Nonlinear Vibration
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Lecture - 24
Analysis of chaotic system

So, welcome to today class of Non-linear Vibration. Today class we are going to study regarding these Chaos or Chaotic Responses in the system. So, already you have learnt that we have four different types of responses in the system. So, they are fixed-point response, periodic response, quasi-periodic response and chaotic response.

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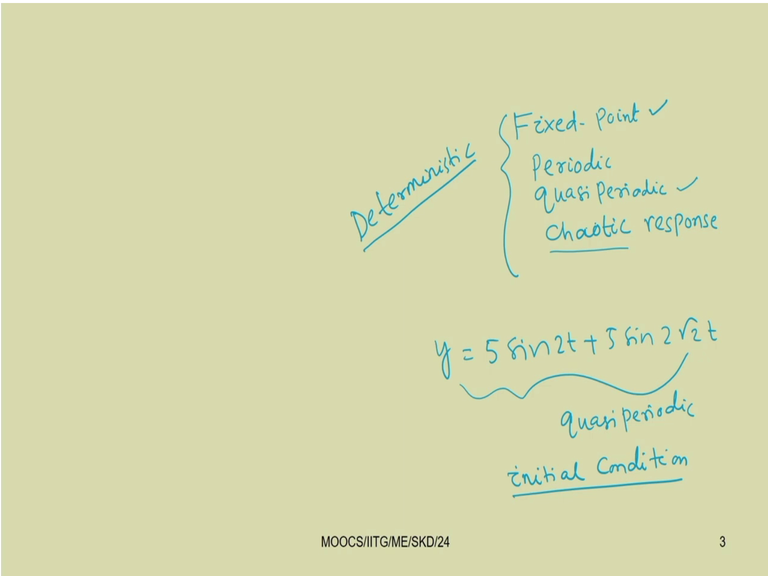

So, four different responses you have studied. So, first one is the fixed-point response, second one periodic response, then quasi-periodic response then chaotic response. So, in case of fixed-point response so, you have a fixed number fixed and then in case of periodic response the response is repeated with a particular time interval and in quasi-periodic response particularly these are aperiodic response so, where the frequency is the ratio of the frequencies are in irrational number.

So, for example, so, you take y equal to 5 sin 2t plus 5 sin 2 root 2t. So, in this case the response will be quasi-periodic. And, the response so, the deterministic response so, all these responses are deterministic response; that means, you can find you can determine. So, what will be the amplitude of the response or what will be the frequency of the response. So, those things can be determined that is why these are deterministic response.

So, out of all these deterministic response so, which are not fixed periodic not quasi-periodic are known as chaotic response. So, there is no standard definition of this chaotic response, but it is written in a negative way that is the response the deterministic response which are neither periodic fixed-point or quasi-periodic are chaotic response.

So, the characteristic of chaotic response is that so, it is very sensitive to initial condition. So, these are sensitive to initial conditions. So, if you change slightly the initial condition so, if the initial condition you change slightly then it will lead to another attractor. Initially you have got one chaotic attractor.

So, now, slightly by changing this initial condition so, you can reach with another chaotic response. So, that is why they are also known as or this effect is also known as butterfly effect.

The sensitivity to initial condition which leads to different type of chaotic attractors are known as the butterfly effect. So, in chaotic responses particularly you will be seeing this butterfly effect. In actual case in all higher dimensional system you can get chaotic response in the system.

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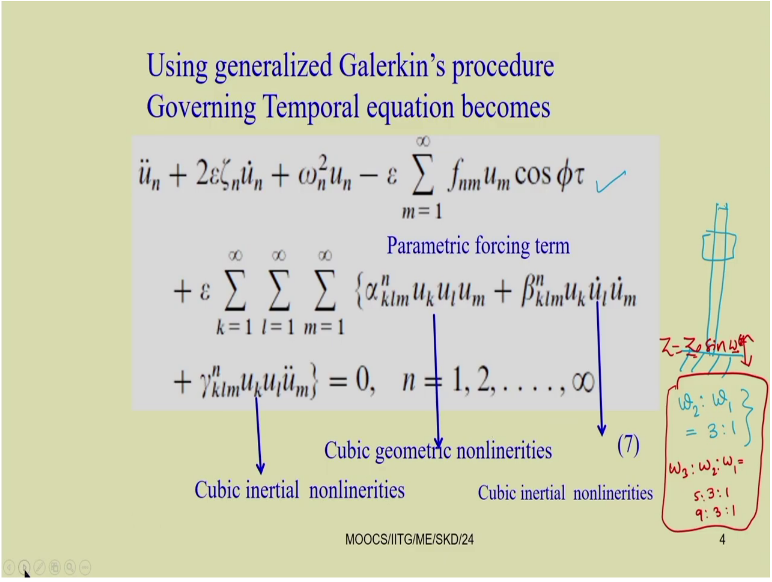

Last time when we have studied a base excited cantilever beam we have studied a base excited cantilever beam whose equation can be written in this form that is u n double dot plus 2 epsilon zeta n u n dot plus omega n square u n.

So, then we have a parametric forcing term that is this f nm u m cos phi tau. Then we have these non-linear terms also. So, geometric non-linear term and these inertia non-linear term. The second order differential equation, so, containing so many non-linear terms and this parametric term actually will leads to chaotic responses.

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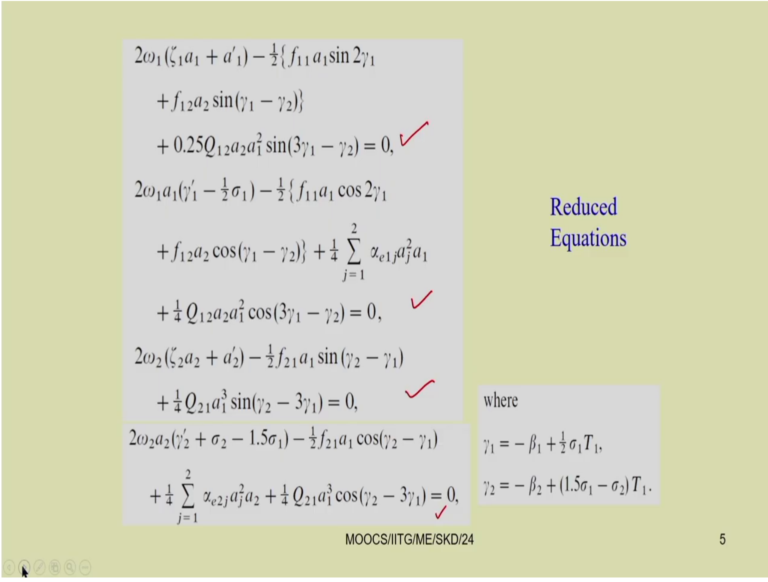

So, in addition to the fixed-point, periodic and quasi-periodic response so, last time we have seen several chaotic responses are there. So, in a general simple systems for example, in the simple spring mass damper system also when you are writing that equation as a Duffing equation by adding this non-linearity.

So, by changing the system parameters you can see that chaotic responses also can be observed in that system. So, in this particular system what we have studied last time that is a cantilever beam base excited cantilever beam and here the mass is put at arbitrary position. So, by adjusting this mass at arbitrary position, so, we have seen so we have found the second mode frequency the ratio of the second mode frequency to first mode frequency equal to 3 is to 1.

We can also by adjusting these things so, we can have other different type of relations. So, that means, omega 3 is to omega 2 is to omega 1 can be of 5 is to 3 is to 1. Similarly, we can get 9 is to 3 is to 1 also. So, 1 3 9 or 1 3 5, so, this way we can generate different type of resonance conditions and here the base is excited.

So, we have taken this base excited z equal to z 0 sin omega t. So, depending on the value of this omega and z 0 that is amplitude and frequency of this basic citation so, we can have different resonance conditions. So, depending on all these resonance conditions, so, we can study or we have we can find so, different type of chaotic responses in chaotic response.

So, this chaotic response also can be observed in n many fluid mechanics or fluid related systems or in any mechanical systems many other mechanical systems biological systems or electronically excited system or electronic systems electrical systems. So, in many systems almost in all real life systems, so, you can find these chaotic response.

And, this chaotic response the application of chaotic response also can be there are many applications of this chaotic response. Those who are interested they can see the journals like these chaos solitons and fractals or these non-linear dynamics, journal of sound and vibration, journal of vibration and control.

So, in all those journals you can find the latest papers related to chaotic field. So, or you can see there are several systems so, where these chaotic responses can be found. Directly by using this Runge-Kutta method so, you can find the response of the system at different value of these foreseeing amplitude and frequency.

So, as I already told so, there will be several resonance conditions so, this particular resonance condition has to be studied to find the response of the system. Particularly, when you are interested in a non-linear system, so, that time, due to the presence of multiple solutions so, you must know so what is the initial condition. So, which is giving rise to what type of or which is giving rise to what equilibrium condition or what fixed-point response.

So, knowing actually this fixed-point response or unstable fixed-point response which may be the root of the other types of non-trivial responses, you can study the responses found in these particular systems. Here when you are using this method are multiple scales so, you just see we have reduced these second order differential equation to 4.

So, this is one, this is the second one, this is the third one and this is the four fourth one; so, four first order differential equation. So, this four first order differential equation can be solved numerically to find the responses of the system. So, already I explained that there will be multiple solutions, multiple fixed-point response; in addition to that there may be quasi-periodic, periodic or these chaotic responses.

So, all these responses may co-exist and these due to these coexistence of different type of responses they may interact also with each other and due to that interaction sometimes some of the attractor will be disappearing or some of the attractor may appear in a bigger way. So, all those things will leads to this crisis in a system. So, we will see how the crisis can occur in the system, how this chaotic response can give rise to crisis when it come in the vicinity of an unstable periodic response.

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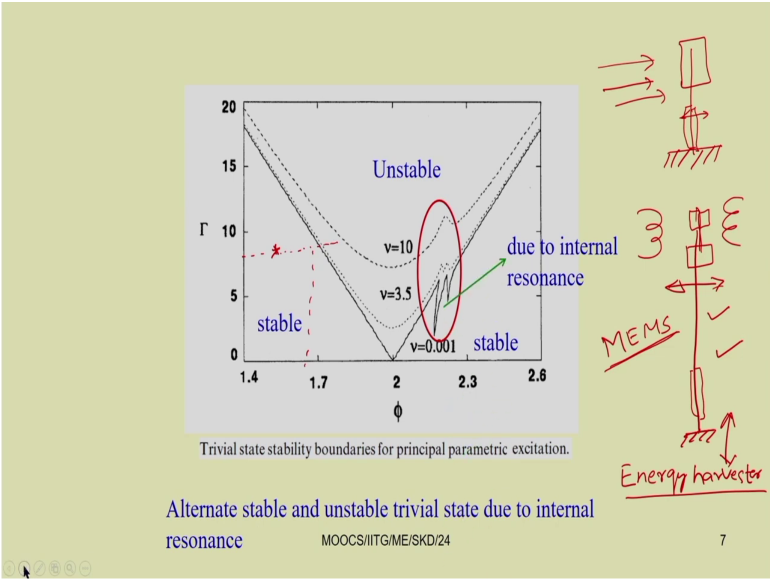

We have already plotted this or we have seen this instability region. So, in this instability region for example, so, if we are taking a point here so, whatever maybe we have taken a point here so, now by increasing the frequency you can see whatever may be the frequency. So, up to this level so, there will be the system will be in trivial state; that means, this cantilever beam will not vibrate. So, it will not vibrate or it will be in it is trivial state.

But if it enters into this zone that unstable zone. So, it starts vibrating in the transverse direction. We are moving the cantilever beam up and down. So, we have a mass a tress pass here. So, it is moving up and down, but the beam is moving in a transverse direction. So, in a parametrically excited system when you are giving a force in one direction the displacement takes place in a perpendicular direction.

So, here what we have observed for certain value of this phi that is the frequency and the amplitude of the base excitation the response may be or the system may be in trivial state or it may be non-trivial state. So, when you are using this cantilever beam as an energy harvester, so, we can use this as an energy harvester.

When it is used as an energy harvester so, we will be interested to run the system so, in this zone so, in this unstable zone so that we can get the non-trivial response and due to the presence of non-trivial response the voltage can be generated. So, if we are putting for example, let us put some piezoelectric patches here; so, by putting some piezoelectric patches. So, we can generate the harvester or we can have.

So, for example, these cantilever beam so, let us keep a magnet fixed a magnet here and let we have some coil here. Due to the presence of this magnet so, there will be change in flux in this coil. So, when it is vibrating so, there is change in flux in this coil and due to that thing this emf will be generated in these coils and this emf can be taken for harvesting the energy.

Similarly, we may put for example, so, in this case we may put this bluff body in this. So, let us put a bluff body and so, when it is oscillating in this direction and due to the presence of the bluff body due to or let us take this simple cantilever beam and this is the bluff body and it is subjected to wind force.

So, when it is subjected to wind force, so, it will start oscillating and this also give rise to. So, if we will have the piezoelectric patch, so, this will also give rise to this energy so or this voltage. So, this way the same system can be used for many different purpose for harvesting the energy.

Also, you can use different type of beams. So, instead of taking a simple elastic beam so, you can take this as a viscoelastic beam or the magneto elastic beam. So, by taking magneto elastic beam so, or viscoelastic beam you can achieve different purpose and in all these cases you can see you can enter to a chaotic regime also.

Also, you can take other different type of systems. For example, the simple system can be reduced in size and it can be used as a sensor that is Micro Electro-Mechanical system Micro Electro-Mechanical systems MEM systems. So, it can be reduced to that of a MEM system so, in which can be used for a sensor purpose for generate making a sensor. So, there are several applications of this type of beam which you can study in great detail ok.

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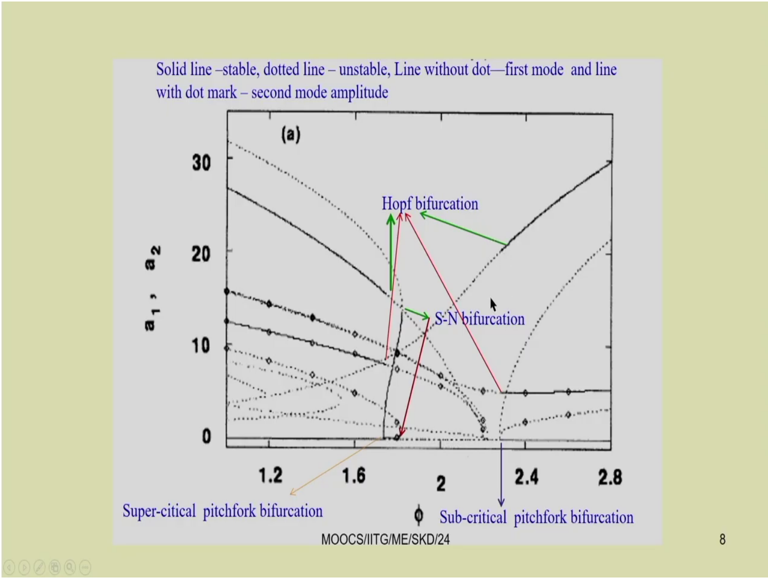

Already we have plotted this response and bifurcations we have studied. So, you just see. So, these are the points for Hopf bifurcation. So, which will give rise to give rise to periodic response, also we have the subcritical and supercritical pitch for bifurcation. Particularly in case of subcritical bifurcation, so, if we reduce this frequency as there is no stable zones stable response near its vicinity.

So, the response so, there is a possibility that it may leads to a chaotic response. The system may jump up to a fixed attractor fixed-point at the infinite or it may have some chaotic attractor. So, near the periodic response so, near the periodic near the Hopf bifurcation so, we have periodic response.

So, if there are some unstable periodic solution also that may leads to or that may gives rise to the chaotic response in that region. So, one should explore the possibilities of periodic, quasi-periodic and chaotic response near the unstable fixed-point response.

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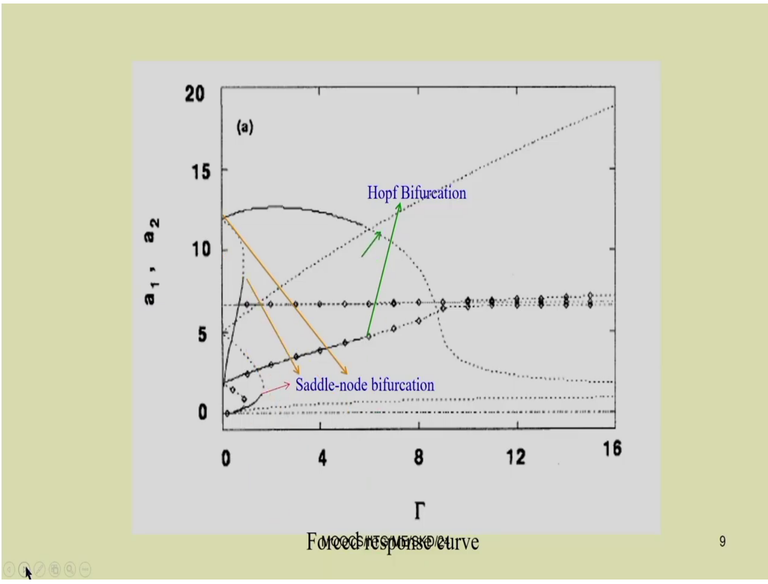

One can plot this response that is amplitude versus this amplitude of the base excitation. So, here also similarly one can study all these bifurcation points.

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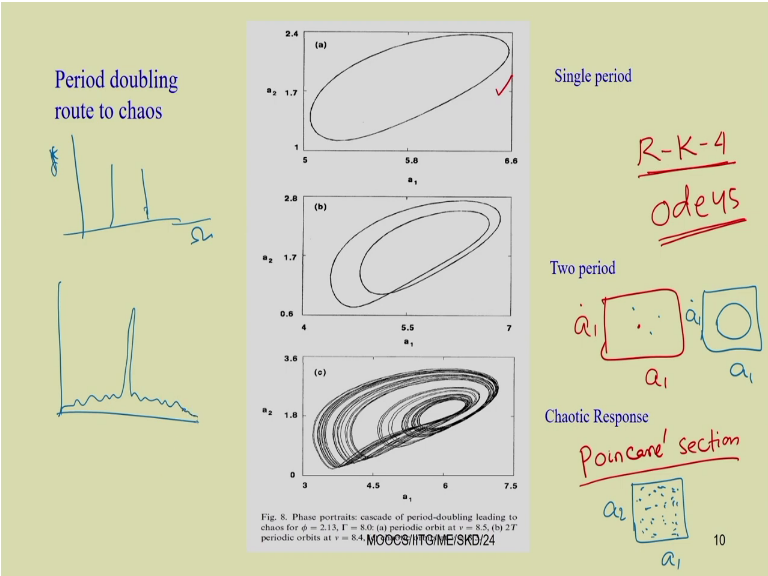

So, here we have seen; so, near the Hopf bifurcation so, we have this period doubling periodic. So, we have a periodic response, then this periodic response by changing the system parameter we have seen or you can check that you can get the two periodic and you can finally, you can get large number of periodic and it will leads to chaotic response.

But how you will generate this thing? So, you have four reduced equation. So, these four reduced equation so, you use this R-K 4 method. So, Runge Kutta method. So, use R-K 4 or Runge Kutta method. So, in MATLAB so, you have this Ode45 you can use. So, using this Ode45 function in MATLAB so, you can generate this periodic response. So, a 1, a 2 so, all those things you can find.

Then you can change the system parameter and check. So, how this periodic response are periodic then two periodic and four periodic, eight periodic and thirty two periodic and finally, it is becoming chaotic response. So, last class we have discussed regarding the Poincare' section; Poincare' section.

If we know the period of this response, so, if we can sample it with the time period then for this single periodic so, we can get only one period one point in this for example, if I will take these a 1 versus a 1 dot. So, there will be a single period.

Similarly, so, if it is two periodic then there will be two points in the. So, there will be two points near this. So, this is one point, then another point will be there. So, there will two points if it is four periodic there will be four points. And, also we have seen so, if the response is quasi-periodic then, so, it will be a close loop. So, you will get a close loop in the a 1 versus a 1 dot curve.

So, this is displacement versus velocity curve or you can plot a 1 versus a 2 plot also. So, there so, here you just see we have plotted a 1 versus a 2 that is the first two modes we have plotted. So, in the first two mode here it is periodic, then this is two periodic and finally, it is chaotic. So, you can plot this phase portrait so, or the state space.

So, when you are plotting a 1 versus a 2 so, these are two different state of the system. So, that is why it is known as so, state space. So, you can plot the a 1 in a 1, a 2 state space.

These are state of the system so, that is why it is known as state space. But when the response is chaotic so, if you find the Poincare' section so, if you find the Poincare' section, you can see that it will filled up the whole space. Initially we have a single point, then it becomes two point, then it becomes four points and finally, it filled of the space.

So, if it is quasi-periodic, then you will have a close loop. When you are considering this chaotic response, then you must have to draw the Poincare' section to take that it filled up the space. Similarly, you can draw or find the spectrum of the response. So, for a single periodic so, you have a so, you have only one spike. So, one frequency it has one frequency; similarly for two so, you will have two spike.

And, for multiple so, if you are plotting the spectrum so, this is your frequency. So, this frequency you can plot and the spectrum here you can plot. For example, you can plot the displacement spectrum the star let me put.

So, in case of fixed-point you have the spikes, but in case of this chaotic response so, you can have a broad band. So, the response may be so, like this. So, the spectrum will be seen like this. So, this is omega verses your d star that is your spectrum plot.

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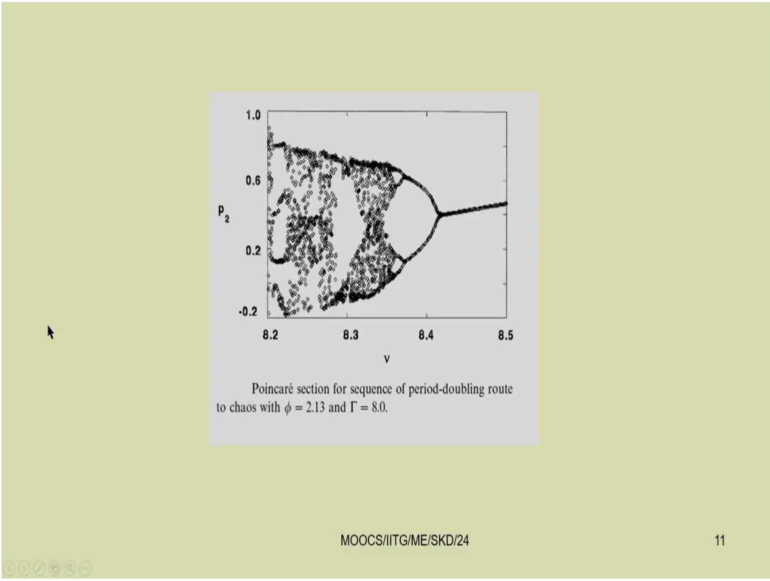

So, if we plot the Poincare' section. So, this is this clearly shows this period doubling the route to chaos. So, initially single period. So, in Poincare' section you have a single point, now two point. So, here you have two points, here you have four points. So, that way you have seen. So, by changing this parameter nu so, we have seen in this case how the bifurcation is occurring.

So, this point is a critical bifurcation point and these point and these and these points are critical bifurcation points you can find the ratio. So, for example, this is this becomes 1 to 2. So, now, this from 2 to 4, then for 4 to 8 you can see. So, this is 2, this is 2. So, after these things, so, you can have 8. So, then 16. You can use there are certain theory or you can find so, when this period doubling occur by using this Feigenbaum number.

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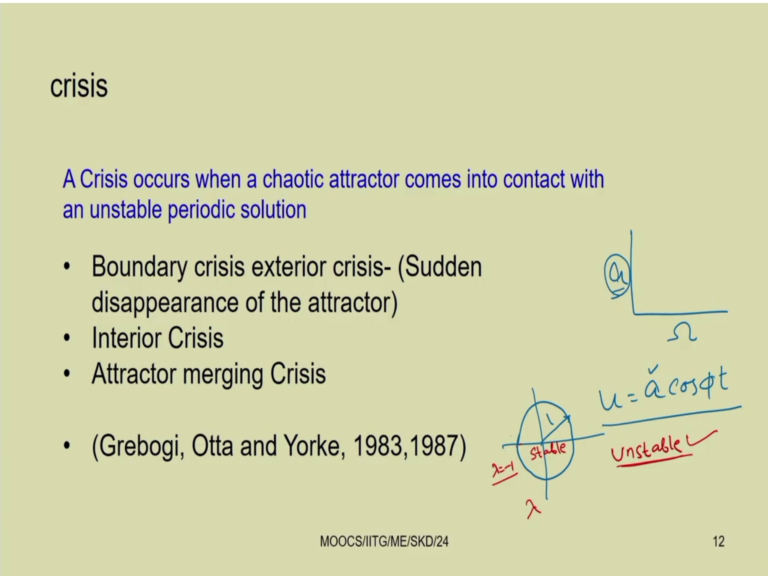

So, Feigenbaum is a universal number that thing we will see after some time. So, sometimes you can get this crisis in this chaotic response. A crisis occurs when a chaotic attractor comes in into contact with unstable periodic solution.

So, later before we have seen in the frequency response curve that in addition to this periodic response, there exists the chaotic and quasi-periodic response. Sometimes this chaotic response may come in contact with the unstable fixed-point or unstable periodic response.

So, you just see when you have plotted this a versus a versus this omega. So, this a is nothing but the amplitude of the periodic response. So, actual response is u which is equal to a cos phi a cos phi t. Here this u that is the response is periodic, but we have plotted only this a versus omega in those curves. So, that is the amplitude of the response.

Out of all these responses what do you have plotted, so, some of them may be periodic and some of them may be stable and some of they may be unstable. So, already you are familiar with the unstable or stable periodic response so, by plotting the monodromy matrix. When you plot the monodromy matrix so, it must be within the unit circle. So, it must be within the unit circle.

The eigenvalue of the monodromy matrix must be within this event circle to have a stable solution. So, if we want to have a stable solution so, it must be unstable. So, if it is outside this thing then it is unstable. So, here it is stable. So, if it is inside this is lamda you are plotting; if it is within this these eigenvalues all the eigenvalues are inside this unit circle, then it is stable and if it is outside, then it is unstable.

So, these are the Floquet multipliers when it is in plus 1. So, then you can have a periodic solution; when it crosses the limits or the cycle by lambda equal to minus 1, then you can have two periodic solutions. So, all those bifurcations so, that period doubling bifurcation so, here when it is leaves the unit cycle through minus 1. So, there will be period doubling and that period doubling again may further double to have this period doubling route to chaos.

If we have a unstable periodic response, we have a chaotic response and unstable periodic response and the attractor the attractor passes through this unstable periodic response or cross this unstable periodic response, then suddenly it may disappear or it may explode.

So, if it disappears suddenly that attractor disappear and the and we are getting some resulting fixed-point response or some periodic response, then that type of crisis is known as boundary crisis or exterior crisis. So, it is known as boundary crisis because later the response will be bounded to a either a fixed-point or a periodic response.

So, this chaotic attractor will suddenly disappear. So, if it suddenly disappear then we tells it as a boundary crisis or exterior crisis. And, when it is coming in contact with an unstable periodic point and the attractor enlarges itself or explodes. And, the original attractor remain inside the newly formed attractor so, then it is known as interior crisis.

Sometimes there will be two attractor existing and when it will come in contact with this unstable fixed periodic response they merge and form a bigger attractor. So, this is known as attractor merging crisis. So, we have three different types of crisis – one is the boundary crisis or exterior crisis, when sudden disappearance of the attractor occurs then we can have the interior crisis.

When suddenly it explodes and the original attractor remain inside the final attractor. So, they and the third one is the attractor merging crisis. In 1983, Grebogi Otta and Yorke, so, they have proposed these three different type of crisis in physical systems so for the system what we have studied before that is a cantilever beam subjected to base excitation.

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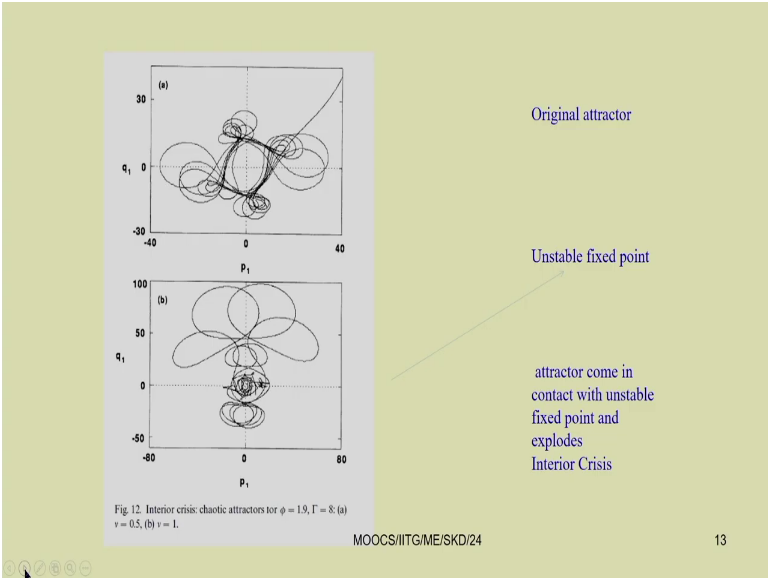

So, here we have observed for example, so, we have observed for nu equal to 0.5 and phi equal to 1.9 gamma equal to 8, we have this attractor. So, this is the this is the attractor present at this point. And, now so, when it comes in contact with another attractor so, slightly by changing this nu to 1 nu to 1 so you can see the same attractor comes inside so, come in contact with a fixed-point trivial response.

So, the 0, 0 is the fixed-point trivial response. So, at that point the trivial response is unstable. So, this attractor comes in contact with the trivial unstable periodic response and it explodes to a bigger attractor. So, that is why so, this is known as interior crisis. So, you can come with a interior crisis.

So, here attractor come in contact with the unstable fixed-point and explodes and gives rise to interior crisis. So, this is the original attractor and this is the attractor which explodes to form the interior crisis.

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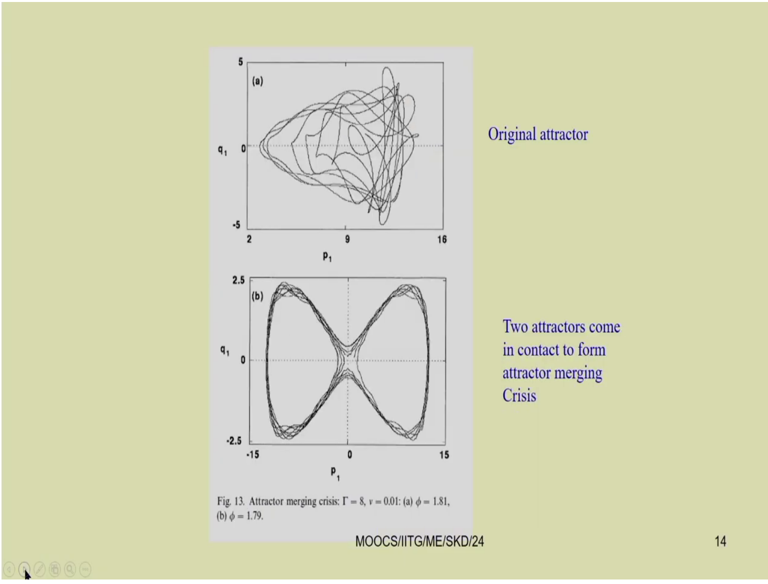

So, similarly we can see some other different type of attractor merging crisis before for example, at nu equal to gamma equal to 8 and nu equal to 0.01, phi equal to so, now, we are changing phi. So, initial curve phi equal to that is the frequency of non-dimensional frequency of frequency of excitation equal to 1.81.

So, when we are slightly decrease that frequency. So, you can see when we come to 1.79, so, suddenly so, it come in contact with this unstable fixed-point. So, these two attractor merge.

So, these two attractor merge to form the chaotic response. So, here it is wandering actually for some times.

So, it will be in the this left side and then suddenly it will go to right hand side and it will stay for stay or graze for some times in this right hand side and then it will come back to the left side. So, this way these two attractor get merged and they form a bigger attractor. So, this chaotic initial chaotic attractor now becomes two chaotic attractor. So, these two, merge to form a bigger attractor. So, this is known as attractor merging crisis.

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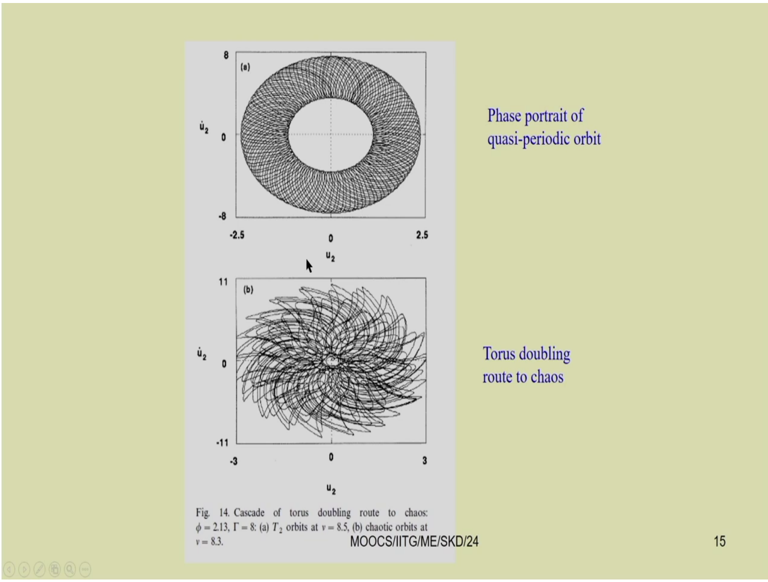

Similarly, there are some other routes to chaos also. So, initially we have a quasi-periodic response. So, here the quasi-periodic response is there for phi equal to 2.13 gamma equal to 8. So, we have a torus at nu equal to 8.5. So, now, when we are reducing this damping that is nu to 8.3. So, these are non-dimensional damping parameter.

Nu is reduced to 8.3. So, from 8.5 to 8.3, so, you can see torus doubling route to. So, this torus here doubled to have the route to chaos. So, torus doubling route to chaos one can observe here. Sometimes we can observe this torus break down to chaos, particularly this occur in case of a combination parametric regions and case and here it is not shown.

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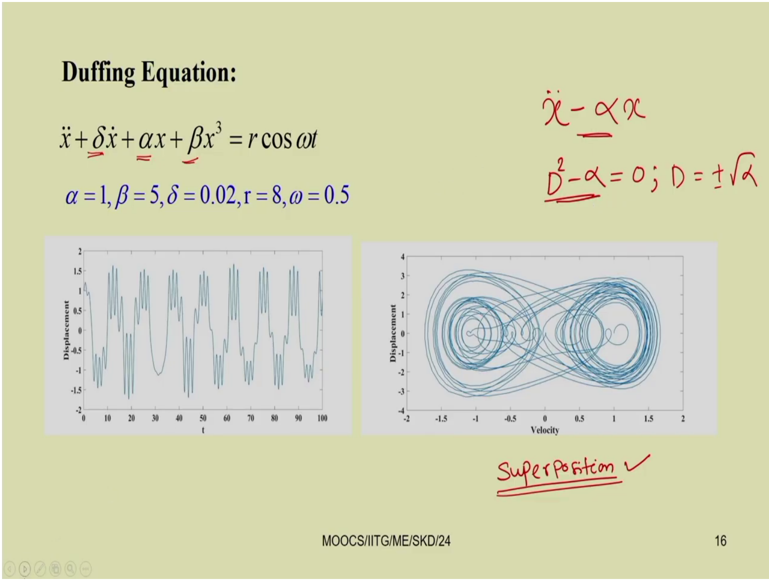

Coming back to the Duffing equation also. So, for example, we have seen this thing; so, in case of Duffing equation by changing the system parameter so, you can see this chaotic attractor. So, this chaotic attractor is nothing but so, it contain many harmonics in this response.

So, by taking this you just see here we have taken this delta alpha is the alpha is nothing but this omega n square, delta is 2 epsilon zeta omega n. So, beta is the coefficient of cubic nonlinearity and r is the forcing that is the amplitude of forcing and omega is the frequency of forcing.

In this case, you just see by changing the system parameters so, you are getting chaotic response. So, that is why you can tell that these non-linear systems are generally or non-linear system generally does not obey the superposition rule. So, it will not obey the superposition rule.

So, by changing slightly the system parameters so, you can have different type of response. So, it may be fixed-point maybe periodic quasi-periodic or chaotic response. You cannot apply or you cannot predict. So, what will happen to the response at next moment when you are studying a non-linear system.

So, you have to properly study this non-linear system and you can see how the displacement velocity are evolving with time. So, here the time response is shown and you just see this this course there is no similarity in this in the different periods. So, the shapes are very irregular and you can see how it wanders so, from one orbit to another orbit.

So, here also it may be attractor merging type of crisis will be there. So, when it is fixed-point response, so you can have the homoclinic orbits. So, two homoclinic orbit, there will be heteroclinic orbit. So, this collapse of these homoclinic and heteroclinic orbit will leads to so, this type of chaotic response.

So, these Duffing type of oscillators. So, easily you can visualize. So, these are the equations where or that of a that of a spring mass system subjected to a forcing where the spring you can take as a non-linear spring. So, if you are taking a non-linear spring with cubic nonlinearity then it will be beta x cube only.

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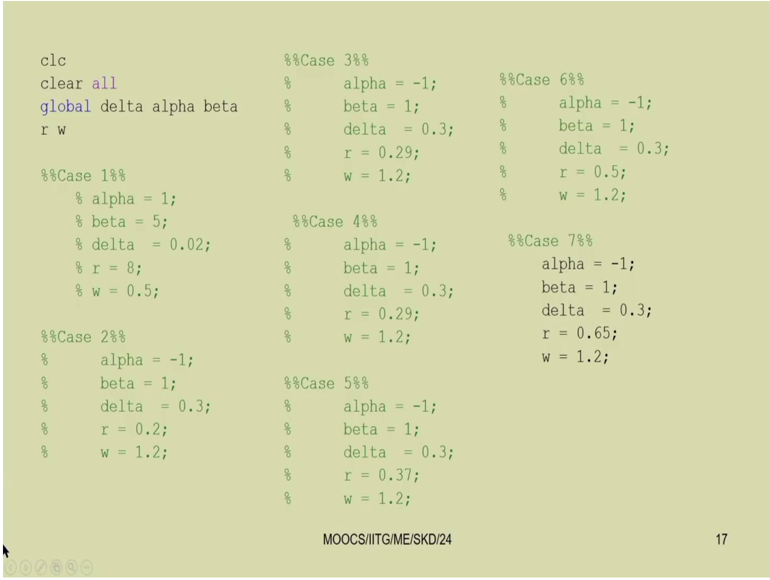

These are the it is a code written to find the responses using this Ode45.

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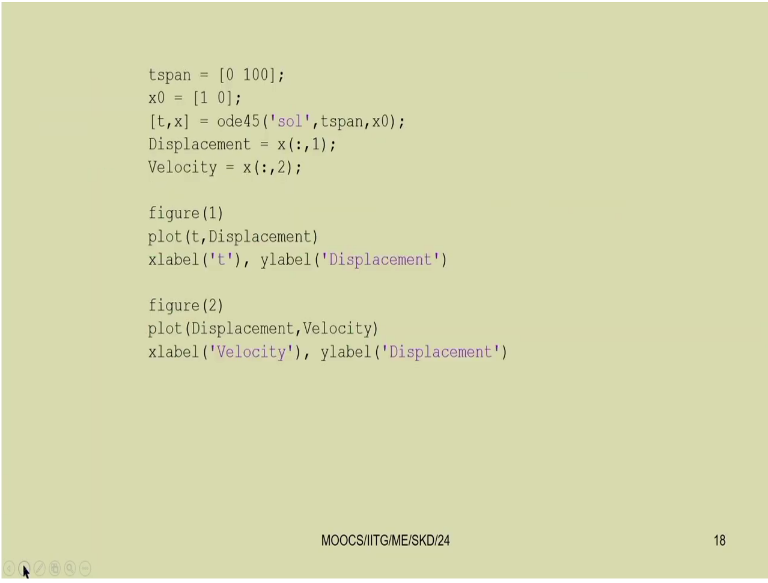

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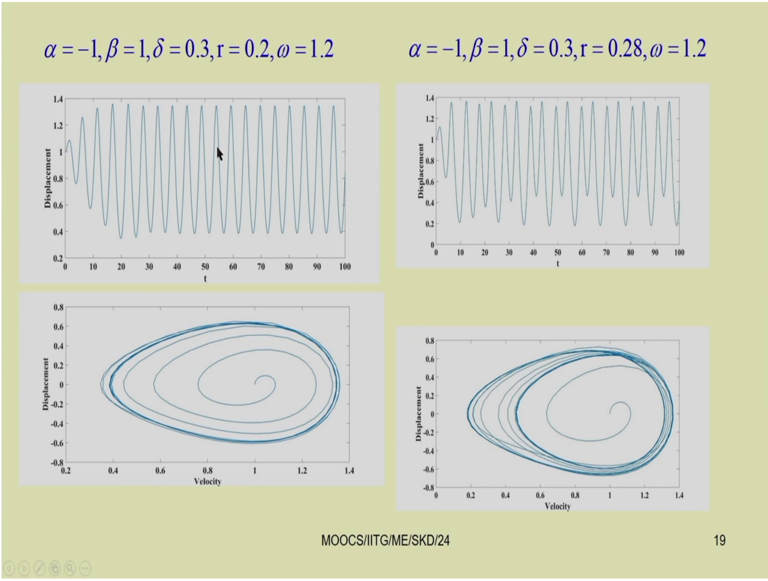

So, initially so, you can have for you just see if you take this alpha equal to minus 1. So, here alpha is coefficient of x. So, if you have a equation x double dot minus. So, this is alpha x. So, here if it is minus alpha x you just see the auxiliary equation becomes D square minus alpha. So, the roots will be equal to by putting it equal to 0 you can find the roots.

So, the roots will be d will be equal to. So, D will be equal to plus minus root over alpha plus minus root over alpha; D square equal to alpha. So, D will be equal to plus minus root over alpha. Here the solution, so, this is real. So, from the solution you can see the system generally is so one of the solution must be greater than 0 and other solution must be less than 0.

So, due to this thing there the system will be generally unstable if you are taking a linear system with this, but when there is non-linearity present in the system with the same Duffing type of equation with alpha that is negative stiffness taking the negative system negative stiffness the response is found to be bounded.

So, you just see the response is bounded, so, here omega equal to 1.2, r you are changing from 0.2 to 0.28. Here it is changing from a periodic response to a two periodic response. So, here it will be only single periodic and in this case it is two periodic. So, you can get two periodic response.

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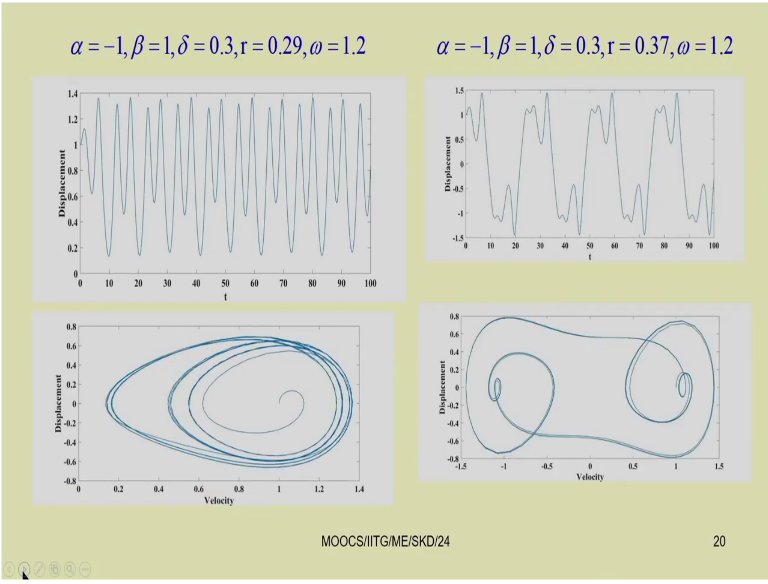

So, then by changing this r further by changing this r, you can see you can see period doubling occurs and finally, you can have a chaotic response.

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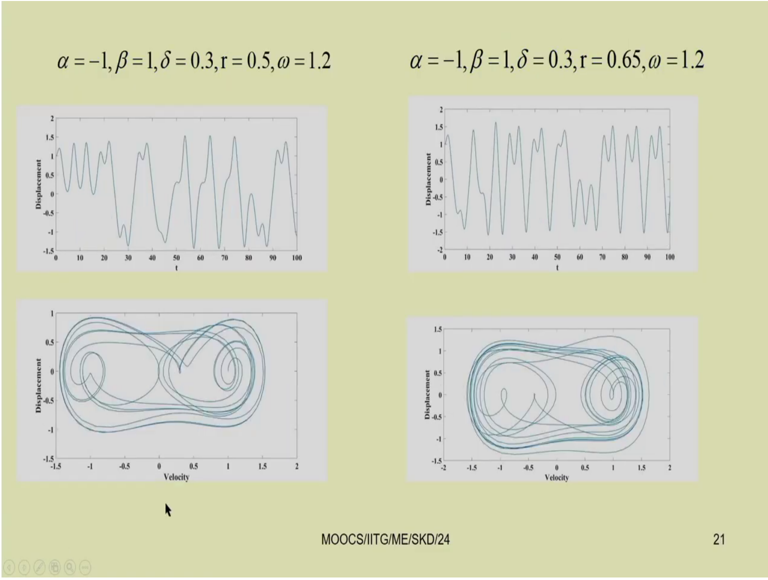

So, this way you can with a simple Duffing equation also you are getting this chaotic response.

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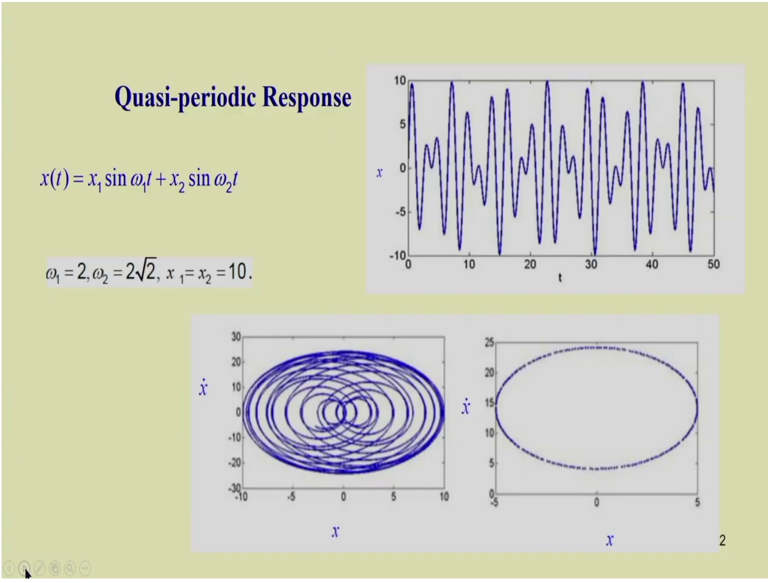

So, similarly from the quasi-periodic response also so, you can get the chaotic response.

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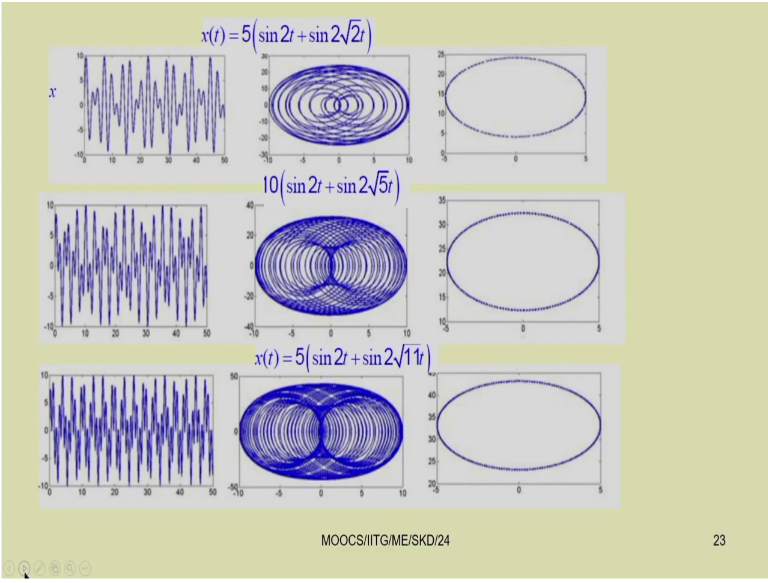

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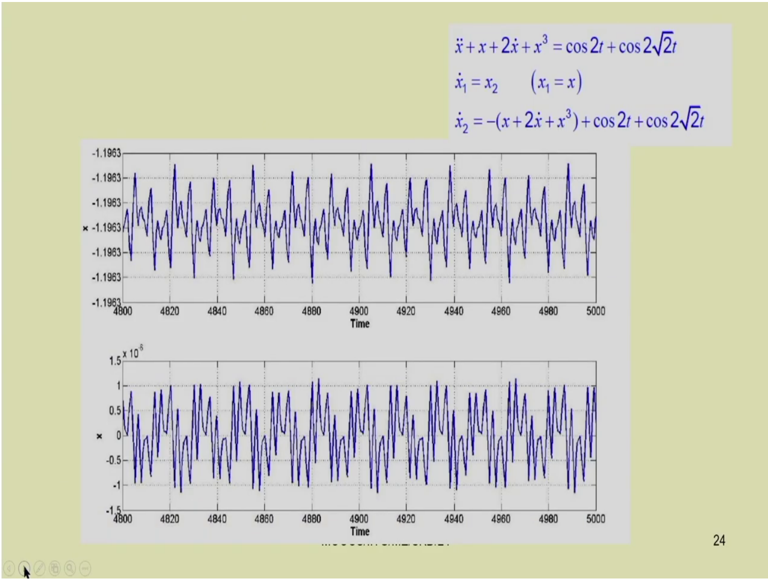

So, when this breakdown or it doubled either torus doubling or torus breakdown both will lead to chaotic response. Here we have taken a simple non-linear spring with two so, let us put two forcing. So, two harmonic forcing term so, one is cos 2t, one is cos 2 root 2t. So, in this case you just see by we can write down these two by two first order equation and here the ratio of the frequency is root 2. So, you are going to get quasi-periodic response.

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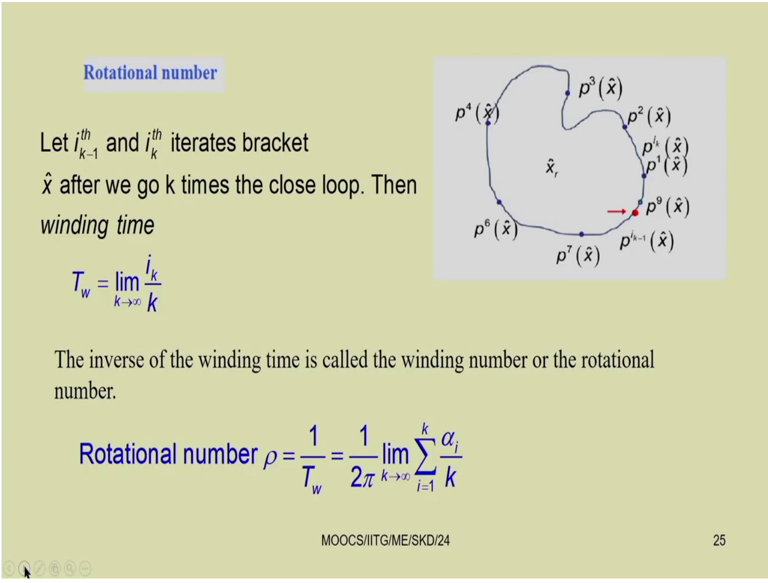

So, as it is non-linear so, you can have multiple solutions also, so, you have to study which solutions you have to take for this multiple thing.

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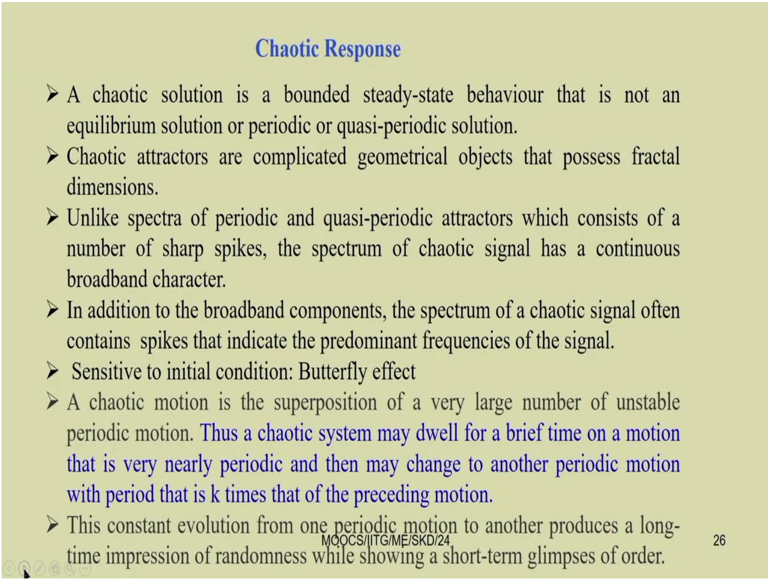

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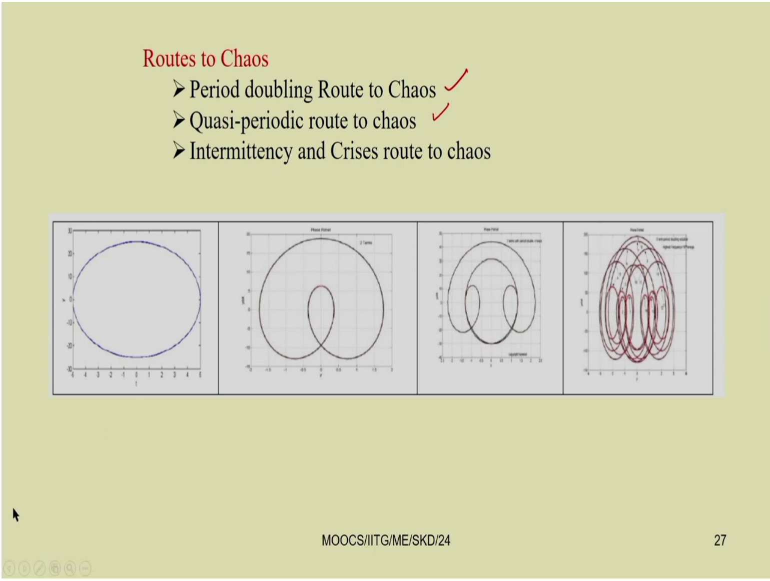

So, already we know. So, we have period doubling route to chaos. So, we have quasi-periodic route to chaos and then we have this intermittency. So, we can particularly in case of fluid mechanics or fluid these turbulent flow you can find this intermittency type of turbulence. So, then we have this crisis route to intermittency route to chaos or this crisis route to chaos. Particularly, if you study or take a delay differential equation you can find chaos also.

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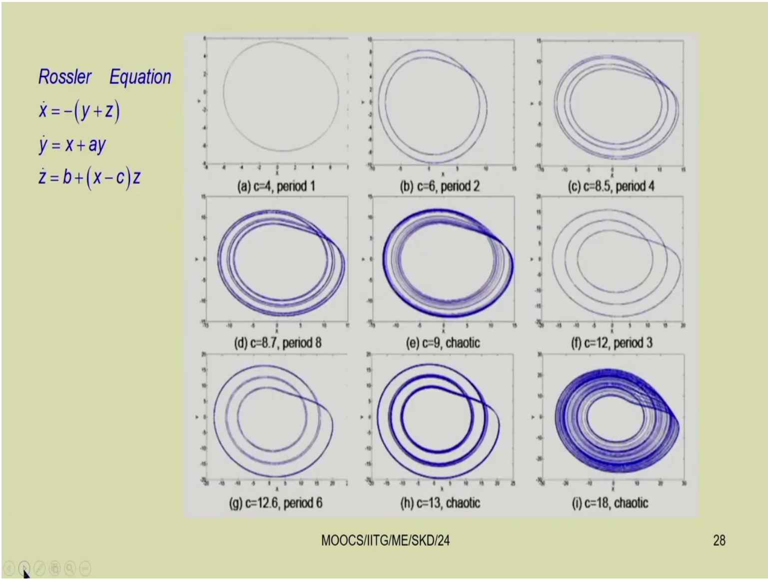

So, in addition to the physical systems what you have seen. So, we can have some other system for example, this Roessler equation. So, in this Roessler equation this is a very simple equation. So, here this first order equations can be written x dot equal to minus y plus z; y dot equal to x plus a y and then z dot equal to b plus x minus c into z.

So, you just see by taking different parameter of c taking parameter of c, one can get this initially this periodic then it becomes two periodic. So, c equal to 8.5, so we have period 4; then c equal to 8.7 period 8 period. So, period 9, so, period 12. So, different periods we can get. So, actual case this is a three dimensional system.

So, x, y, z, but you can plot it using this two dimensional also that is it is plotted for example, x y or it can be plot y z or x z. So, this is in two dimensional, it can be plotted in two dimensional or in three dimensional itself.

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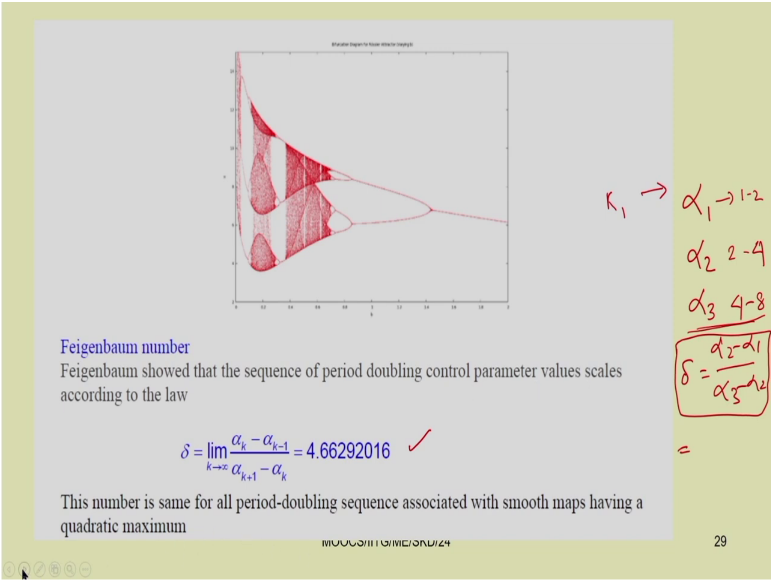

So, we can define a number actually in this period doubling route to chaos. So, we can define a number so, which is known as Feigenbaum number. So, this Feigenbaum number can be defined by this alpha limit k tends to infinite alpha k minus alpha k minus 1 divided by alpha k plus 1 minus alpha k.

So, this is the k-th bifurcation. So, if you are taking so, for example, period doubling so, let period doubling start alpha. So, let us take k equal to 1. So, we can take k equal to 1. So, when the it change from periodic 1 to periodic 2. So, when it is changing from periodic 1 to 2, so, let us take this alpha 1, alpha. So, we can take alpha 1, alpha 2, alpha 3.

So, this three let us take. So, alpha 1 when it is changing from period 1 to 2, so, then when it is changing from period 2 to 3. So, this is alpha 3 and period 3 to. So, when it is changing from 1 to 2 that is alpha 1 and 2 2. So, it will double. So, this becomes 2 to 4 then so, it will becomes 4 to 8. So, it becomes 4 to 8 that is alpha 3.

According to this Feigenbaum number so, delta will be equal to. So, actually we have taken this first one we have not taken k tends to infinite. Generally, once you take k tends to infinite so, delta can be written as so, by using these things you can write this is alpha 2 minus alpha 1 by alpha 3 minus alpha 2.

So, physically you can realize this Feigenbaum number like this. So, that is delta equal to alpha 2 minus alpha 1 by alpha 3 minus alpha 2. So, this number is a constant number. So, that value equal to 4.66292016. So, this number is same for all period doubling sequence associated with a smooth maps having a quadratic maximum. You can verify so, I have shown you several period doubling route to chaos.

So, for example, I have shown a parametrically excited system where you can find the period doubling and also in this Duffing oscillator we have seen and in this Roessler map also we have seen this period doubling route to chaos. So, in all these cases you can verify the Feigenbaum number. So, this is left as an assignment, so that you can verify this whether this Feigenbaum number is working there or not.

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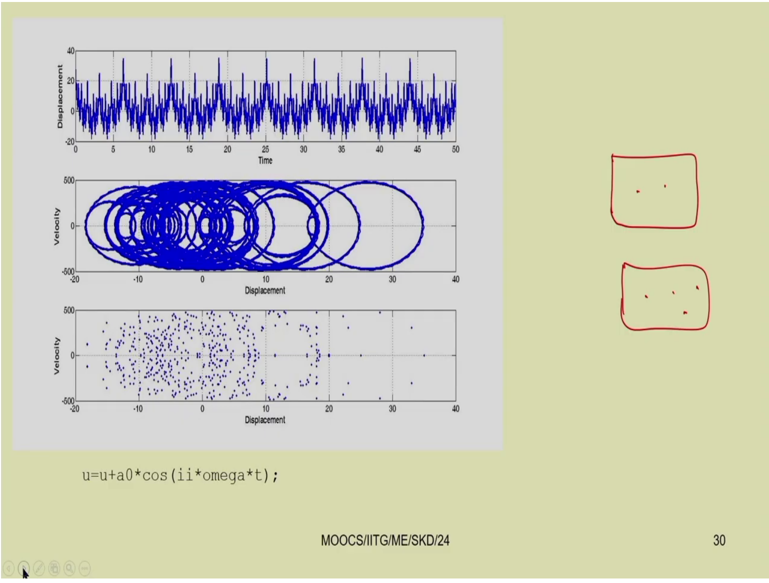

So, you can generate a period doubling route to chaos by or you can generate a chaotic response, for example, by simply writing your own code.

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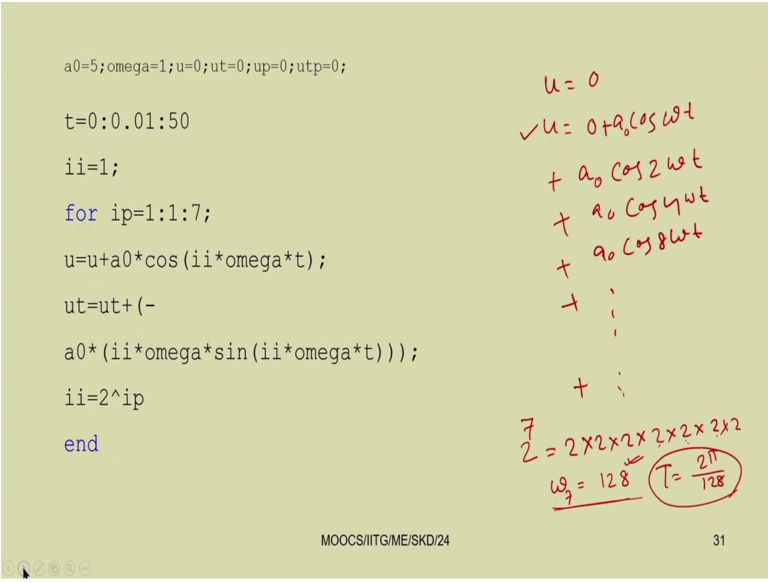

So, for example, you just take u equal to. So, this code is given here you just take initialize. So, you have initially you have taken u equal to 0 initially you have taken u equal to 0 then u you take u plus a 0 cos ii into omega t, where this ii initially you have taken for first iteration it will be 1.

So, for initially so, u equal to 0, then the next step becomes u equal to 0 plus a cos so, here your i ip equal to. So, we have taken ii equal to 2 to the power ip that is it will be initially it will be 1. So, it will go on multiplying ii equal to 2 into ip initially it is 1, then next time when it comes. So, then this becomes 2 into 1. So, 2 into 1 that is 2, 2 to the power 1 equal to 2 then next time then ip becomes 2 so, this will be 2 to the power 2 that is 4.

So, if you go on adding this thing that is u equal to a cos for example, a cos omega t plus a, a 0 I have written. So, a 0 cos 2 omega t, then you will have a 0 cos 4 omega t a 0 cos 8 omega t you go on adding and for example, you go on adding and you plot this thing with respect to time. So, this will give rise to a curve. So, here we have plotted up to. So, in this case we have plotted up to 7.

So, we have taken this to 7 7. So, you can get a response like this. So, this response clearly shows the response is chaotic. You can plot this thing you can plot the Poincare' section. So, here you know what is the maximum omega you have taken. So, maximum omega for example, you have taken 2 to the power. So, as ip you have taken 7. So, you have taken maximum this is 2 to the power 7. So, 2 to the power 7.

So, 2 into 2 into 7 times you can multiply this 2 and you can find 6 1 2 3 4 5 6 and 7. So, this is 16, 32, 64 and 128. So, you have taken 128. So, your by taking here we have taken omega for example, omega we have started with 1. So, then finally, this is omega equal to so, omega 7 equal to 128 you have taken. So, the time period so, this will gives to least time period. So, this will t will be equal to 2 pi by 128.

So, by sampling the response with 2 pi by 128 so, then you can plot the Poincare' section. So, this Poincare' sections will look like this. So, it fills up the whole space. So, depending on the time up to which you are taking. So, you can verify or you can visualize that the Poincare' section is filling of the whole space in the phase portrait. So, that is why the response is to be chaotic.

So, in case of for example, you just take only one or two terms so, if you take only two terms then you can see you have only two points. So, if you take four so, then you have four points and if you can take more than that thing more than 16 and 32, then it will completely fill up the space and the response will be chaotic. And, here you can check your displacement is bounded, velocity is bounded that is why this chaotic response is a deterministic response. So, here the response is bounded.

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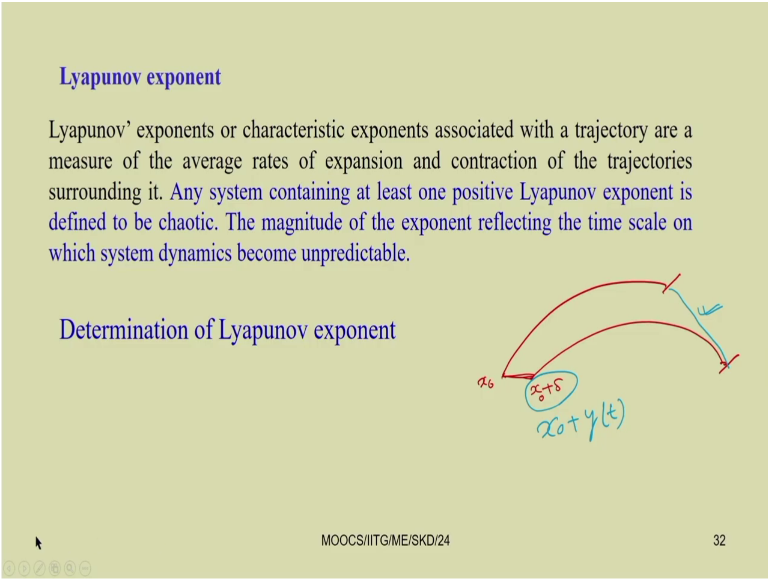

Also you can characterize this chaotic response by using this Lyapunov exponent and this Lyapunov exponent or characteristic exponent associated with a trajectory are a measure of the average rate of expansion and contraction of the trajectories surrounding it. Any system containing at least one positive Lyapunov exponent is defined to be chaotic for the chaotic response or the chaotic system at least one of the Lyapunov exponent must be positive.

And it can be shown that for a fixed-point response so, the existence of a fixed-point response you can verify or you can find the Lyapunov exponent and you can check that it is to be 0. So, the magnitude of the exponent reflecting the timescale on which the dynamic dynamics becomes unpredictable.

So, what we can do? So, how to find this Lyapunov exponent, we can see what we can do. So, let us start with a initial point x 0. So, we have started with the initial point x 0 and we know with time, so, this will grow and the this is the response of the system. So, now, what you can do? So, you just start the same system with another points very close to it. So, this is x 0 plus delta you just take and then you find the solution.

So, if you find the solution so, you just see initially this is the difference between this thing initially this is the difference between this initial position and with time t so, you have seen the difference grows. So, this is the difference now the difference is this. So, you have seen the difference as grown. So, the difference has grown and so, this is the next this is the difference.

So, here the difference is delta, but here the difference is very large. Any growing function can be written by using an exponential form also. So, by using some exponential form so, you can write the function and by taking that way, so we can determine this Lyapunov exponent.

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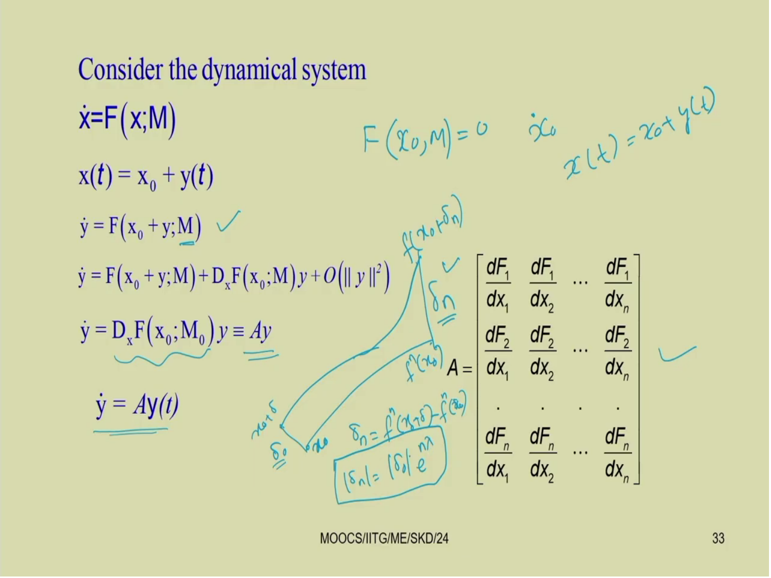

So, let us see for example, consider the dynamic system. So, let us have a dynamic system x dot equal to F x mu. So, with time so, it will grow and the response can be written. So, you can find this x t equal to x 0 plus or you can perturb this x so, let initially it has an equilibrium solution.

So, this equilibrium solution can be written as x 0, what is equilibrium solution? So, it satisfy the this equation. So, that is at x dot equal to F x M. So, if x 0 is the equilibrium solution then F x 0 M equal to 0. So, now you just perturb this x t. So, take x t equal to x 0 plus y t. So, the previous case I have taken this thing as delta. So, initially you have this is the equilibrium position.

So, now, you have taken perturb it and you have written this thing x 0 plus you just write this is y t. So, because this delta will grow and you can write this as y t also. So, here by substituting this x t equal to x 0 plus y t. So, you can have this y dot. So, you can find. So, now, you substitute it in this equation.

So, this becomes x 0 dot plus y dot. So, it will be equal to F x 0 plus y M minus F x 0, M as x 0, F 0 M equal to 0. So, you know. So, if 0 is the equilibrium position. So, F x 0, M so, it is equal to 0. So, by perturbing this thing so, you got this y dot equal to F x 0 plus y and M M is the critical parameter what you are changing.

So, this way you can find this y dot. So, y dot equal to so, by expanding that thing using this Taylor series. So, you can write this y dot equal to F x 0 y M plus D x F x 0 M. So, into y so, this way you can write, but this part equal to already you know this part equal to 0. So, this y dot becomes D x F x 0 M 0 and you can write this thing into y. So, this is equal to Ay.

So, this part you can write using a matrix. So, that is a. So, by using this A matrix. So, this is nothing but similar to your this Jacobian matrix. So, this Ay you can find. So, y dot can be written as y dot. So, what is y? y is the deviation. So, how this deviation is growing? So, that is the by starting that thing so, you can find the Lyapunov exponent.

So, this is the A matrix. A matrix is nothing but del F 1 by del x 1 del F 1 by del x 2 and del F 1 by del x n. So, this is for the first function. So, your F may contain n number of function. So, if it is n-dimensional equation. So, for example, in this Roessler equation, so, you have three equations. In that case you your n will be equal to 3 and you have 3.

So, your A matrix will be a 3 is to 3 matrix. So, after finding this y dot equal to A y t then taking different initial conditions. So, you can plot. So, how much deviation it is having what is the deviation. So, knowing that a deviation so, you can find this Lyapunov exponent. For example, initially it is x 0, now this becomes x 0 plus delta let you take.

So, let this is the function. So, now, a time t so, this has come x 0 plus delta n here you have this with time. So, this becomes F n x n x 0 F n x 0. So, you can write this deviation delta n. So, this deviation delta n can be written as F n x 0 plus delta minus F n delta. So, you can write this way and these or this delta n you can write equal to delta 0 into e to the power n lambda x 0. So, this is delta n equal to delta 0.

So, any growing function can be written by using this way that is this delta n. So, finally, this deviation let it is delta n. So, this delta n can be written so, F x 0 delta n minus F n x 0 that function. So, initially this is this now it is going to this form. So, this is your delta 0. This delta n can be written using this delta 0 by using this function that is delta n will be equal to delta 0 e to the power n lambda.

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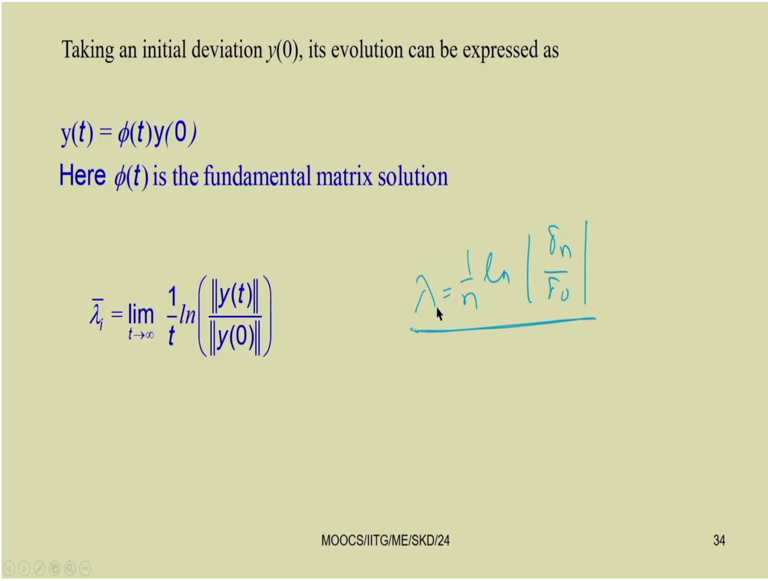

So, here this lambda is nothing but the Lyapunov exponent which can be written as you can you have seen these things. So, you can write that lambda equal to 1 by n ln so 1 by n ln so, delta n by delta 0. So, physically you I hope you have understood that this is nothing but so how it is either converging or diverging so, that part we are checking.

So, if it is converging so, this delta n will be less and if delta n is more or it is growing then if it is diverging then delta n will be very high and delta n by delta 0 will be higher value and if it is converging, then you can have a low value. Now, by taking this y t equal to phi t into y 0.

So, where this phi t is the fundamental metric solution so, you can find this lambda i. So, that is limit t tends infinity 1 by t. So, either you write in terms of this n. So, n step I have written this n is the n after n-th iteration or after n-th time. So, you are taking some time increment. So, for example, let you are taking time increment h.

You have starting with time t equal to 0. So, after n-th iteration it will be 0 plus n into h. So, that is why so, after n-th iteration so, you can get what is the deviation. So, after knowing that deviation so, let the deviation is delta n and initially it is delta 0 lambda equal to 1 by n ln delta n by delta 0. In this way you can find the Lyapunov exponent.

In the next module, so, we are going to study regarding the numerical methods used for used in this analysis of dynamical system. So, there we are going to study more on how to use or how to find this Lyapunov exponent. So, this you will find the code written for finding this Lyapunov exponent.

So, here in this example I have given the time period is known to you, but generally in case of the experimentally obtained times time series or obtained the time response. So, the time period are not known. So, in that case from the experimental data how you can determine this Lyapunov exponent also you can learn that time.

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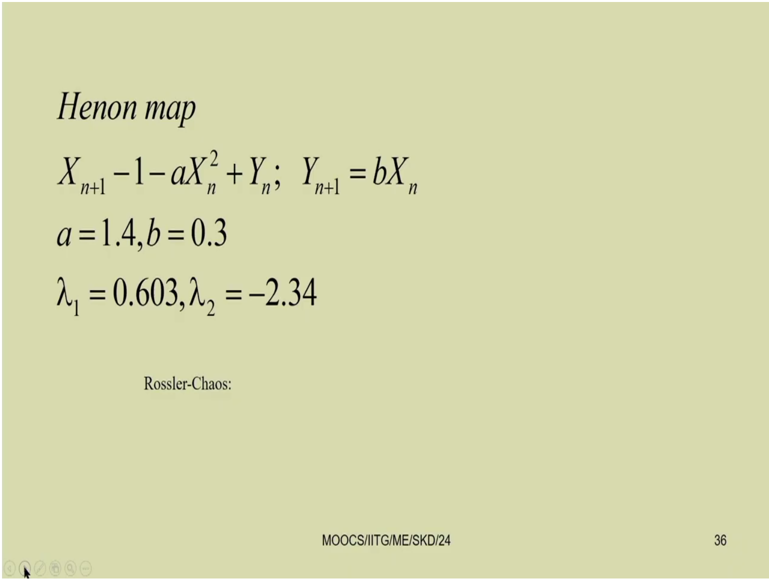

So, in the next module you will learn all the numerical methods. So, here we have plotted this or you can take a Henon map to find the chaotic response.

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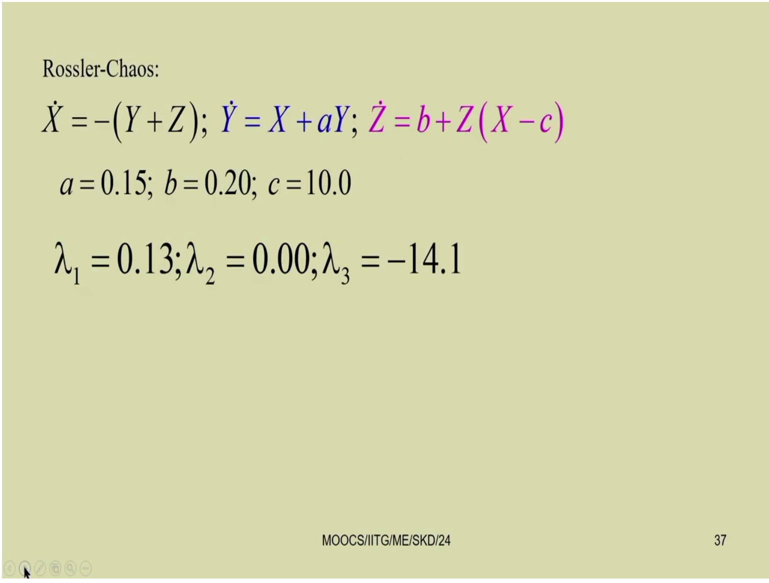

Rossler map already have shown.

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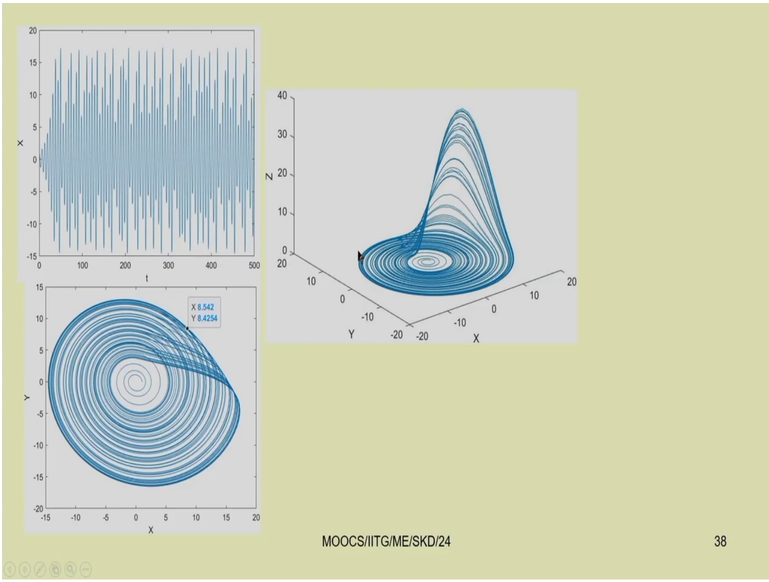

So, here this three dimensional plot is also plotted. Previously, I have shown you this X versus Y. This is the time response in case of a this time response for this Rossler map Rossler equation. So, here you just see the time response it is bounded the response is bounded.

So, it is within plus minus, this is this side it is minus 15, this side it is plus 20, but the response is not periodic or quasi-periodic and it is not a fixed-point also, it is not going to a fixed-point that is why this response is chaotic. So, if you plot in X-Y, so, this is the X-Y plot and if you plot in XYZ. So, this is the plot in XYZ direction. So, you can find the Lyapunov exponent in this case also.

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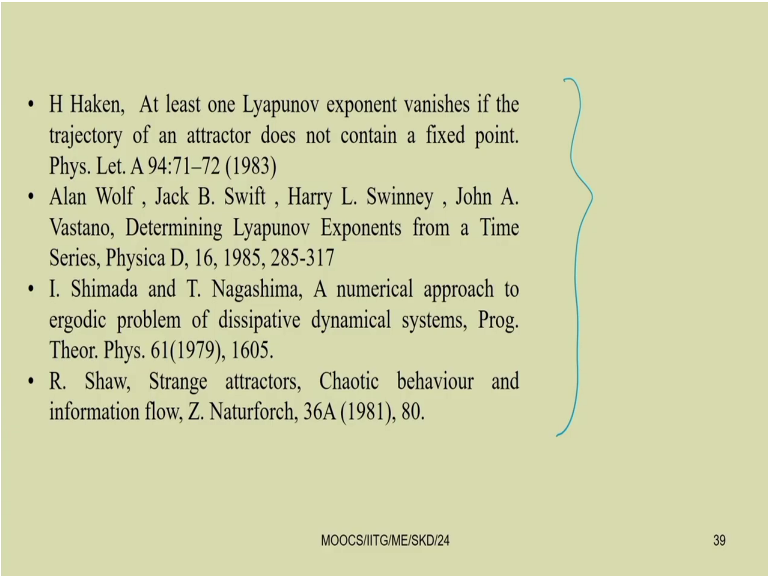

So, you can find the Lyapunov exponent and we will see you can get the you can check that one of the Lyapunov exponent must be positive. So, we can see all these papers. So, for example, this is a very good papers. So, by H Haken, so, at least one Lyapunov exponent vanishes if the trajectory of an attractor does not contain a fixed-point. So, at least one of the Lyapunov exponent vanishes if the trajectory of an attractor does not contain a fixed-point.

Similarly, this is a very good paper so, where this determining Lyapunov exponent from a time series. So, it is published in Physica D in 1985 by Wolfs, Swift and Swinney and Vastano. So, generally it is known as Wolf's method for finding the Lyapunov exponent. Also you can see this is another paper a numerical approach to ergodic problems of dissipative dynamical systems you can find it in this journal that is theoretical physics.

Then, another one that is strange attractor, chaotic behavior and information flow. So, these are very good before in this field of chaotic response. So, now, as you know how to generate this chaotic response or you have seen in this physical system chaotic responses are there. So, you must know how to control this chaos. So, this controlling this chaos, determining the characteristic of the chaos are a subject of its own. We will study or we will give some example as an assignment to know how you can control this chaos.

So, there are several methods to control the chaos. So, now, what do you have seen? So, by using this exterior crisis or this boundary crisis. So, if you can bring the attractor near to a fixed-point response or near to one unstable periodic response, then you have seen that it disappear.

So, that is one method you can use for controlling the chaos that is known as the OGY method. Similarly, you can use some feedback control loop you can use some feedback control loop to vary the system parameter and bring it to a domain where the response no longer is chaotic. So, it may be fixed-point or periodic.

So, the basic purpose is to change the system parameter in such a way that so, it will be out of the chaotic domain. So, you have seen so, in between the chaotic region also you have a window of fixed-point response or window where you can have a fixed-point periodic response.

So, somehow by changing the system parameter, using some adaptive control mechanism so, if you can shift or you can take that attractor to these that window where it is not periodic then you will be able to control chaos. Sometimes this chaos are also necessary in many places this chaotic responses or chaos is necessary. So, later we will see where this chaos are necessary when we will study the application of these non-linear vibration systems.

So, next class we are going to study the next model next module where we can study the numerical methods and in this numerical methods particularly, we will be interested to study how to find the roots of the characteristic equation, how to solve if we have a number of algebraic equations are there, how you can use this Runge-Kutta method or there are some other methods, other methods for solving this differential equations. So, all those methods we will study in next three classes by using different numerical methods.

Thank you very much.