**Nonlinear Vibration**
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**Lecture - 23**
**Stability and bifurcation analysis of periodic and quasi-periodic response**

So, welcome to today class of Non-Linear Vibration. So, we are continuing module 7, where we are discussing regarding this periodic, quasi-periodic and chaotic responses.

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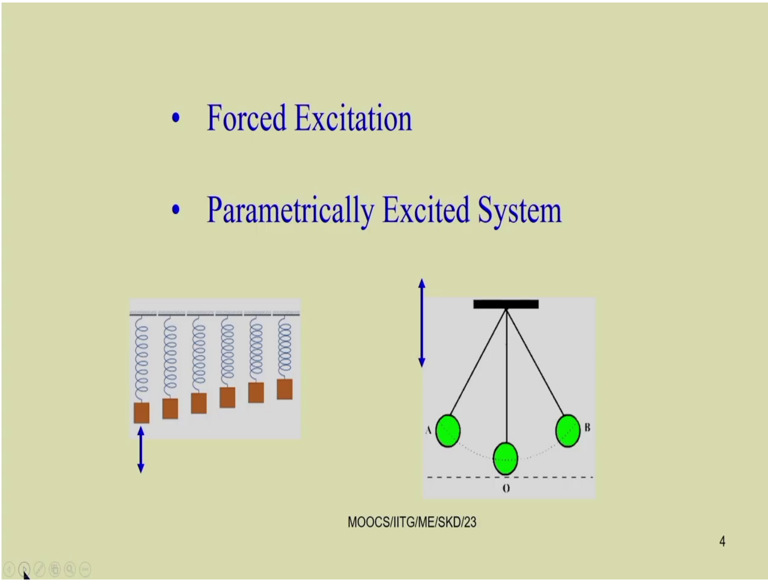


So, last class we have seen different type of response. For example; so, we have seen this system with force excitation and parametrically excited system.

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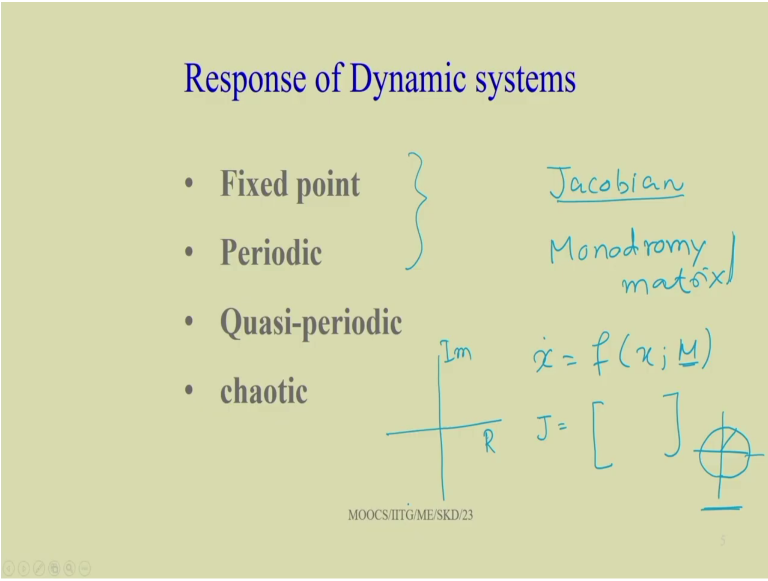


So, in both the cases so we have observed one can have this fixed point response, periodic response, quasi-periodic response and chaotic response. In case of periodic response, the response will reoccur or the response will repeat after a interval of time T.

So, it may be periodic, it may be two periodic or it may be multi-harmonic responses. Similarly, in quasi-periodic response, which we are going to study today and chaotic response we will see how the behaviour of the system is represented by this quasi-periodic and chaotic responses today.

So, these three type of responses already we have seen this fixed point and periodic response. And today class particularly we will be interested to study the stability and bifurcation of periodic response. So, already we know how to study the stability of the fixed point response by finding the Jacobian matrix. We can find the eigenvalue of the Jacobian matrix to study the stability and bifurcation of the fixed point response.

Similarly, we have to study the eigenvalues of the monodromy matrix to study the stability of the periodic response. So, this Jacobian matrix, so, for example, let us have a equation. So, in this form that is x dot equal to f x comma semicolon M.

So, M is the parameter, in this case we can find the Jacobian matrix by finding this first derivative of this f. So, we can have the Jacobian matrix and after finding this eigenvalue of the Jacobian matrix, then by checking whether it is in the left or right side of the s plane. So, this is the real axis, this is the imaginary axis.

So, by checking the eigen values we can tell whether the system is stable or unstable. So, for a stable system the eigenvalue must lie in the left hand side of the s plane. And if some of the eigenvalues are lying on the right hand side of the s plane then the system is unstable. Similarly, we can find the monodromy matrix.

So, to find the monodromy matrix so, starting from some initial point so, we have to find the response after one cycle. So, after getting this response after one cycle, then we can construct the monodromy matrix and by finding the eigenvalue of the monodromy matrix so, we can find the stability of the system.

So, there we are studying or we can find a unit circle if the eigenvalues or the Floquet multiplier are lying within the eigen within this unit circle, then the system is stable otherwise the system is unstable. So, this part we will see today again and from these things we will study how to carry out the bifurcation analysis in case of these periodic response.

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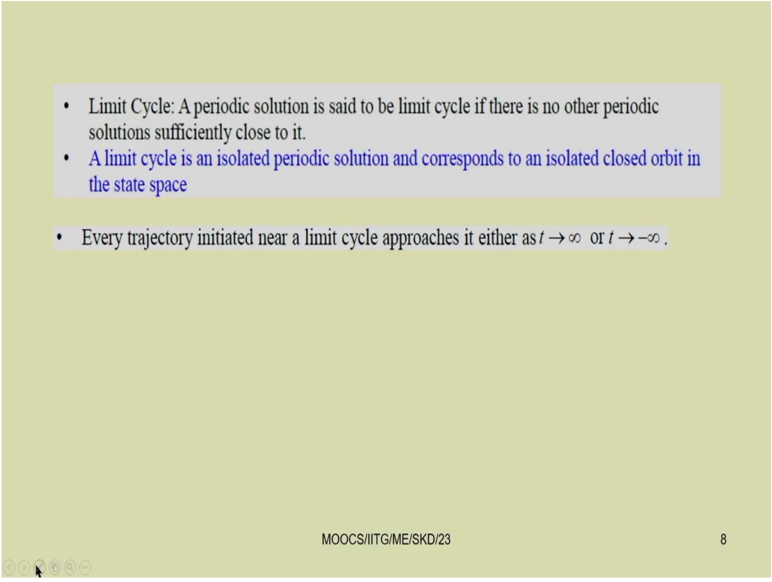


So, already we know the bifurcation of the fixed point response. So, we have static bifurcation and dynamic bifurcation. So, in case of static bifurcation, so, we know so, we have this saddle node bifurcation, pitchfork bifurcation and transcritical bifurcation.

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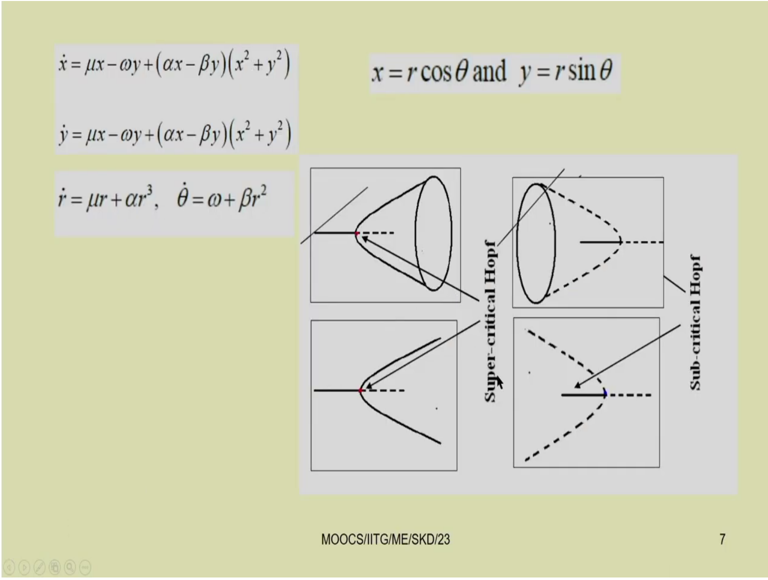


And, in case of this dynamic bifurcation we have the Hopf bifurcation, so, where we may have the supercritical Hopf bifurcation or the subcritical Hopf bifurcation.

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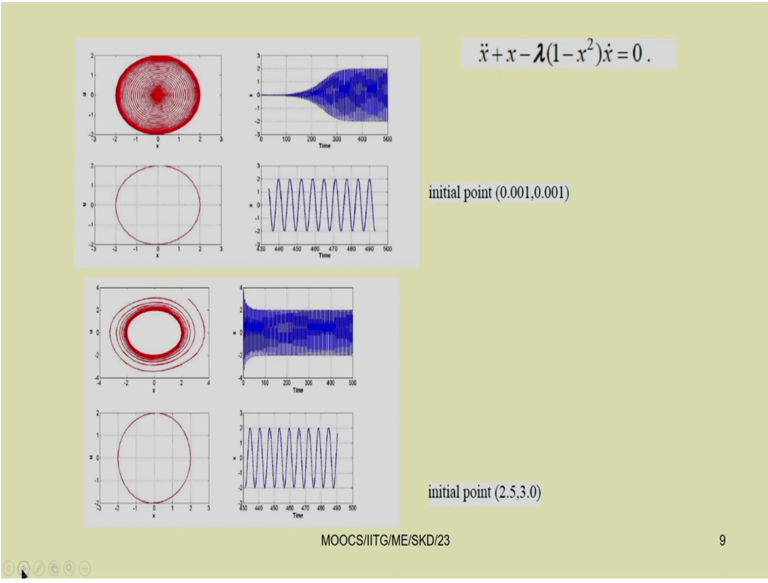


So, in periodic response also we have understood regarding the self excitation, self excited system like this Van der Pol equation so, where it leads to a limit cycle. So, either you start from the outside or from the inside of the circle. So, always it will go to this limit cycle.

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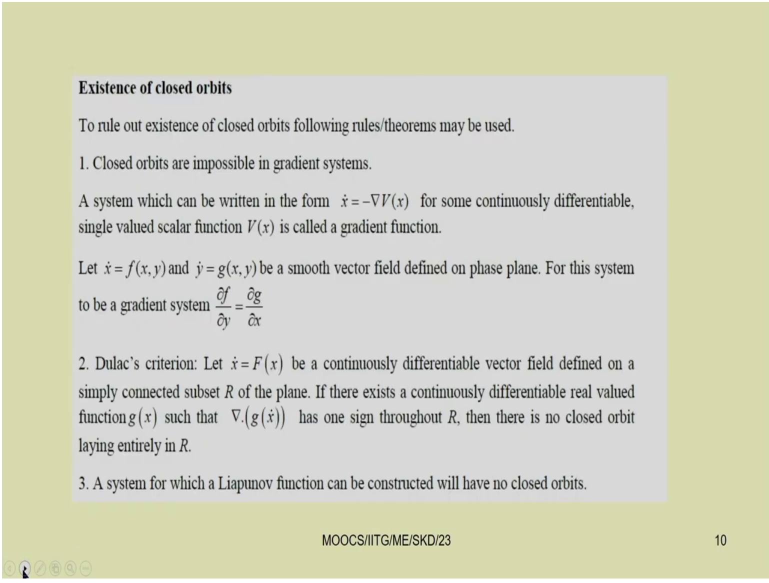


Also further we know, so, we can see what is the conditions for the existence of the closed orbit. To rule out the existence of the closed orbit following rules or theorem may be used. So, closed orbit are impossible in gradient systems. You should note that the closed orbits are impossible in gradient system. A system which can be written in this form.

So, in this form if you can write a system that is x dot equal to minus del V x for some continuously differentiable single valued scalar function V x it is called a gradient function. So, let x dot equal to f x, y and y dot equal to g x, y be a smooth vector field defined in the phase plane.

So, for this system to be a gradient system then del f by del y. So, del f by del y equal to del g by del x. So, in that case a system which can be written in the form of x dot equal to minus delta V x for some continuously differentiable single valued scalar function. So, it is called a gradient function and the closed orbit are impossible in case of the gradient function.

Similarly, another condition that is known as the Dulac's criterion. So, here this if x dot equal to F x be a continuously differentiable vector field defined on a simply connected subset R of the plane. If there exist a continuously differentiable real valued function g x such that del dot g x dot has one sign throughout r then there is no close orbit laying entirely in R.

A system for which a Liapunov function can be constructed will have no closed orbit. A system for which a Liapunov function can be constructed will have no closed orbits. So, these are the conditions you may have to check. So, where the closed form solution or close loop cannot exist or you cannot find a closed orbit in all these cases .

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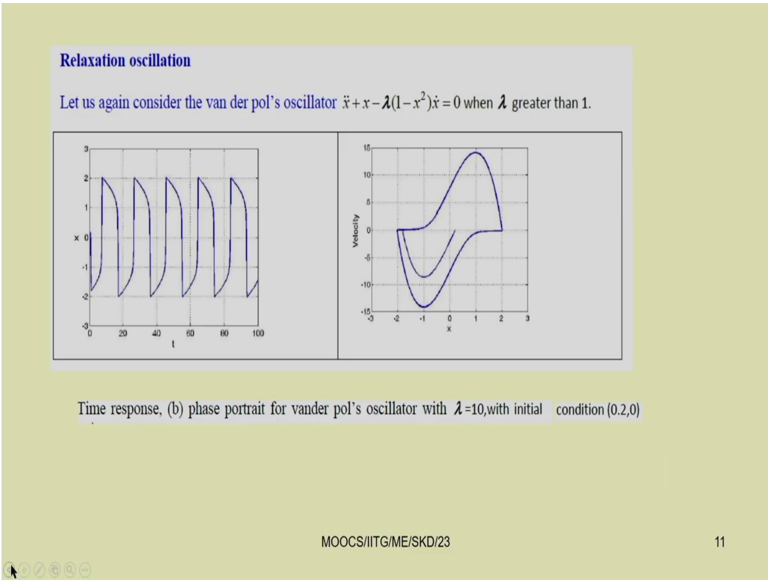

Again, we have studied regarding this relaxation oscillation. So, in case of the relaxation oscillation we will have two different time period. So, in one time period it will slowly move and in the second case it will move with very high speeds. So, it can be very very clear from the velocity diagram.

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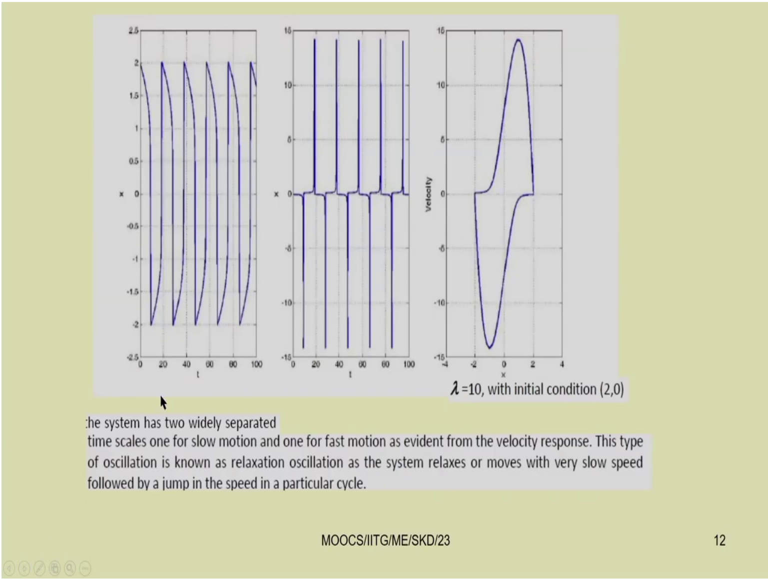


So, for example, in this velocity diagram; so, you can find two parts. So, in one part it is slowly moving and in another part it move with fast motion. So, two time scales are there, with one with slow motion and one with the fast motion. These are known as relaxation oscillation. So, in case of the Van der Pol oscillator by taking this lambda value very high value so, we can get this relax relaxation oscillation.

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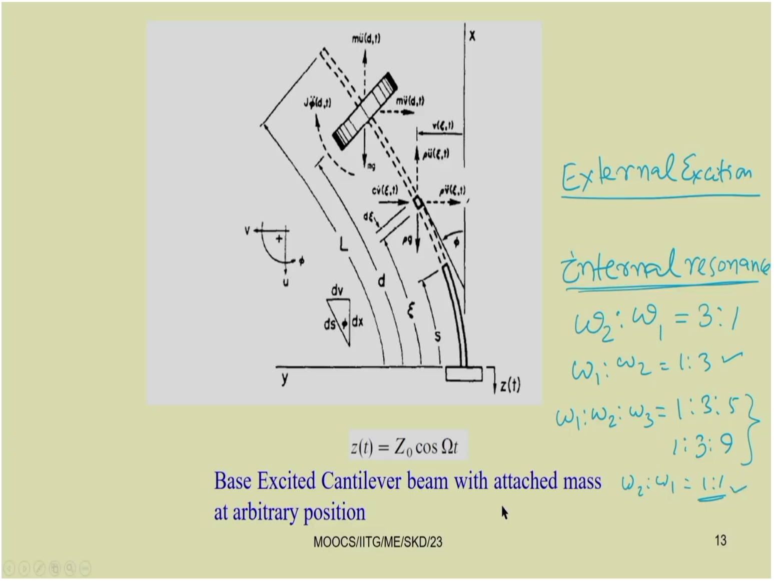


Similarly, we have studied regarding the system with internal resonance. So, though we are very much acquainted with external excitation leading to external resonance condition, external excitation so, this leads to resonance condition.

So, in this case we have studied this internal resonance. So, in case of internal resonance the second mode in this particular case we have studied to be so, the relation between the second mode and first mode or the ratio between the second mode and first mode is 3 is to 1. So, that is why it is known as 3 is to 1 internal resonance condition or 1 is to 3 internal resonance condition.

So, if you write it in terms of 1 to 2, so, it will be 1 is to 3. So, generally it is possible in case of system containing cubic non-linearity. The system containing cubic non-linearity also you can have for example, if you take multimode for example, omega 3. So, let you take this three mode, then you can have 1 is to 3 is to 5, 1 is to 3 is to 9.

So, these type of internal resonance conditions also you can have. So, these are known as three mode interaction. So, this is two mode interaction, this is three mode interaction; sometimes in many paper you can find this omega 2 is to omega 1 equal to 1 is to 1 also. So, in that case it is 1 is to 1 internal resonance conditions particularly 1 and 1 is to 1 internal resource conditions will occur.

So, if you have a coupled beam or you can have a square beam. So, in case of the square beam so, in both the direction so, it is natural frequency will be same. So, this omega 2 is to omega 1 maybe can be written in terms of 1 is to 1 in that case. In this base excited system so, last class we have studied we can observe many different type of bifurcations.

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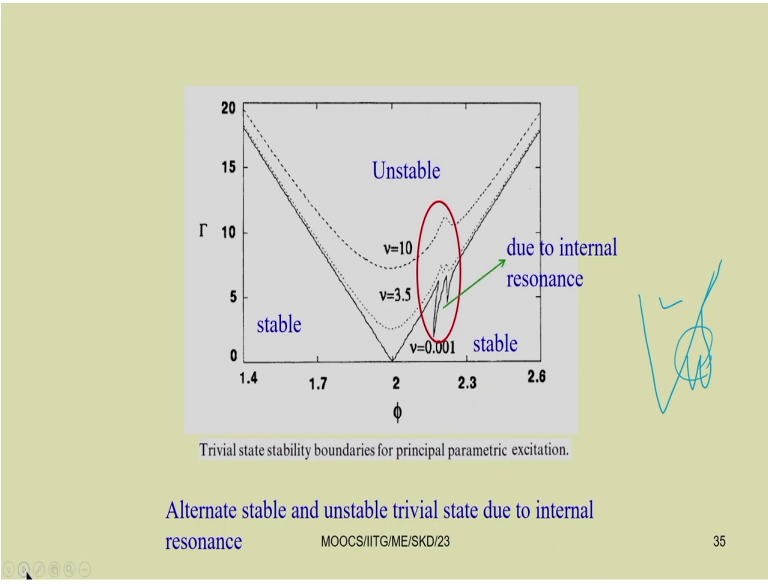

For example, so, we have seen in this principal parametric resonance conditions, we have seen the parametric instability region, where this additional kings are observed due to the presence of internal resonance otherwise you can have only one loop.

So, due to the presence of internal resonance condition so, this portion, so, in this portion, so, you can have several wings and then it moves like this you have observed several or alternate stable and unstable regions of instability in this case due to the presence of internal resonance condition.

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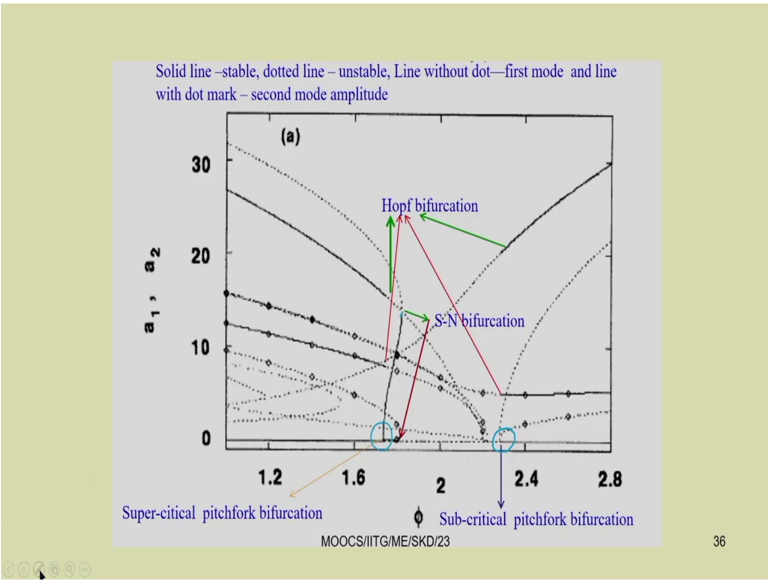


So, then you have seen different let us see different bifurcations, origin of different bifurcations also you can note, we can have these supercritical pitchfork bifurcation. So, you just see this is the pitchfork bifurcation. So, in the bifurcation point another branch generally starts. So, this is the starting point or the root for the non-trivial; non trivial branches.

So, here in the super critical and sub critical pitchfork bifurcations you can get the non trivial branches out of the trivial branch. So, in case of the saddle node bifurcation, so, you just see so, we have two saddle node bifurcations here. So, this saddle node bifurcation, so, here you can observe stable point or stable solution gives rise to unstable solution.

And, in case of the Hopf bifurcation you can get, so, from the stable periodic or stable fixed point response you may get stable periodic response or unstable periodic response. So, depending on that thing so, you can have the subcritical pitchfork bifurcation or supercritical pitchfork bifurcation.

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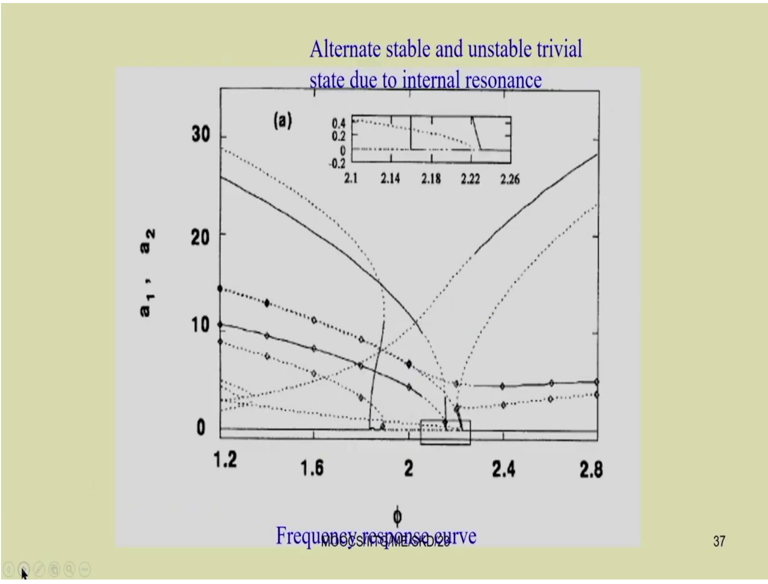

You can plot the frequency response or you can plot the force response plots also.

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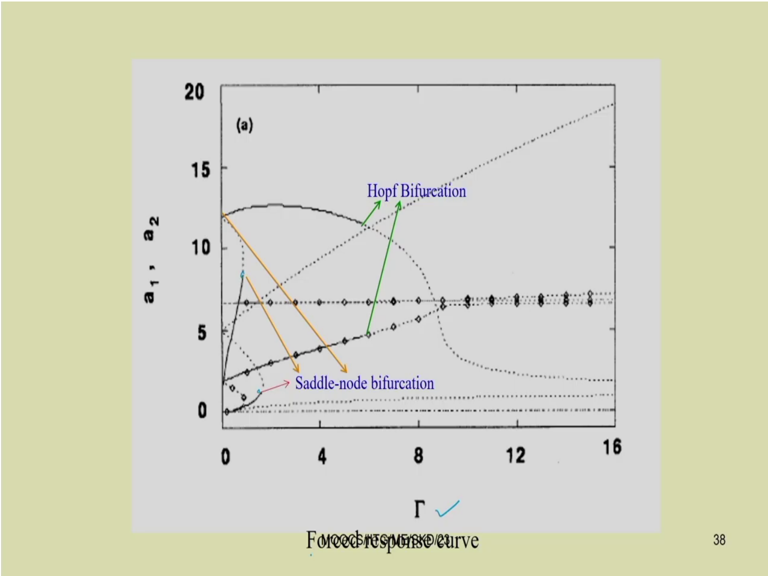


So, this is a force response plot where you have plotted the response amplitude versus the amplitude of the forcing. When you are plotting the amplitude of forcing so, you can have different. So, here you can have different bifurcation points also, you clearly you can observe different bifurcation points. This is saddle node, this point is saddle node and this is also a saddle node bifurcation point.

So, you can distinguish from the shape itself or by doing this eigenvalue analysis also you can distinguish the saddle node bifurcation point from other bifurcation point. At the bifurcation point the eigenvalue must be 0 and after the bifurcation point. So, you can have either a the real part may be negative or positive depending on the system whether it is going to be stable or unstable.

So, here you can have Hopf bifurcation point. So, at the so, you can see this is the Hopf bifurcation point. So, we we are having different bifurcation points and you have seen in the presence of Hopf bifurcation point or due to the presence of Hopf bifurcation point. So, we have the periodic response emanating from that position.

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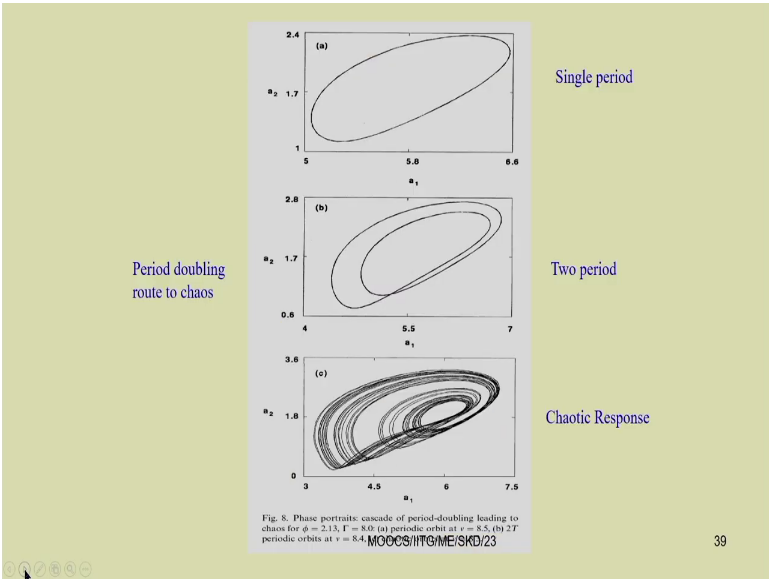


So, we may have a single period, two period or this period doubling route to chaos. So, today class we will see how this period doubling route to chaos and what is the relation between these two periods also.

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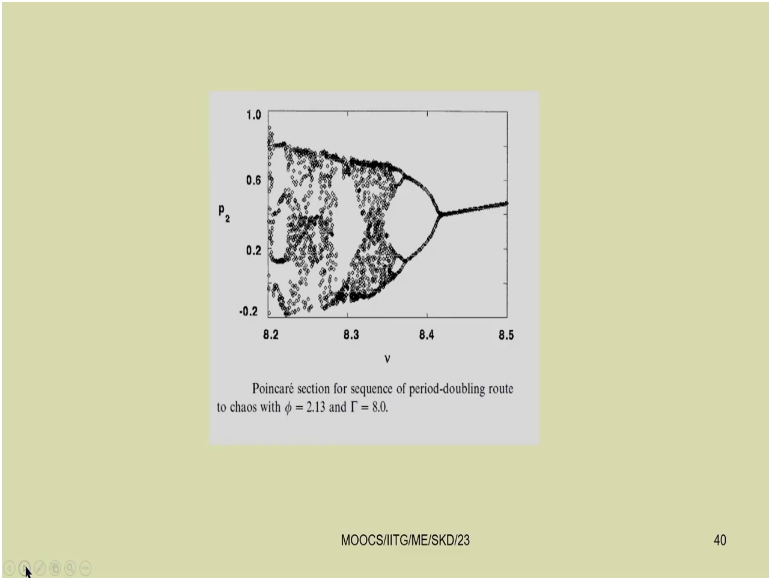


Today also, we will study how to plot this Poincare section. So, in this case this Poincare section has been plotted for the periodic response and clearly you can see this is single period, then we have two period, then four period and this is the period doubling route to chaos. So, finally, a chaotic response has been observed in this type of system.

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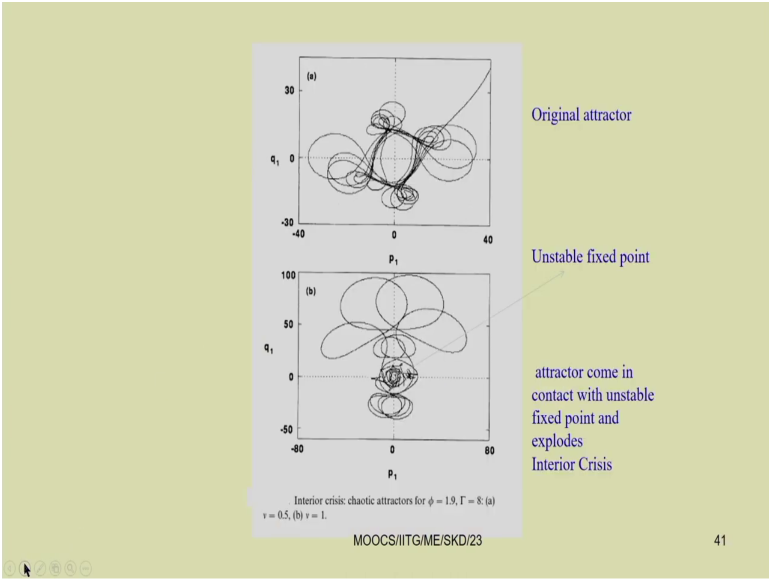


We will see this attractor merging crisis and also these in this system. So, you just you have noted that we have observed this quasi-periodic response also.

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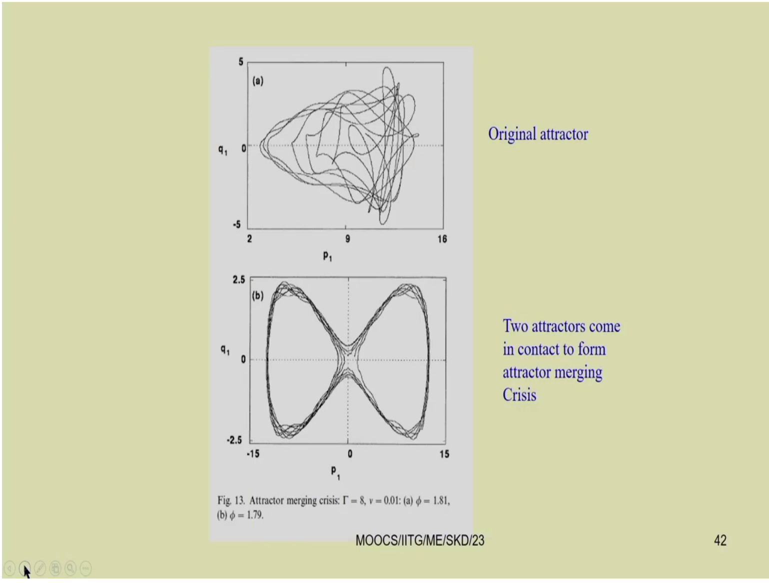

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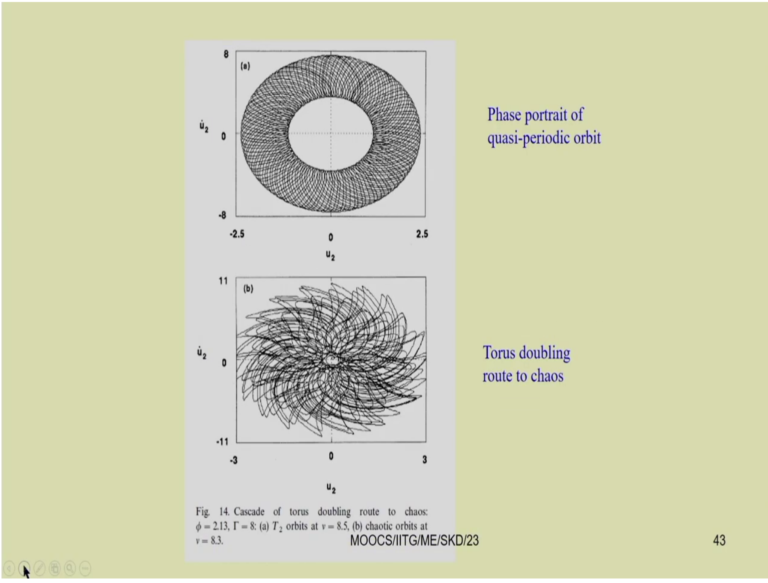

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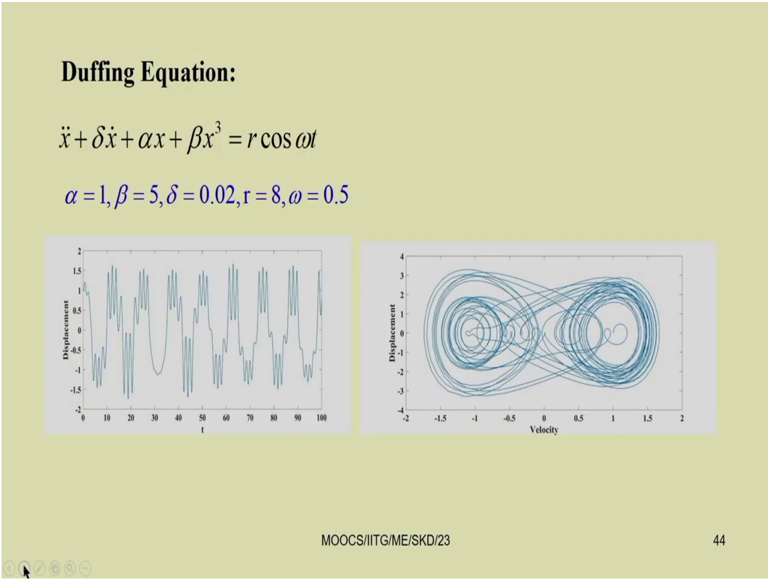


And, in Duffing equation also, last class we have studied regarding the periodic and also chaotic responses [noise].

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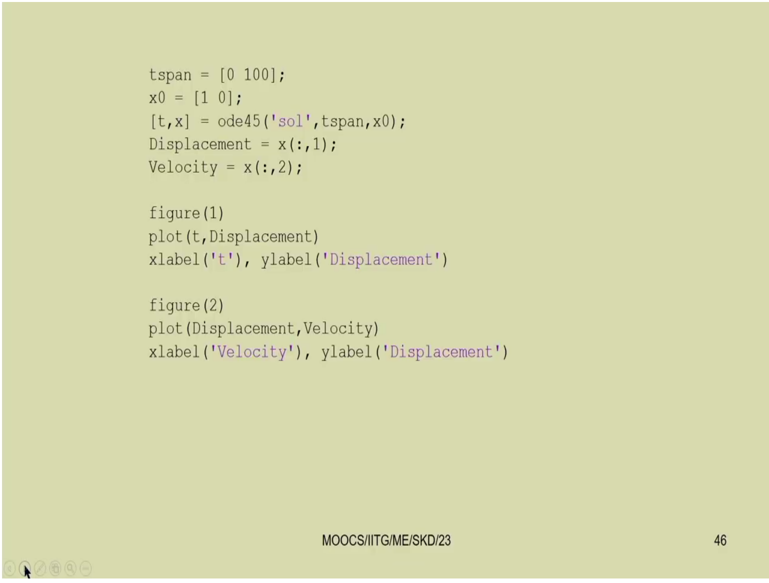

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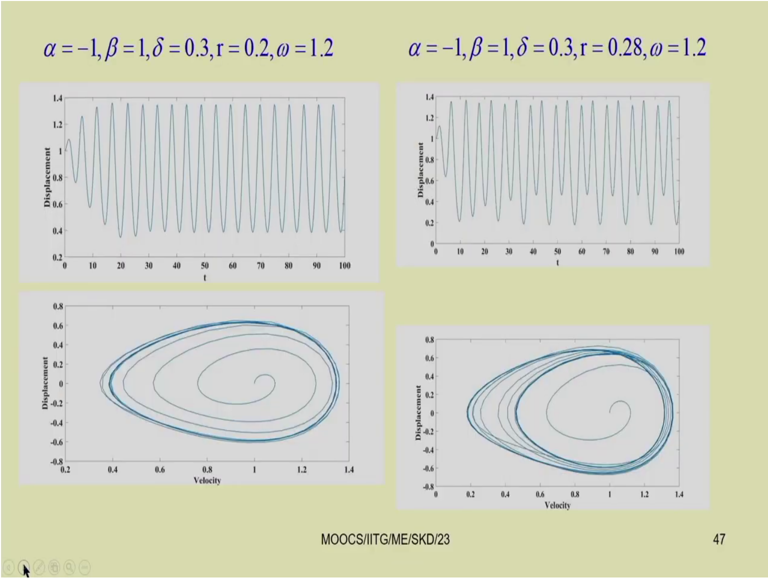

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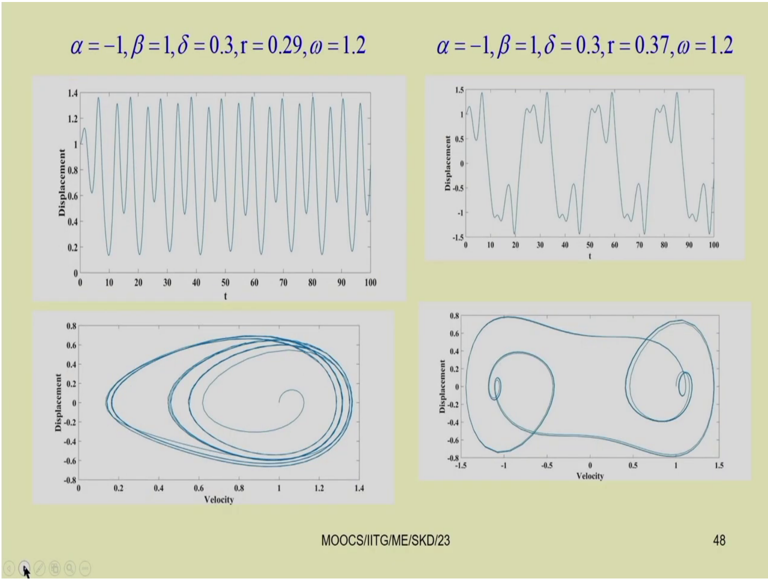

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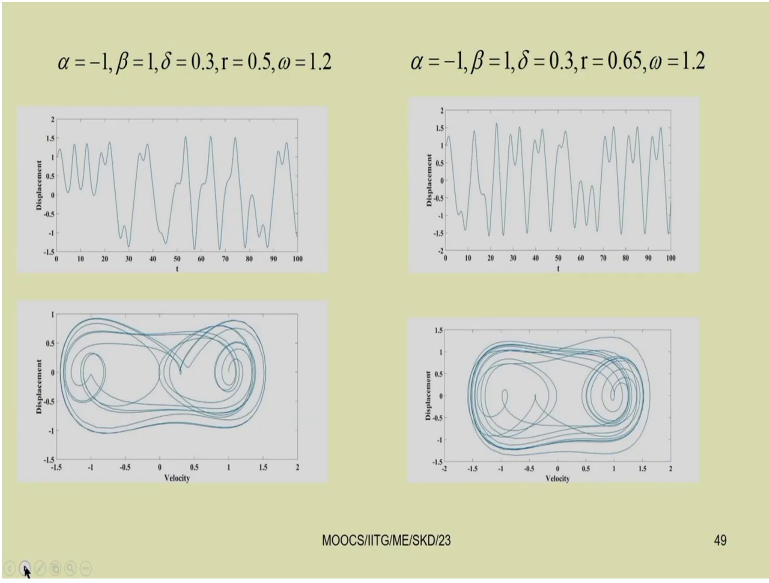

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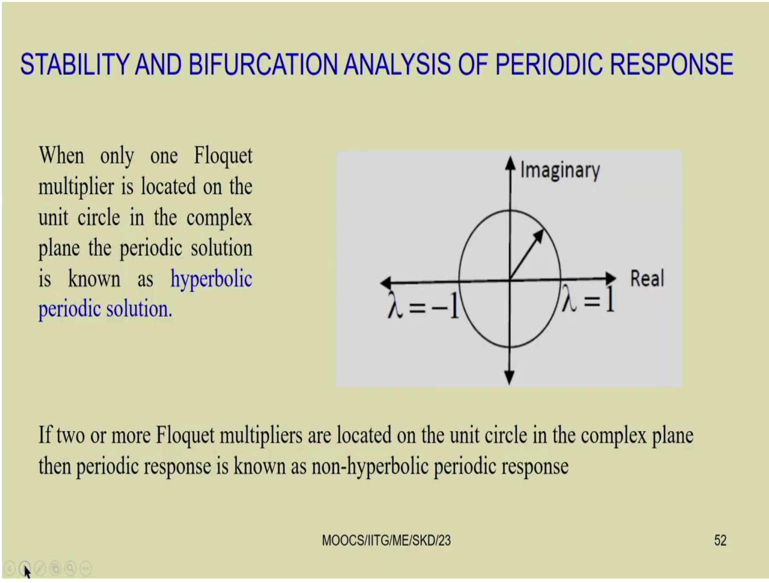


So, today class particularly we will be interested to study the stability and bifurcation analysis of periodic response. Previous class, you have seen different type of fixed point, periodic, quasi-periodic and chaotic responses also. So, let us study the stability and bifurcation analysis.

So, already you know how you can tell whether a system is stable or not. So, if you slightly perturb the system, if slightly perturb the response of the system it becomes unbounded the response grows and it becomes unbounded. Then the response is unstable, but if it come backs to the original position, then the system is stable. In case of the periodic response like the fixed point response we can have the bifurcation also.

So, in case of bifurcation or we tell a bifurcation occur if there will be qualitative or quantitative change either in the number of solutions or in the qualitative that is the stability change. So, if there is change in number or change in the quality of the solution then we can tell there is a bifurcation.

So, number change means so, initially let it has a single solution and after the bifurcation point it leads to two or three solutions. So, in that case we can tell the system has a bifurcation point. Similarly, if there is a change in stability, so, initially it is stable now it becomes unstable the branch becomes unstable that critical point also we can tell a bifurcation point.

So, similar things we have observed in case of the fixed point response and in periodic response already we are familiar with the Floquet theory and by using the Floquet theory we know that we can study the stability of the system.

So, here by finding the eigenvalue of the monodromy matrix, so, we can draw the unit circle. So, this is the unit circle drawn here with this real and imaginary parts of the Floquet multiplier. So, that is lambda, we have three different conditions here or we will see what are the difference conditions. So, if the root that is lambda eigenvalue lies inside this unit circle, then the system is stable.

So, if it is outside the unit circle the system is unstable; if it is on the circle then either it can be plus 1 or minus 1. So, if it is plus 1, then we have period t and if it is minus 1, so, we have a period 2t depending on the when it is on the circle we can tell that is on the transition curve. Taking this transition curve we can plot the parametric instability region.

So, already I have shown one such parametric instability region in case of a base excited cantilever beam. When one of the Floquet multiplier is located on the unit circle in the complex plane, the periodic solution is known to be hyperbolic periodic solution if only one.

So, if only one of the Floquet multiplier so, out of depending on the degrees of freedom so, we can have number of Floquet multiplier; if one of the multiplier is located on the unit circle then it is called hyperbolic periodic solution. So, if two or more Floquet multipliers are located on the unit circle in the complex plane then the periodic response is known as non-hyperbolic. So, this is known as non-hyperbolic periodic response.

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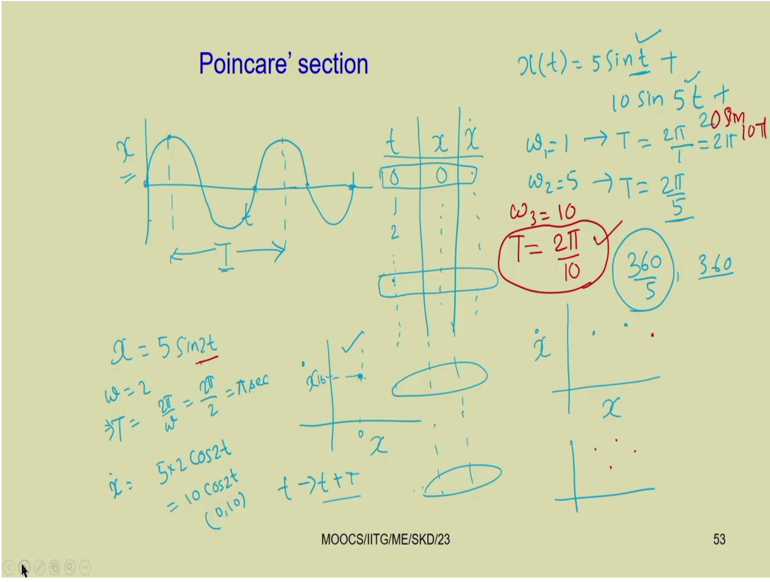


Before going further, we should know how to construct the Poincare section. So, what is the Poincare section? How we can section and we can reduce the dimension of the system. So, for example, we know let us have this x versus t curve that is response and we have a sin curve. In this case let us start with this point. So, let we have started this at this point. So, we know this to this is one period .

So, this is a periodic system, sin if it is written sin or cosine. So, this is a periodic system. So, for example, if I will write let x equal to 5 sin 5 sin 2t. So, in this case this omega equal to 2 implies this time period equal to 2 pi by omega. So, 2 pi by omega that is 2 pi by 2. So, this becomes pi. So, this time period becomes pi second in this way you know the time period.

So, after you know the time period this curve you have plotted this x versus time response plot you have plotted. So, how you have plotted? So, you have taken different value of time and you have found the response x. So, you have found the response x and you have plotted this response plot.

So, similarly if you want to plot this velocity response, so, you can plot this x versus t, x and x dot. So, this t x by taking this t x you are plotting this diagram. So, now, you know. So, let us start. So, t equal to for example, let us start t equal to 0. So, 0 second, 1 second, 2 second. So, that way so, we can have different time we can write by finding the time response at different points.

So, similarly is corresponding to x equal to if you have sine, so, this is 0. So, that way you can have different or let me write this is. So, I can have different time and I can find for a given value of omega. So, I can find these t x and x dot. Let me start for a t equal to 0 as the starting point. So, this is the starting point.

So, that means, this is the starting point and after one cycle, so, after one cycle the t I know what is the cycle of this. So, after one cycle again I can find this point. So, corresponding to this, so, I can find after one cycle let this is after one cycle. So, this is the response after one cycle. Then I can find this x and x dot. So, if I will plot this x versus x dot taking all the points after one cycle.

So, this is after one cycle, this is point after one cycle. Similarly, you take all these points after one cycle what you can get? So, in this particular case, so, you just see. So, you will get only this point 0, 0 point. x equal to for example, x equal to this. So, you can get this x dot equation will be equal to 5 into 2 cos 2t. So, here this is equal to 10 cos 2t.

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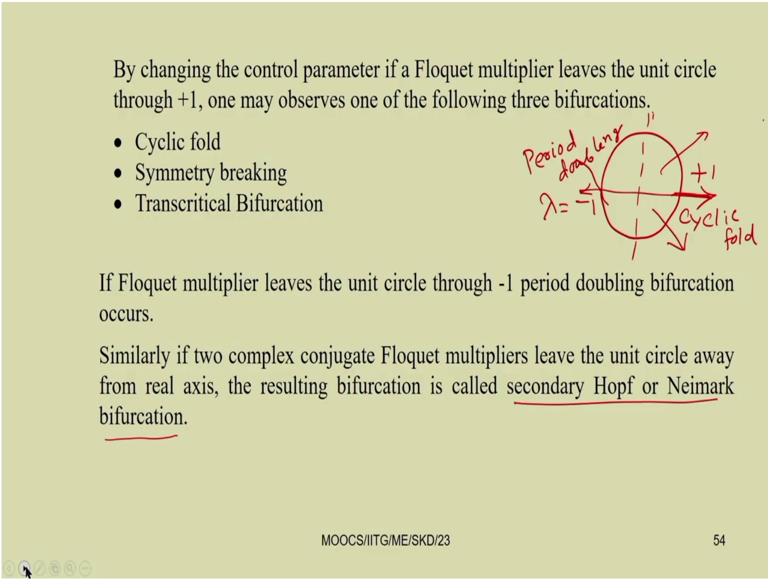


So, that means, we are finding the point. So, we are finding the locus of all points which are an interval of time t; so, that means, t we are replacing. So, replace this t by t plus T. So, again another cycle, again another cycle. So, that way you can go on finding here what you have you have seen. So, sin. So, this is equal to in this case sin 0 equal to 0. So, after one cycle also it is 0; after one cycle again also it will be 0.

So, let us see the third. So, this is again 0, but what will happen to the x dot? Similarly, x dot will be. So, at t equal to 0 x dot becomes. So, cos 2t this is 0, so, this becomes 10. Similarly, after one cycle it will be 10. So, your x will be. So, if I am starting this is let this is the 0 point. So, then this is 0 and this is 10.

So, this is the point you are getting. So, every time you are sampling it so, you will get the single point 0 and so, you will get the single point 0 and 10. This is known as the Poincare section. So, that means, you are sectioning this thing. So, you are sectioning or sampling this response at a time period t.

So, let us take another example. So, in which let our x can be written or x t equal to x t equal to 5 sin t plus 5 sin t plus 10 sin 5t. So, in this case we have two time period. So, we have omega 1 equal to so, we have two omega. So, we have omega 1 equal to omega 1. So, here omega 1 equal to 1 and omega 2 equal to 5 omega 2 equal to 5.

So, in that case as omega 1 equal to 1 and omega 2 equal to 5. So, this gives rise to T that is time period equal to 2 pi by omega that is 2 pi by 1. So, 2 pi by 1. So, in the first case it is 2 pi by 1 2 pi by 1 equal to 2 pi; in the second case, this becomes 2 pi by 5. So, in the second case this becomes 2 pi by 5. So, it is T equal to 2 pi by 5. So, you just see you have two time periods. So, one is 2 pi and other one is 2 pi by 5. So, this is 360 degree.

So, 2 pi is 360 degree by 5. So, one is 360 degree by 5 and other one is 360 degree. So, what you can do? So, you can take the lowest time period that is T that is 2 pi by 5 and sample it. When you sample it ah 2 pi by 5, so, what you can find? If you plot this x versus x dot x versus x dot you will get two points on this thing. So, these two periodic, so, now, you just see, so, you have a response with two periodic.

So, these two periodic response if you plot by using this x versus x dot, so, you just see it is reducing only to two points. So, instead of representing the whole time period whole time the response with whole time we can represent the same value or same thing, same time response by using this Poincare section with only just two points. So, this is the Poincare section. So, here also we have plot the Poincare section if we have a single period; here we have plot the Poincare section with two period.

So, when we have two period, we have only two points on the Poincare section. Similarly, if we have a response, so, let me add another number. So, let it is 20 sin 10T 20 sin 10T. So, in that case another time period I have added and this T 3 or now the T will be. So, here the omega equal to 10 as omega equal to 10 as omega 3 equal to 10. This is becomes 2 pi by omega. So, that is 2 pi by 10 you can sample it.

So, you can 2 pi by 10. So, you can sample it with 2 pi by 10. So, previously initially in the first time you have sampled it with 2 pi. So, t equal to, so, in this first case if you have sin 2t, so, here you have sample it with t pi if it is only t then you have sample it by . So, only t sin t let you have only 5 sin t then in that case you have to sample it omega equal to 1.

So, time period will be 2 pi by 1 that is 2 pi and now, when you have two then you have to take the minimum time. So, out of these three, so, this is the minimum time you have taken. So, 2 pi by 10. So, by taking this minimum time so, if you sample it, so you can see so you have one additional. So, you have one additional point. So, here you got three points. So, this is three periodic similarly you may have four periodic also.

So, for four periodic four points you will get. So, later will see if the ratio instead of this relation so, here what you have seen this omega 2 by omega 1 equal to 5. Similarly, omega 3 by omega 1 equal to 10; omega 3 by omega 2 equal to 2 here the ratio are integers. So, when the ratio are integers so, you are getting periodic response.

So, you can get periodic response, but when the ratio are irrational number, no longer it will be periodic. So, in that case we will see the response will be quasi periodic. So, like the fixed point response, so, in this case also so we have three different bifurcations if it is crossing the limit cycle or crossing the unit circle through plus 1. So, if the roots are crossing these unit circle through plus 1.

So, this way so, if it is crossing through plus 1, so, this is plus 1. So, we will have three different type of bifurcation – one is cyclic fold bifurcation, symmetry breaking bifurcation, then the transcritical bifurcation. So, we have cyclic fold, symmetry breaking and transcritical bifurcation. Cyclic fold bifurcation is similar to the SN that is saddle node bifurcation. So, we will see that thing if the Floquet multipliers leaves the unit circle through minus 1.

So, this is lambda equal to minus 1. So, lambda equal to minus 1. So, if it crosses this unit circle through minus 1, then you can have in this case you can have this period doubling bifurcation. So, period doubling bifurcation. So, if it is moving through plus 1, so, either we can have this cyclic fold cyclic fold or symmetry breaking or transcritical bifurcation.

Also it may leaves the unit circle through this as complex conjugate, it may leave the unit circle through complex conjugate. So, in that case we can have the secondary Hopf or Neimark bifurcation. So, if two complex conjugate Floquet multiplier leave the unit circle away from real axis, the resulting bifurcation is called secondary Hopf or Neimark bifurcations.

In case of periodic response similar to the fixed point response we have different type of bifurcation. So, you have seen so, when it is leaving through this plus 1 so, then it is known as cyclic fold, symmetry breaking or transcritical bifurcation. So, when you see the shapes so, clearly you can understand these different type of bifurcations.

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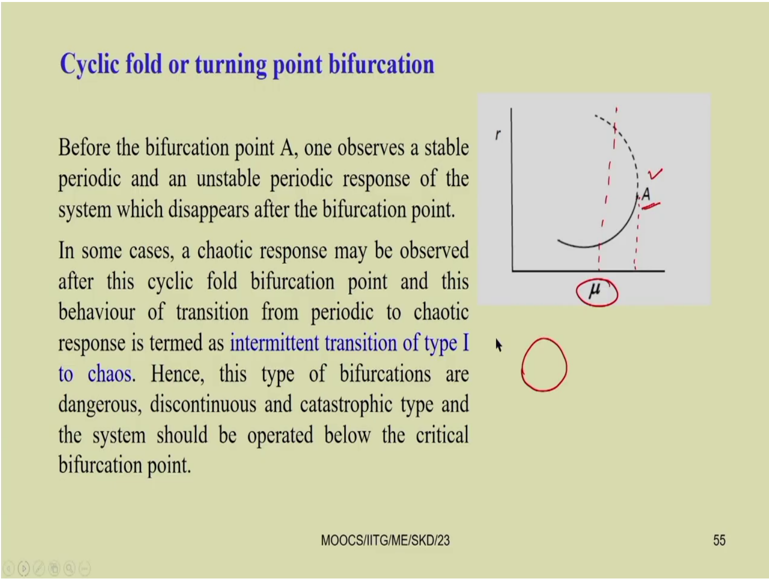


So, let us see the cyclic fold bifurcation. Here you are just plotting the amplitude either you plot the amplitude of the response or you just plot take the Poincare section and plot it. Before the bifurcation point let A is the bifurcation point. So, before the bifurcation point A, one observe a stable periodic and unstable periodic response of the system.

So, you just see you just take any point here so, this is before A. So, this is before A point. So, before A point so, you have two periodic response. So, one is the stable periodic and one is the unstable periodic. So, you have plotted only the amplitude part of this thing.

So, amplitude part so, you just see here you have this is the amplitude; that means, so, you have a periodic response if you plot the phase portrait corresponding to different different points you can see what is the response or how the periodic responses are there. The size of the periodic so, this shows the amplitude of the periodic response every point on this thing shows the amplitude of the periodic response.

So, before this bifurcation point you have both stable and unstable periodic response and after this bifurcation point A, you just see so, there is no neither stable or unstable periodic response exist after a in this type of solution or in this type of cyclic fold. So, this is cyclic fold or turning point bifurcation cyclic fold or turning point bifurcation, after the bifurcation. So, the solution disappears. So, there is no solution.

If there is no solution after this bifurcation so, the response so, if you increase this parameter that is the system parameter further, then it will jump to an infinite attractor at infinite. Sometimes it may leads to chaotic response also. In some cases a chaotic response may be observed after the cyclic fold bifurcation and this behaviour of transition from periodic to chaotic response is termed as intermittent transition of type 1 to chaos.

So, this is intermittent transition from periodic to chaotic or from so, if you reduce the value of mu then from chaotic to periodic you can get. So, this type of transition is known as intermittent transition of type 1 to chaos. Hence this type of bifurcation are dangerous discontinuous and catastrophic type and the system should be operated below this critical point. So, it should not be operated after A, which will leads to chaotic response.

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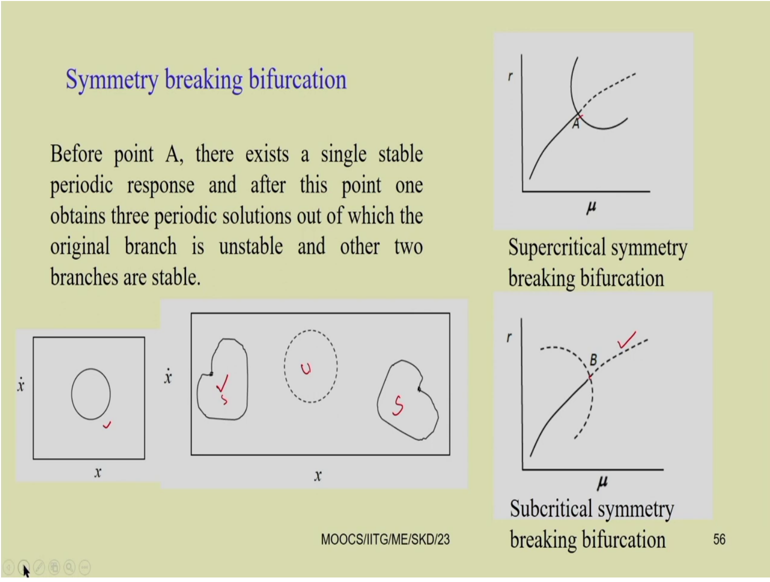


Similarly, we can have symmetry breaking. In symmetry breaking bifurcation, so, we can have two different type of bifurcation – one is super critical and super critical symmetry breaking and second one is the sub critical symmetry breaking. So, in case of super critical symmetry breaking so, before the bifurcation point, so, you just see A is the bifurcation point. Here we are plotting the amplitude versus the system parameter mu; mu is the system parameter.

So, before bifurcation point A, there exist only single stable periodic response. So, this is single stable periodic response and after bifurcation so, we have three responses are there. So, three periodic solutions will be there. So, out of which two are stable and one is unstable.

This stable periodic solution will continue as an unstable periodic solution and at a point two more stable periodic solution will be emanating from point A or starting at this point. The resulting so, you can see. So, before bifurcation so, you have a periodic response and after bifurcation so, this is the phase portrait we have plotted. So, after this bifurcation, so, if you see so, we have three different periodic responses.

So, for periodic responses it is a closed loop. So, if you plot this x versus x dot it is a closed loop. After the bifurcation so, we have two stable. So, this is stable. So, this is stable periodic solution and we have one unstable periodic solution. So, as from a stable branch we are going to another stable two stable branches this is known as supercritical symmetry breaking bifurcation.

So, this is a continuous bifurcation, but we may have the other type also. So, initially we will have three type of solution three – one is stable and other two are unstable periodic solution and after the bifurcation point we have only a single unstable periodic response.

Initially, we have a stable periodic and two unstable periodic response and finally, after bifurcation point B we have only unstable periodic response. So, in that case we called it as subcritical symmetry breaking bifurcation and the subcritical bifurcations are always dangerous bifurcation because this unstable state cannot be achieved.

As we are not able to achieve this unstable state it will jump to an attractor at infinite or it may leads to some chaotic response. This way the symmetry breaking bifurcation. So, you can observe this symmetry breaking bifurcation in this particularly in this multi degrees of freedom system.

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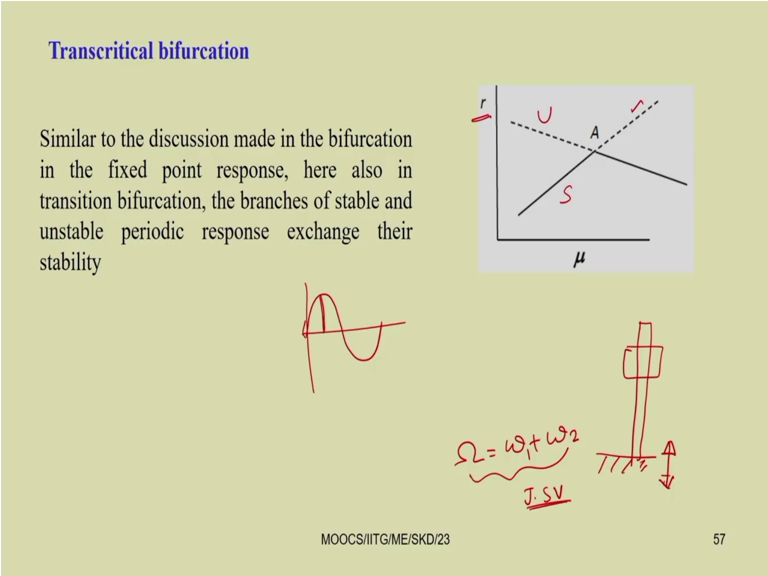


So, in case of multi degrees of freedom systems, for example, in the case of cantilever base excited cantilever beam subjected to base excited cantilever beam with an arbitrary mass attached mass. If we consider the combination parametric resonance condition, that is, by taking this omega equal to omega 1 plus omega 2 so, you can observe this periodic and this type of bifurcations, this type of symmetry breaking bifurcation or and the turning.

So, this turning point bifurcation cyclic fold or turning point bifurcation. So, you can refer the paper refer our paper in Journal of Sound and Vibration on periodic and quasi-periodic response. So, you can find that thing similarly we have a transcritical bifurcation similar to that in case of a fixed point response.

So, in case of the transcritical bifurcation, here if you plot this r versus mu mu versus r so, initially we have this periodic response and stable periodic and this is stable periodic and unstable periodic response. We are plotting only the amplitude; do not get confused between this r. So, r represent the amplitude of the response.

What is amplitude of the response? So, if you have a periodic response so, this is the amplitude of the response. So, the amplitude of the response so, this is amplitude of what the response versus mu; mu is the system parameter critical parameter based on which you are changing the system parameter.

Before the critical point A, so, the system has both stable and unstable periodic response and after the bifurcation point. So, we can note that the stable branch change to unstable branch and the unstable branch change to stable branch. So, we have both unstable and stable after the bifurcation. What they interchange? There the branches interchanging their stability conditions qualitatively.

So, you just see the number remain same number before the bifurcation we have two solution and after the bifurcation also we have two solution here. The number remaining same, but the quality of the solution that is the stability of the solution changes after point A. A stable branch becomes unstable and one unstable branch becomes stable branch after bifurcation. So, this bifurcation is known as transcritical bifurcation.

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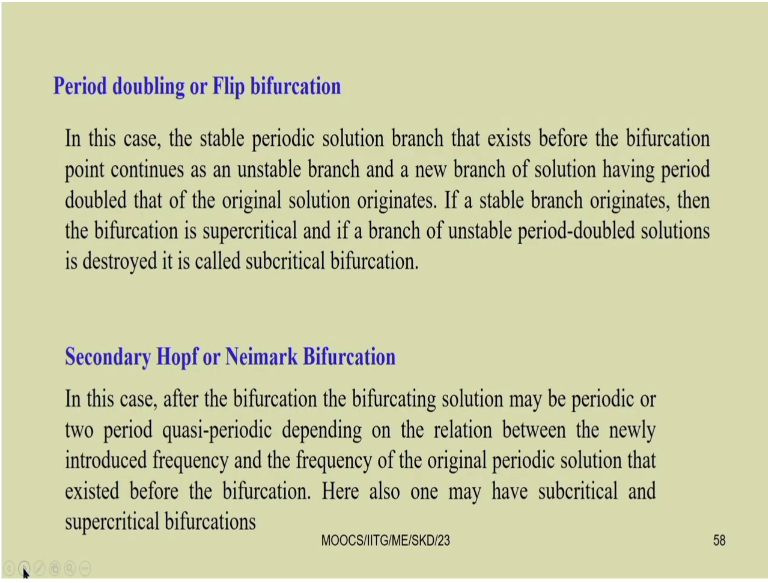


Similarly, you can study the period doubling or flip bifurcation. If it is passing through the lambda equal to minus 1, so, if the roots are passing through the unit circle through minus 1, so in that case we can have the period doubling or flip bifurcation. A stable periodic solution branch that exists before the bifurcation point continues as an unstable branch and a new branch of solution having period doubled that of the original solution originates.

If a stable branch originates, then the bifurcation is supercritical and if a branch of unstable period doubled solution is destroyed, it is called subcritical bifurcation. In case of the subcritical so, unstable double period solution will be destroyed and in case of supercriticals a stable periodic solution will originate. So, in that way so, you can study the period doubling or flip bifurcation.

Similarly, this period doubling so, with further change in the system parameter generally undergoes another period doubling. So, one can get four period. Similarly, one can increase the 4 period to 8 period and 8 period to 16 period and that leads to period doubling route to chaos.

So, we can have the secondary Hopf or Neimark bifurcation. So, in this type, after the bifurcation, the bifurcating solution may be periodic or two periodic quasi-periodic depending on the relation between the newly introduced frequency and the frequency of the original periodic solution that existed before the bifurcation. Here also one may have sub critical or super critical bifurcation.

In this way you can have different bifurcations in case of the periodic solutions also. So, in given a periodic solution, so, you have to find the Floquet multiplier by finding the eigenvalue of the monodromy matrix and then you can study by changing the system parameter what type of bifurcations will occur.

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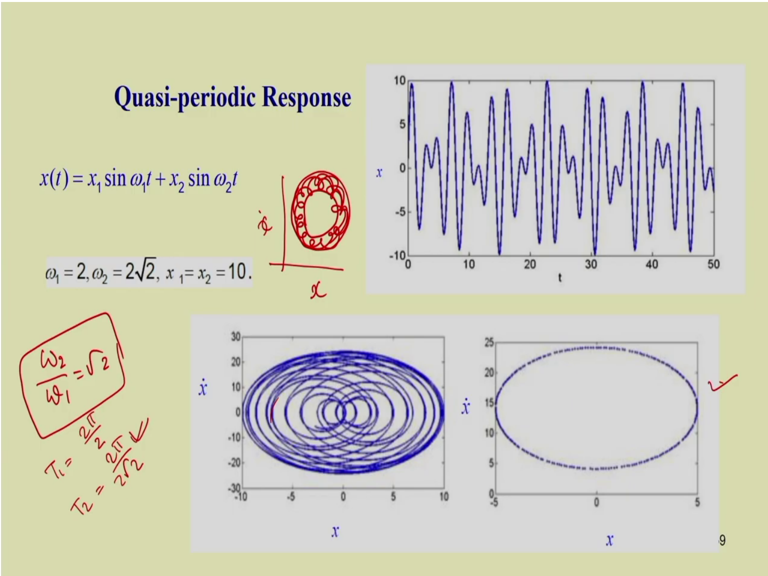


So, let us see what we mean by quasi-periodic response. So, let us take this example for x t equal to x 1 sin omega 1 t plus x 2 sin omega 2 t. So, if we are taking for example, omega 1 equal to 2 and omega 2 equal to 2 root 2, here the ratio between omega 2 and omega 1 is root 2, omega 2 by omega 1 equal to root 2. So, this root 2 is an irrational number.

So, if you plot this x versus t, so, you can clearly observe that the response are not periodic. So, they are a periodic. So, they are known as a periodic response or quasi-periodic response. In this case if you plot the phase portrait, so, you can x versus x dot so, you can get a curve similar to this.

Sometimes you may get a curve similar to this which is known as torus. So, you will have two circle and these two circles are connected by so these two circles are connected by this, this way. These are known as torus. So, if you plot this x versus x dot so, you can get the torus also .

If you draw the Poincare section so, in this case how to draw the Poincare section? So, here you just see. So, you have two frequency. So, time period so, T 1 will be equal to 2 pi by 2 and this is T 2 will be equal to 2 pi by 2 root 2 and taking this lower time period so, we can sample the response. Taking the lower time period we can sample this response and that sample things if we plot then so, you can get this plot.

So, you just see the Poincare section is a closed curve. So, in case of the quasi-periodic response the Poincare section is a closed curve. Previously we have seen the Poincare section for a periodic response contain few numbers. So, if it is single periodic only single point will be there, if it is two periodic two points will be there and if it is multi periodic multiple number of points will be there, but it will not be these points will not be placed in a closed curve.

But, in case of a quasi-periodic response the points will be placed in a closed curve and easily you can distinguish between this periodic quasi-periodic and later we will see in case of the chaotic response so, it will fill up this space. If I have a chaotic response, the Poincare section will fill up the whole space. In that way you can distinguish between periodic quasi-periodic and chaotic response. Let us take another example. So, here we have taken 2 and root 2 x 1 x 2 we have taken equal to 10.

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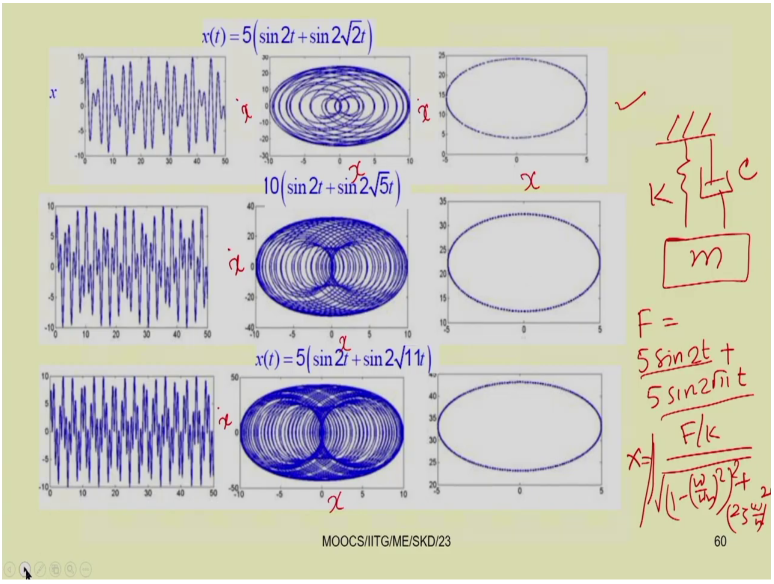


Similarly, if we can take this for example, x t equal to 5 sin 2t plus sin 2 root 2t. So, similar curve we can get and you can see this Poincare section.

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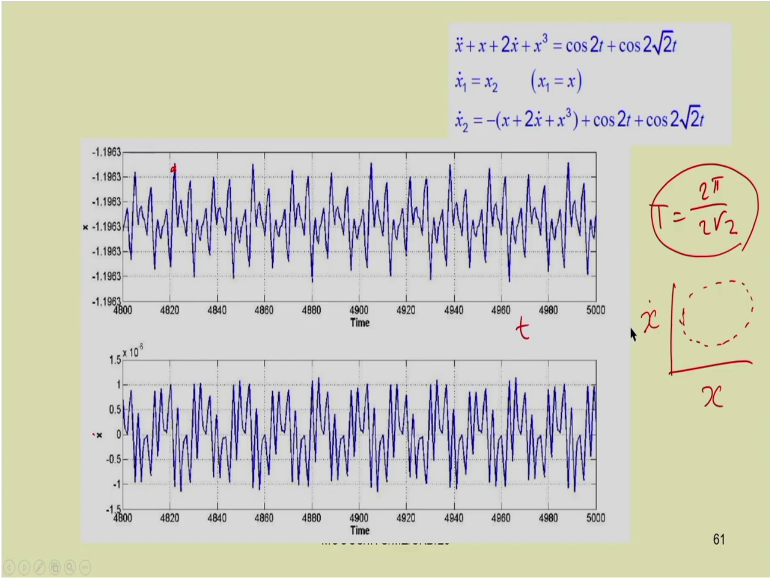


So, the Poincare section between x and x dot so, you can find this. Similarly, you just take. So, here you have taken the ratio equal to root 2 also in this curve you just see you have taken the ratio equal to 2 root 5 by 2; so, that is equal to root 5. So, here the ratio is taken to be root 5. So, that itself is an irrational number also. Similarly, here it is plotted 2 root 11 by 2. So, this is root 11. So, that is also an irrational number.

So, you just see you are getting different type of curves, different type of phase portrait. So, this middle line is x versus x dot. So, x versus x dot is the phase portrait. So, if you plot this phase portrait, so, you just see you are not getting a curve similar to what you get in case of this periodic response. So, this is x versus x dot. Similarly here so, you have x versus x dot.

So, how to generate this type of system in vibrating system particularly? So, if you are interested in a vibrating system for example, let us take the spring mass damper system or the simple spring mass system spring mass damper system subjected to multi harmonic excitation.

So, this is m; let it is subjected to F equal to F equal to 5 sin 2t plus 5 sin 2 root 11 t. So, you know by applying the superposition rule. So, we can find the response. So, this is K, this is C as this is a linear system. So, for first we can find for 5 sin 2t what is the response and then we can find for 5 sin 2 root 11 t. So, in case of 5 sin 2t, so, you know the solution will be F by K.

So, that is solution is equal to F by K divided by root over 1 minus r square 1 minus r is omega by omega n 1 by omega by omega n square whole square plus 2 zeta omega by omega n whole square. So, this is the formulas we already you know, first we can find for 5 sin 2t and then we can find the response for 5 sin 2 root 11 t.

So, finding both the relays both the response so, the solution will be. So, this is the; this is the x. So, x equal to this; so, we can have x 1. So, for the first one, we have x 1. So, we have x 1 sin 2t plus 5, 5 1 then for the second part this part. So, this will be x 2 sin 2 into root 11 plus 5 2. So, that way we can find the solution. So, where 5 1 and 5 2 are also can be obtained from this response. This way physically you can realize the response.

So, though I have given you the example of a linear spring and damper system, so, already we have studied regarding this Duffing equation. So, in case of the non-linear systems also you can study similar thing. So, in that case so, instead of you just note that in case of the non-linear system, so, as you have different resonance conditions so, by just changing the simple this response forcing amplitude or other system parameter so, you can see your resonance conditions will be different.

As resonance conditions are different, so you can have different solutions. So, the solution will not be unique like in case of the linear system. So, you cannot apply the superposition rule so, in case of a non-linear system. So, when it is a linear system so, you can apply the superposition rule, but when it becomes non-linear so, you have to particularly check what type of resonance will be there and based on the resonance condition so, you have to find the solution or the response.

So, let us see another curve also. So, the same thing what I told you just now; so, this is a system with cubic non-linearity x double dot plus x plus 2x dot plus x cube equal to cos 2t plus cos 2 root 2t. So, you can divide into two first order equation. So, for the first order equation x 1 dot let us take equal to x 2 and second equation x 2 dot equal to minus x plus 2x dot plus x cube plus 2 cos 2t plus cos 2 root 2t.

And, taking that thing so, we can plot this x versus t and x dot versus t. So, that is x dot versus so, x dot versus t. So, this is the displacement versus time, this is the velocity versus time. So, from that thing, so, we can plot we can plot the phase portrait and the time response phase portrait and the Poincare section.

This is given as an assignment to you to plot the phase portrait and these Poincare section of this system. So, Poincare section so, you can see how to draw this Poincare section? So, for drawing the Poincare section you can start for example, start at some position you can start at here.

So, then after t equal to 2 pi by 2 pi by 2 root 2. So, you can take the time 2 pi by 2 root 2 so, one time period equal to 2 pi by 2 another is 2 pi by 2 root 2. So, the lowest one is 2 pi by 2 root 2. So, by taking this lowest time period so, we can sample it. So, you can note that by sampling this thing you may not find the same point after this time period 2 pi by 2 root 2.

How many points you are getting after this thing? So, you have to plot it. So, you can sample that thing. So, by sampling this thing so, you can plot this x versus x dot which will give the Poincare section in this case. So, you can see in this case, you will get you are going to get a closed curve ok.

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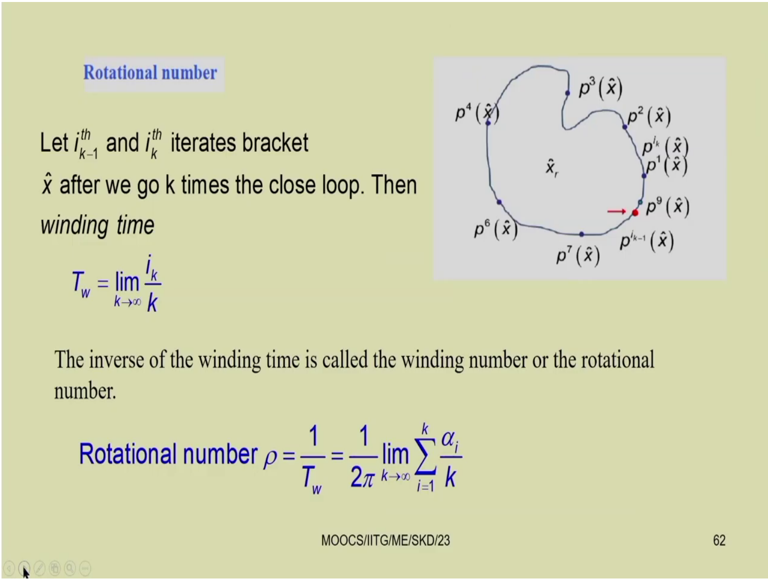


So, this way you can study the quasi-periodic response also. So, particularly you may be interested to know the rotational number. For example, just now we have seen. So, we can take a starting point for example, this is the starting point we have taken. So, let us go round.

So, this is the Poincare section. So, we have plotted the Poincare section. We have taken the Poincare section of Poincare section in x and x dot plane. So, let i k minus 1th and i k kth iterates bracket this x after we go k times the close loop. So, you go after going for example, after going first time we may not get this point again. So, second time we may not get this point again. So, late after 9th iteration, we are getting the same point.

So, then we can tell that so, then in that way so, we can find the winding time. So, this winding time will be equal to limit k tends to infinite i k by k. So, limit k tends to infinite i k by k. So, let us find all the times all the iteration for which we are bracketing this point.

So, after finding for example, as k tends to infinite, so, this i k by k then we can find the winding time. So, after getting this winding time actually this rotational number rho equal to 1 by T omega rotational number equal to 1 by T omega. So, this is this can be given by 1 by 2 pi limit k tends to infinite, i equal to 1 to k alpha i by k. So, this way you can find or simply you can find this 1 by T omega.

So, from the angle also you can find so, i k. So, you can take this point and this point find this angle which is bracketing this. So, from that thing this alpha angle can be obtained and from that you can study this rotational number. In a simpler way, so, rho equal to 1 by t omega you can find, which will give you the rotational number.

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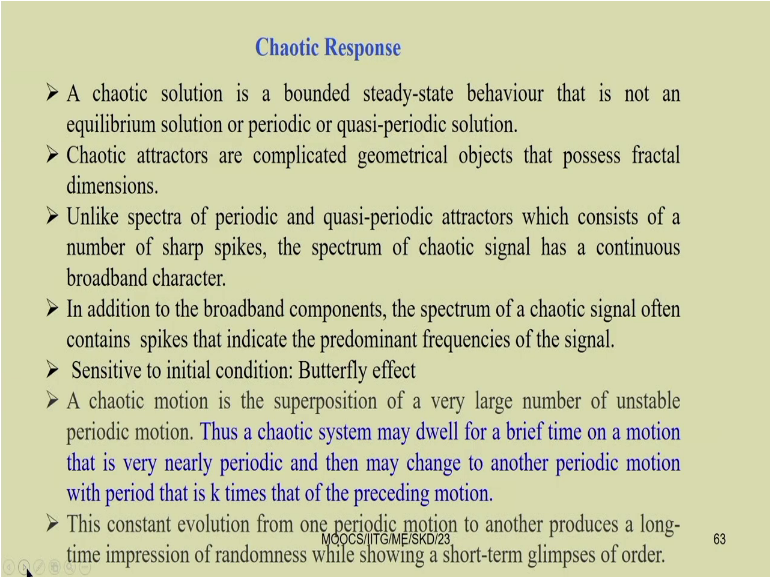


So, let us see what we mean by this chaotic response. So, briefly we will study this chaotic response and next class we will fully devote for the chaotic response. A chaotic solution is a bounded steady-state behaviour that is not an equilibrium solution or periodic or quasi-periodic solution. So, it cannot be a fixed point response, periodic response or a quasi-periodic response.

So, chaotic attractors are complicated geometrical objects that possesses fractal dimensions. So, we can see the shapes of chaotic attractor so, it can be loops like fractal self similar objects. So, unlike spectra of periodic and quasi-periodic attractor which consists of a number of sharp spikes, the spectrum of chaotic signals have a continuous broadband character.

So, in case of periodic response we will get single periodic will get one spike, for two periodic we will get two spikes. So, that way we can have the sharp spikes in case of the periodic and quasi-periodic spectrum. But, in case of chaotic signal a continuous broadband character will be available.

So, in addition to the broadband components the spectrum of a chaotic signal often contains spikes that indicate the predominant frequencies of the signal. So, it is very sensitive to the initial condition. So, that is known as the butterfly effect. So, if you change this initial condition so, you can have it will go to a different chaotic character.

A chaotic motion is the superposition of a very large number of unstable periodic motion. So, thus a chaotic system may dwell for a brief time on a motion, that is, very nearly periodic and then may change to another periodic motion with period that is k times the preceding motion.

So, by changing this initial condition this k also can be changed, so, you can reach to another chaotic attractor. This constant evolution from one periodic motion to another produces a long time impression of randomness while showing a short term glimpse of order.

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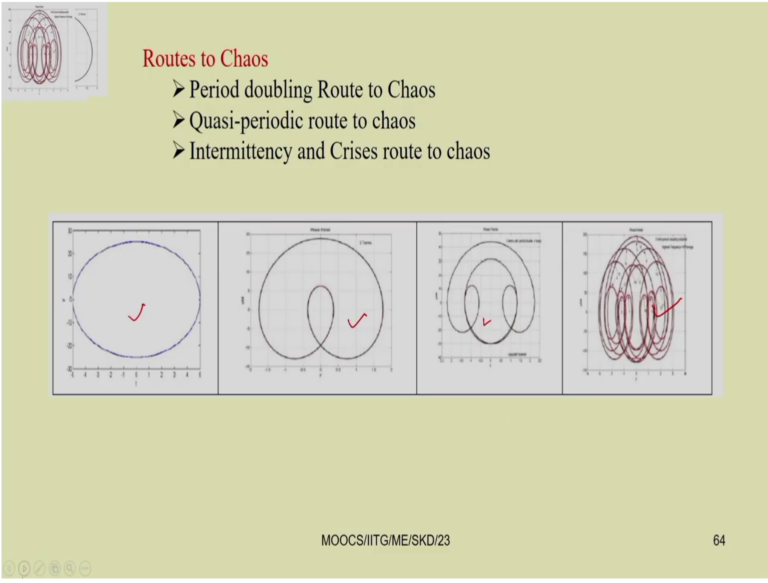


So, we can see for example, let us take this period doubling route to chaos. So, we have seen in case of for example, you just take the Duffing equation or the case what I have shown in case of a parametrically excited system. So, initially we have single period, then it is two period, then this is four period and then eight and sixteen and then it will do two chaos.

So, we have different route to chaos. So, one such route just now I have shown you. So, that is known as period doubling route to chaos, then quasi-periodic route to chaos and intermittency and crises route to chaos. So, all these things we will study in the next class.

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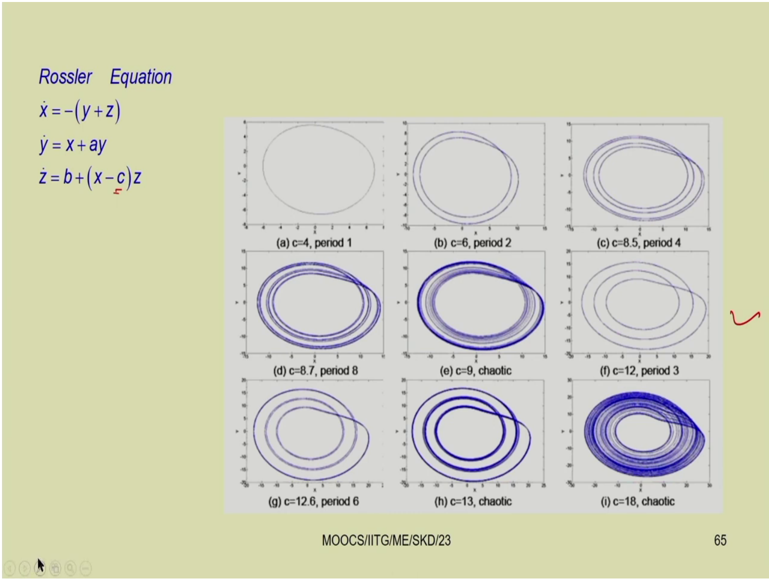


So, in the meantime, so, you can plot these Rossler equation. So, it is plotted here. So, here also you can see for different system parameter. So, it is periodic, then two periodic, then the period goes on increasing and finally, for this value of c for this value of c so, you can see the response is chaotic. So, next class we will see how what is the relation between so, when this bifurcation occur in case of the period doubling bifurcation.

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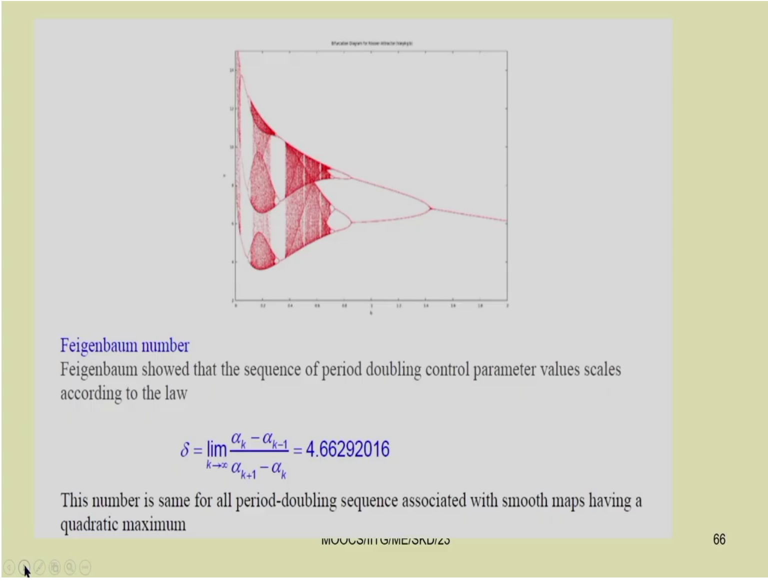


So, particularly I will tell you about this Feigenbaum number. So, this is a constant number universal number that is that value is 4.66292016. We will study regarding this Feigenbaum number and also we will study other different routes to chaos. So, next class will study more on these chaotic responses.

Thank you.