solution which is lying in a isolated region, and that is the whole objective that we should prefer the solution from the less crowded region as compared to the solutions which are lying in the crowded region.

So, when we are selecting in this way, so in generation by generation with EC techniques, we will be preferring these diverse solutions and these diversity will be maintained throughout the generation. And at the end, we can expect that our EC technique will be able to give us a well distributed set of a solution on the Pareto-optimal set.

Now, here the aim of this particular method is to promote solutions lying in the least populated region of the search space. Now, the what is the critical point about this particular method is the sigma share. Why? Because this value has to be chosen by us.

The other point which we can understood is that we have seen in the current example that the niching or the sharing function we applied in the objective space which is f 1 versus f 2, this can also be done in the variable space. So, we are free to use this niching either in the objective space or in the variable space. Generally, in multi-objective optimization, we prefer the fitness sharing or niching approach in the objective space.

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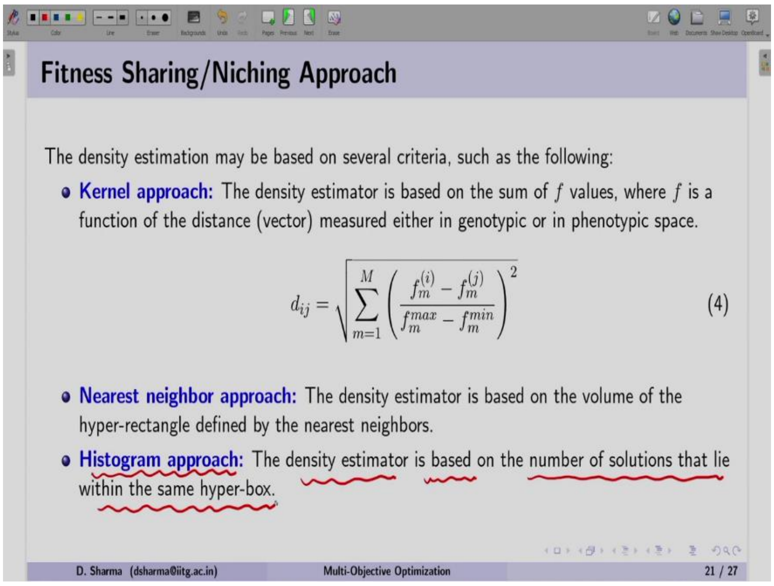

$$d_{ij} = \sqrt{\sum_{m=1}^{M}\left(\frac{f_m^{(i)} - f_m^{(j)}}{f_m^{max} - f_m^{min}}\right)^2}$$

Under the fitness sharing approach, we have different ways to perform the sharing fitness or if we can find the niching approach. One of the approach is called the Kernel approach. In this particular approach, the density estimator is based on the summation of f values where f is the function of the distance vector which is measured either in the genotype or in the phenotype.

As per our discussion in this particular course, that genotype will be defined in terms of the variable, and phenotype what we see physically. So, we can perform this Kernel approach anywhere. Now, as you remember that in the previous method we were calculating this d ij as an Euclidean distance that as also can be calculated as given in the equation number 4 where we are subtracting the objective function value between i and j, and we are dividing with f m of maximum value f m of minimum value.

We are taking the square and taking the summation, so that is going to give us the distance between the two solution i and a j. We have another approach called nearest neighboring approach. In this case, the density estimator is based on the volume of the hyper rectangle defined by the nearest neighbors.

So, in this case, we will be finding who are the neighbor of the say solution i, and then we will be finding the hyper volume. Now, the solution which will be having less hyper volume meaning that solution is more crowded, so that will give us the indication of d ij with this approach.

And at the last we have the histogram approach. In this particular approach, the density estimator is based on the number of solution that lie within the same hyper box. So, in this particular approach, we are going to define the hyper box or hyper space. Inside that hyper space, if any solution is lying on more than solution more than one solution is lying, so accordingly we will say that the estimator will be based on the number of solution line in a one hyperspace.

So, what is the difference between histogram approach and nearest neighbor approach? In the nearest neighbor approach, we calculate the hyper volume. And here we have already defined a hyper volume or hyperspace. And in this case, we are looking how many solutions are lying. And accordingly the fitness will be shared among the solutions which are lying in the same hyper space, so that is called fitness sharing or niching approach.

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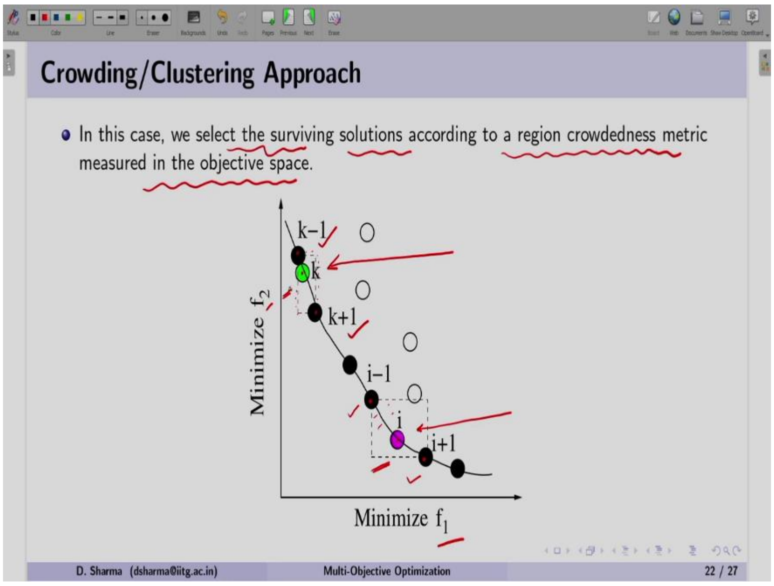

Now, we are moving to the another approach called crowding and a clustering approach. As we can see here that we can select the surviving solution according to the region crowdedness metric measured in the objective space. So, let us take an example here. We have we want to minimize f 1, we want to minimize f 2. Let us take two solution as solution k and solution i.

Now, if we want to find what is the crowdedness of this solution. So, let us take solution number k. So, what we will do is, we will find the neighbor. So, the neighbor of solution k is k minus 1 and k plus 1. Now, we will see how far they are. Like k minus 1 and k plus 1, how much they are far from each other.

So, this is small rectangle which you can see here this particular rectangle will tell the gap between k minus 1 and a k plus 1. Similarly, for solution i, we have neighbor say i minus 1 and i plus 1. And then if we see how they are far say i minus 1 and i plus 1 though this particular rectangle will help us to find it out.

Now, looking at these two rectangle, we can say that the solution i is less crowded as compared to the solution k. Now, since it is two objective the way we have discussed the crowdedness, we can easily or we can visually identify that the solution k is more crowded as compared to solution i. But we have to remember that when we are going to implement in the code or in our programming, then we need a certain value based on that we can say which one is less crowded.

So, identifying the neighbor and coming up with a concept like the boxes of the neighbors, then we can identify or calculate some value. Based on that, we can say that the solution i is less crowded as compared to solution k. When we have to select one solution between the two, the solution which is less crowded that is the solution i will be selected over solution k.

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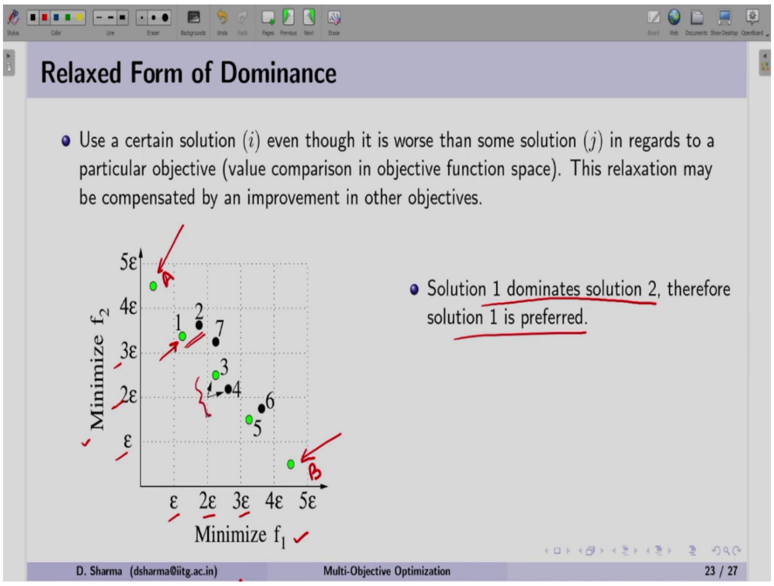

The another way to preserve the diversity is the relaxed form of the dominance. Now, here use a certain solution say i even though it is worse than some solution j in regards to a particular objective. This relaxation may be compensated by an improvement in the objectives in other objective.

So, what it means that? So, let us take a case. And in the figure on the left hand side, we want to minimize f 1 and f 2 , and we have divided this design domain in terms of as you can see in an x-axis epsilon, 2 epsilon, 3 epsilon and so on; similarly, the f 2-axis, epsilon, 2 epsilon, and 3 epsilon and others.

So, we have divided this f 1 and f 2 space into the square boxes. Now, as we have understood the solution which are lying alone in the boxes as I have mentioned say suppose this is solution A, this is B, these are independent these are isolated solutions. So, anyway these solutions are going to be selected, so that is why these are in green color.

Now, let us compare the second box where we are comparing solution 1 versus 2. By comparing them, we can find that solution 1 is dominating 2, therefore, solution 1 is preferred. And therefore, we are going to select 1 over 2.

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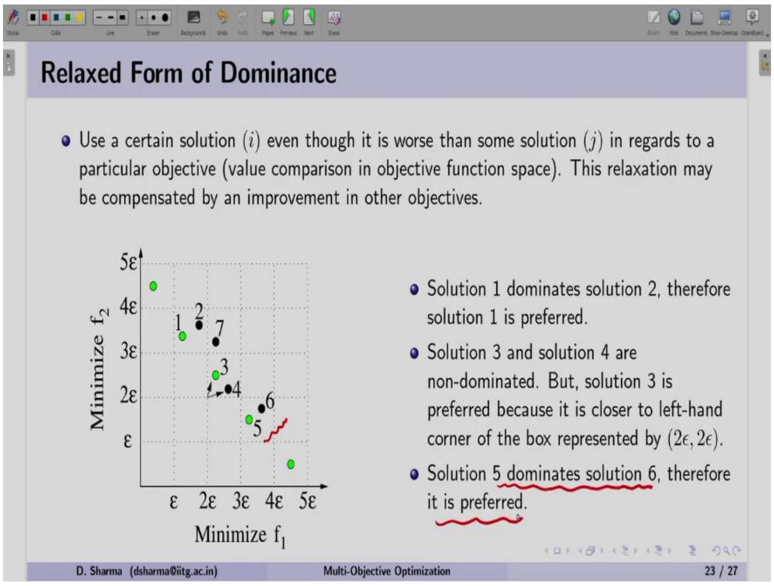

Now, let us take us case of solution 3 versus 4. Now, looking at this 3 and a 4, we can find that these solutions are non-dominated. So, in this case, which particular solution we are going to select? So, we will select solution 3, why? Because it is closer to the one of the vertex which is 2 epsilon and 2 epsilon. So, therefore, based on this particular corner we are selecting solution 3. Now, if we compare solution 5 and a 6, it is clear that 5 is dominating 6. Therefore, 5 should be preferred.

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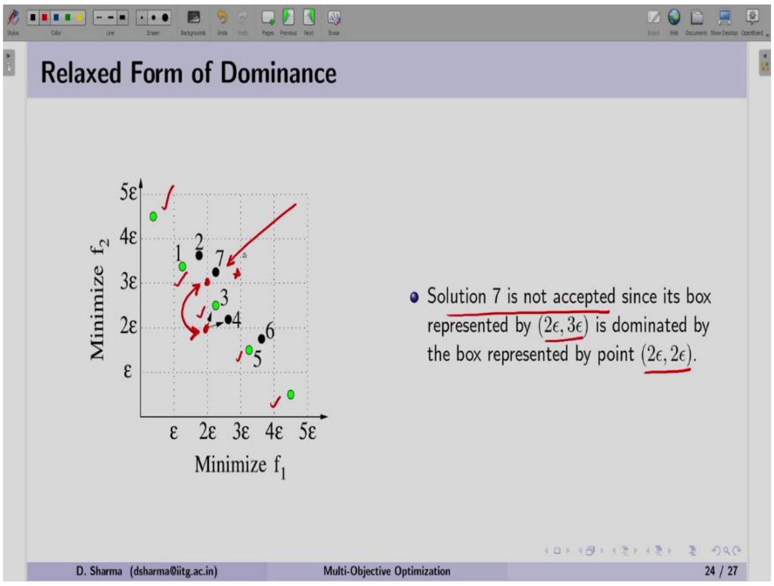

Now, in this particular approach what we have identified is we are selecting one, one solution from the boxes. Now, if we look at the solution number 7, this solution is also an isolated solution in its box. Even though we are not selecting this particular solution, it is only because the box which is represented by 2 epsilon, 3 epsilon is actually dominated by the corner 2 epsilon e epsilon.

What I mean by that? This particular 2 epsilon, 2 epsilon by 2 epsilon, this is the corner which we have taken, and a solution number 3 is already selected. Now, since the solution 7 is lying in the isolated box. So, the corner which we have is 2 epsilon and 3 epsilon. If we are going to compare these two corners and using the concept of a dominance, we know that this particular box will be dominating this box. So, therefore, we are not going to select solution 7 even though it is lying in the isolated box.

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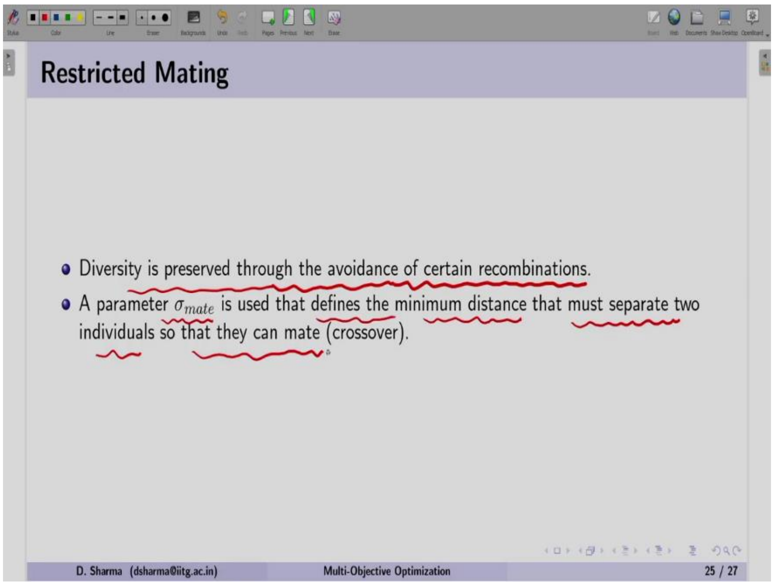

One of the interesting way to keep the diversity among the solution is the restricted mating. What is that? The diversity is preserved through the avoidance of a certain recombination. So, as we remember we perform say crossover with genetic algorithm. So, we will perform the crossover in such a way that it should enhance the diversity among the solution.

So, in this case, the parameters sigma mate is used that defines the minimum distance that must separate two individuals, so that they can mate. So, basically, in a crossover, we can use it. So, meaning that if the two points which are very close to each other, then the new solution will be quite close to them. So, we are not generating any solution that will be little diverse from them.

Therefore, we are defining certain distance say sigma mat. So, if it is far according to the sigma mat, then only we are going to perform the crossover between the two solution. So, in this particular session, as a part of introduction to multi-objective optimization, we focused on convergence and the diversity which are the two goals for multi-objective optimization.

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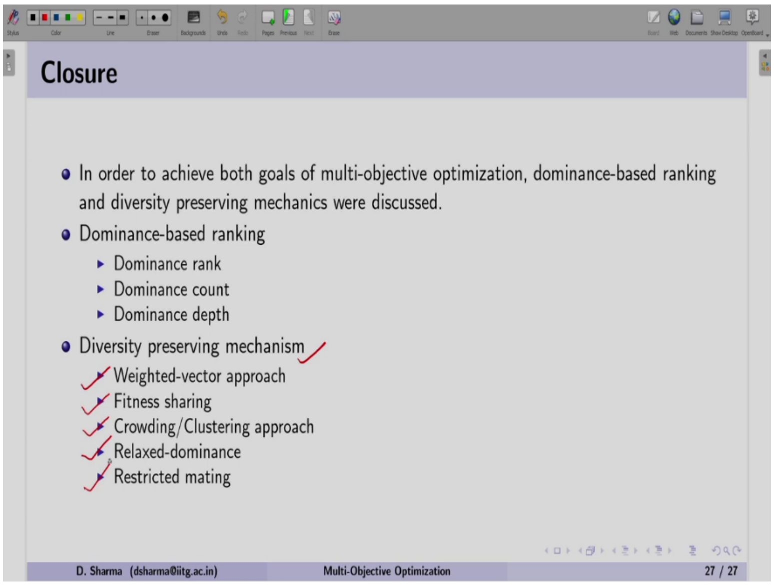

So, these two goals we have achieved using the dominance-based ranking and the diversity. Now, in under the dominance-based ranking, we discussed the dominance-rank dominance count and dominance depth. So, we restricted our discussion only on the dominance-based.

Once the goal-1 will be satisfied, then we have to focus the second goal which is the diversity. So, the diversity preserving mechanism we have gone through. We started our discussion with the weighted-vector approach, and then we discussed the fitness sharing. And thereafter, we have this crowding and clustering approach where we look the neighbors and find the crowdedness.

We also discussed the relaxed dominance in which we are selecting a solution from one of the boxes and finally, the restricted mating that will be helping us to maintain diversity in the given population. Within, with this understanding on the multi-objective optimization and achieving both the goals on convergence and diversity for multi-objective optimization, I conclude this session.

Thank you very much.