Nonlinear Vibration
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Lecture - 22
Analysis of periodic, quasi-periodic and chaotic systems

So, welcome to today class of Non-linear Vibration. So, we are going to start the module-7. So, in this module, so we are going to study regarding the periodic, quasi-periodic and chaotic responses. So, we will study the stability and bifurcation of the periodic response, different type of periodic responses also we are going to study. Then regarding the quasi-periodic response and chaotic response we will briefly study.

So, we will characterize or we will see how we can distinguish between this periodic, quasi-periodic, and chaotic responses also. And we will study the time response, phase portrait, and also this Poincare section and the lyapunov exponent to characterize the chaotic response. Today class we are going to study briefly the introduction to periodic, quasi-periodic, and chaotic response.

So, already we have seen so different type of systems, for example, we have seen the duffing oscillator we have seen this Van der Pol oscillator also, and we have taken this Mathieu Hill type of equations in the previous classes.

Today class so we will see how we are generating these periodic response, quasi-periodic response, and chaotic response in different type of systems. Particularly in the previous classes, we have seen about the fixed point response. Now, our objective is to see this periodic, quasi-periodic and chaotic response.

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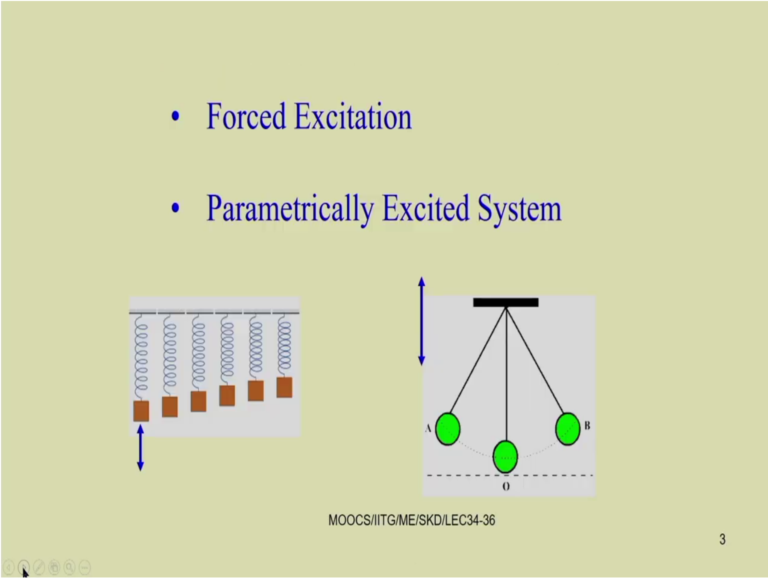

We have discussed regarding this forced excitation and also parametrically excited system. For example, in case of the forced excitations, we have a spring mass system. So, in the spring mass system, so if we have applied a force, so this is the longitudinal vibration of the spring we have seen.

Similarly, we have taken a pendulum. So, the motion of the pendulum itself is a forced excited system. But if the platform of the pendulum is moving up and down, in that case we will get a equation which is similar to that of Mathieu Hill type of equation. And those equations are known as parametrically excited system.

So, in case of forced excited system, the force and the displacements are taking place in the same direction, but in case of what parametrically excited system the force application of force and the direction of motion so are perpendicular to each other or are orthogonal to each other. In these systems, in case of forced excitation, so we have seen particularly in non-linear cases, so we have taken weak excitation and hard excitation. And also we have studied different type of resonance conditions.

For example, so we know regarding the simple resonance condition. So, in addition to that, so we have these we have the sub harmonic and super harmonic resonance condition. For parametrically excited system, so we have principal parametric resonance conditions, combination parametric resonance condition.

So, in case of principal parametric resonance condition so the resonance occur when it is when the excitation frequency is nearly twice the natural frequency particularly by taking a continuous system so where we have infinite number of natural frequency.

So, we can get a number of parametrically excitation or parametrically resonance conditions. Also in case of parametrically excited system, so we have combination resonance. So, this occur when the natural when the excitation frequency is the sum or difference of the other model frequencies.

For example, so we may have these external frequency equal to omega 1 plus omega 2, so this is the combination resonance of some type of first mode and second mode. Similarly, we may have omega equal to omega 3 minus omega 1. So, in this case, it is combination parametric resonance of different type. We have used different perturbation methods. Also last class we have seen we can use these two method to find the parametric instability region.

And also by using different methods, so different numerical methods also we can find the response of the systems we can plot the free vibration. In case of forced vibration, so we can have these frequency response plot, and the forced response plots for different system parameters. So, given a system, so we can taking the different system parameter, so we can have different responses.

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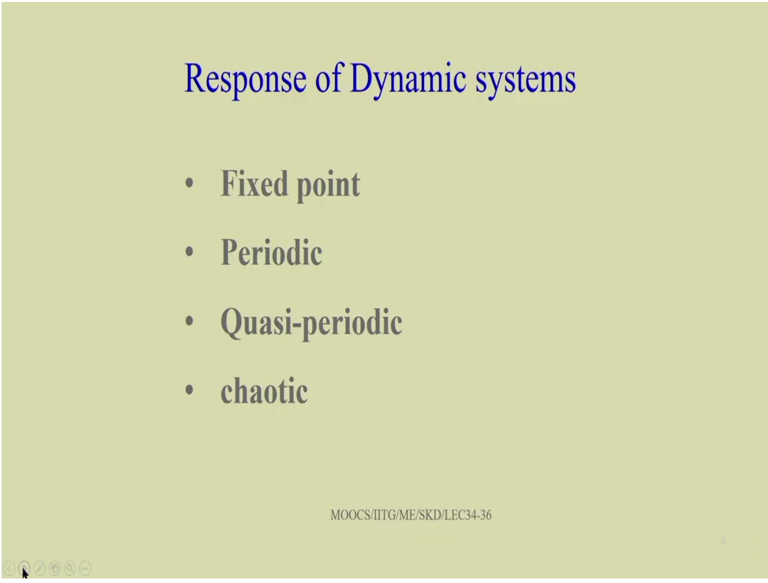

So, out of this all these responses, we are more familiar with the fixed point response. Today we are going to study regarding this periodic, quasi-periodic, and chaotic responses of the system.

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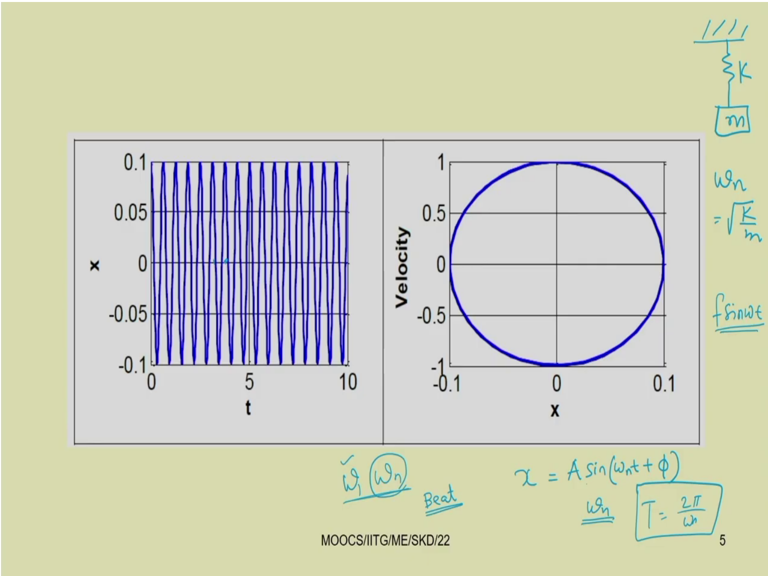

Let us start with the simple example of the spring mass damper system, spring mass system. So, if we have a simple spring and mass system, so this is simple spring and mass system. We know system will have a response so here omega n equal to root over K by m.

So, depending on the initial condition we know so we can write the response x equal to response x equal to A sin omega n t plus phi or you can write this thing equal to A sin omega n t plus B cos omega n t. Here it will oscillate with a frequency of omega n that is the natural frequency of the system.

These type of vibration, so there is no damping. So, we are assuming there is no damping in the system or damping is very very negligible. And there is no external force acting on the system. And if we start oscillating the system or pull the system slightly downward and leave it, so it will continue to move with a frequency that is equal to omega n. So, similarly for a damped case, so we can derive this solution and we can find the response of the system.

Here you can note or you can see the response of the system is periodic. So, this is the time response. So, the time response is written in terms of the sin only the sin or you can write in terms of cos also.

So, this is the periodic response because it repeats. So, the motion repeats with a time period t equal to 2 pi by omega n. So, you can see the time period for example so you can start from here. So, now, it has gone down, then come up and this, so this is one period. So, this to this is one period.

Similarly, so you can find the period between two similar points on the time response plot. So, you take two similar point having the same phase, so then you can find the time period. So, after so this time period can also be written as t equal to 2 pi by omega n. So, here I have shown the response for a free vibration response of the system, also you are familiar or you know that if it is excited by a force for example, it is excited by a force f sin omega t and let us have the damping also in this system.

So, if there is no damping, so when it is a excited with a force of x f sin omega t, the response will contain two frequencies; one frequency is the external frequency of the system, and the other frequency is the natural frequency of the system. So, it will contain two frequency. And if the frequencies are very very close to each other, so already you are familiar with the system that it will experience a beating phenomena where the beating frequency is the difference in the frequency of these two that is omega and omega n.

If there is damping, then the free oscillation part or the transient part of the oscillation will died out. And in steady state, the system will oscillate with a frequency with that of the excitation frequency of the system. These parts already we are familiar with in case of the linear vibration. And also when we have added the non-linearity in the system, so we have studied the duffing equation and we have seen the response of those system.

So, let us see now you know that we can have periodic response. So, in this periodic response, so it repeats with time. And then also we know so it may be multi period also, so it can be 2 period, 3 period, 4 period. Depending on the number of frequencies, we can have different periods. So, we can have different periods in this case.

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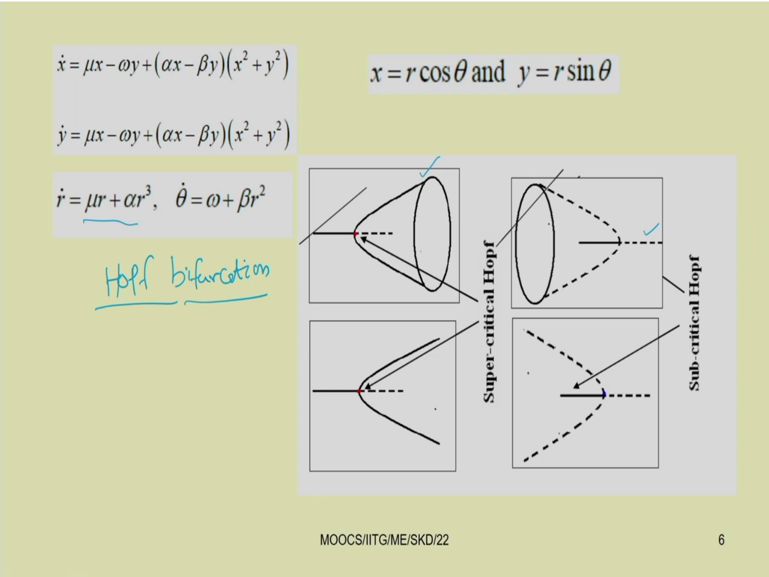

So, let us see. So, already we have discussed regarding the bifurcation of the fixed point. So, in case of bifurcation of the fixed point and when we have discussed regarding this Hopf bifurcation, and the generic form of Hopf bifurcation can be written in this way that is x dot equal to mu x minus omega y plus alpha x minus beta y into x square plus y square y dot equal to mu x minus omega y plus alpha x minus beta y into x square plus y square.

So, by taking this r x equal to r cos theta and y equal to r sin theta, so these two equations can be conveniently reduced to a very simpler form which is r dot equal to mu r plus alpha r cube, and theta dot equal to omega plus beta r square. So, you just see if I will put these r dot and theta dot equal to 0, we will have a solution where I can take common r here. So, it will be r into mu plus alpha r square; r equal to 0 is a solution also. So, r equal to 0 is the trivial state.

So, we can have the trivial state and non-trivial state here also. So, you can see due to this Hopf bifurcation the trivial state in the first figure trivial state becomes unstable. So, the trivial state become unstable at this point, and resulting in this supercritical Hopf bifurcation so where a periodic response is generated.

So, this periodic response is a stable periodic response we will see how we can determine. So, whether the response periodic response is stable or not by using this procreate theory, and which we will study in the next class.

Similarly, in case of the Hopf bifurcation, so this unstable fixed point response gives rise to a stable fixed point response by decreasing. So, here we are decreasing the system parameter. Otherwise, we can tell a stable periodic response initially we have a stable periodic response, and a stable fixed point response, and after bifurcation it yields an unstable fixed point response. In this case, the response is known to be sub critical Hopf bifurcation.

So, we can have supercritical Hopf bifurcation or sub critical Hopf bifurcation depending on the response what we are going to study. Here you can see the fixed point. So, here the fixed point is going to become unstable here and the resulting so we are a resulting periodic response. So, here we have a periodic response and a fixed point response. And the resulting solution after the bifurcation it is a unstable one. So, as it is a unstable one, so we will see, so this is known as sub critical pitchfork.

So, in case of supercritical, it is a continuous bifurcation; but in case of sub critical bifurcation, so it is a discontinuous bifurcation. Because after the bifurcation the as the system is not stable, there is a chance that it may the response may jump up to infinite or it may jump up to some other type of responses, for example, it may leads to a chaotic response also.

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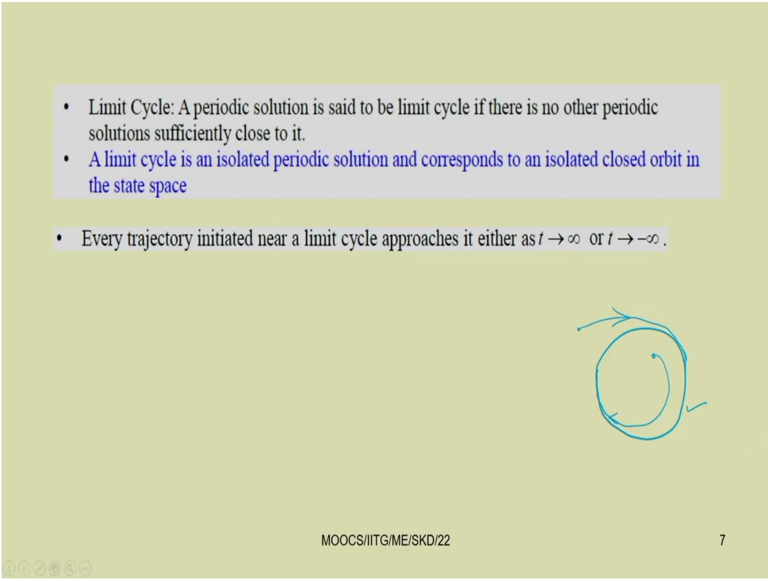

Let us see the other type of periodic response. Here we can see what we mean by this limit cycle. A periodic solution is said to be a limit cycle if there is no other periodic solution sufficiently close to it. A limit cycle is an isolated periodic solution and corresponds to an isolated closed orbit in state space. So, every trajectory initial initiated near the limit cycle approaches it either a t tends to infinite, or t tends to minus infinite.

That means, if this is a periodic this is a limit cycle, so if you start from, so for example, let us start from this position. So, the response will grow and finally, it will come to this periodic orbit at t tends to infinite. Similarly, if we are starting from a point outside this thing, outside this trajectory, so it will come back, so it will come back to this periodic orbit. So, this will come back to this periodic orbit at t tends to infinite.

A limit cycle, so this is a periodic solution the response is a periodic response, or it is a periodic solution. And there is no other periodic solution sufficiently close to it. So, you cannot find another periodic solution close to it.

So, with different initial condition, so always it come back to this periodic trajectory or this periodic orbit. So, this type of solution we have seen in case of the Van der Pol oscillator. So, this Van der Pol oscillator is a self-excited system.

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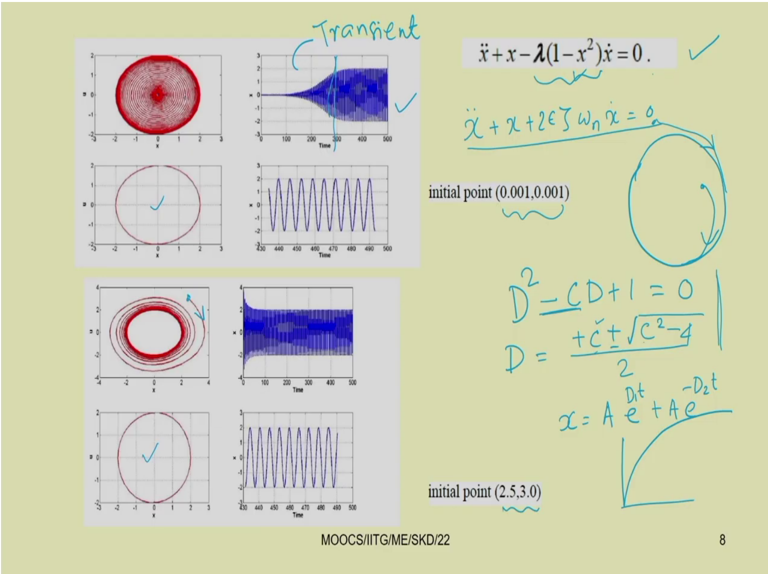

So, here you just see in case of the spring mass system. So, if we are giving some forcing term, then only it will continue to have its motion with a frequency omega. But if there is simple damping in the system, the response died out and it becomes 0. But in case of a Van der Pol oscillator, so this is the Van der Pol oscillator.

So, you just see x double dot plus x minus lambda into 1 minus x square x dot equal to 0. So, let us examine these term. So, these term. So, we have a x dot term. So, this is a, so this x dot term that is similar to a damping term. So, these damping here the coefficient of this damping equal to lambda into 1 minus x square.

So, for example, let us take mod x less than 1. So, if it is less than one. So, in that case, so what will happen? So, in that case, this 1 minus x square is also less than 1 lambda into 1 minus x square into x dot, this is the damping term 1 minus x square into x dot. If we are starting a point inside this one, so if we are starting a point inside this orbit, then so this is a negative damping.

So, the damping become negative say 1 minus x square is less than 1. So, as 1 minus x square is less than 1, so we have a negative damping here. As the damping is negative, then the solution will grow. So, already we know this thing if the damping is negative, for example, so we will have the auxiliary equation D square plus, so let me write a term which is negative D square minus let me write, so as it is negative let me write this is C D plus, so this is x, so this is 1, equal to 0.

So, we can have a root. So, the root will be D equal to minus B that is plus C minus. So, minus B will give rise to plus C plus minus root over B square minus 4 AC, B square becomes C square minus 4 by 2. So, you just see. So, here the if for example, let this C square greater than 4. So, if C square greater than 4, then we have a real root here. Real roots will have two roots also here; one will be C plus root over C square minus 4, another one will be C minus root over C square minus 4.

So, due to the presence of this plus C term, so as this plus C part, so this is a the real part of the solution is positive. The response will grow. As you know the response can be written x equal to A e to the power, so we can write this is D 1 t plus A e to the power minus D 2 t.

So, one of the root that is either D 1 and D 2 the real part of that thing will be positive or the real part is positive, then the response will grow. So, the response will exponentially grow, the response will exponentially grow, and the system will be unstable.

Now, by starting at this point within the circle, the response will grow and it will move to this limit cycle. Similarly, if we are taking a term outside this thing that is x square is greater than 1 or x is greater mod x is greater than 1, so then these terms becomes negative. So, as these terms becomes negative, then this whole term becomes positive that is we have a positive damping here. So, the equation will be similar to x double dot plus x plus 2 epsilon zeta omega epsilon zeta omega n x dot equal to 0. So, here the response will die down and it will come to so finally, it will come to this trajectory.

So, this is similar to that of a damped oscillator, so under damped oscillator. So, in that case, it will come down. In one case, it is growing. So, if you have started this motion from the inside this circle, then it will grows and reach the circle. And if we are starting outside the circle, it will come back to the circle. So, that is why this is self-excited oscillation.

So, you can see this thing in this case. So, we have started with initial point 0.001 and 0.001. So, you have started with a very initial point inside this thing. So, as we have started initial point inside this thing, so you can see after sometimes the response grows and it reaches a steady state response. So, this part you can see up to this part the response is changing, so that is the transient part. So, this is the transient solution. So, this is the transient part, and this is the steady state part.

So, we can see. So, we have the transient part and the steady state part of the oscillation. To only plot the steady state part, so here only the steady state part is one. So, you can plot, for example, you just see.

So, we have taken the time for 430 to 500. So, here time is taken from 0 to 500, but here the time is taken only the last several cycles have been taken. So, you can see in the last several cycles if you plot, then clearly you can see the response to be periodic or it is either in the form of a sine or cosine curve.

So, if you plot this x versus x dot or this u x versus the velocity, so this is displacement versus velocity plot, so that is known as phase portrait. When you are plotting this displacement versus time, this is your displacement, variation of time or time response plot.

So, variation of displacement or velocity with time will give you the time response plot. And this variation of this displacement with this velocity will give the phase portrait. So, now you know how to plot the time response, and phase portrait in case of the Van der Pol type of equation.

So, you just see in the second case. So, here the initial condition is taken inside the circle. But if we are taking this initial point outside the circle that is 2.5 and 3.0 it is taken, so then you can see, so it goes back to the original. So, it goes back. So, it reduces the orbit reduces. And finally, so it from the time response, it will be clear. So, it reduces and finally, it reaches the steady state solution.

If you plot only the last part of this thing, so you are getting the same phase portrait or the same periodic response. So, these are different way you are generating. So, these periodic response. So, here you have seen for the self-excited system that is the Van der Pol oscillator so where you are generating the periodic response.

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Existence of closed orbits

To rule out existence of closed orbits following rules/theorems may be used.

1. Closed orbits are impossible in gradient systems.

A system which can be written in the form $\dot{x} = -\nabla V(x)$ for some continuously differentiable, single valued scalar function $V(x)$ is called a gradient function.

Let $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$ be a smooth vector field defined on phase plane. For this system to be a gradient system $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$

2. Dulac's criterion: Let $\dot{x} = F(x)$ be a continuously differentiable vector field defined on a simply connected subset R of the plane. If there exists a continuously differentiable real valued function $g(x)$ such that $\nabla.(g(\dot{x}))$ has one sign throughout R, then there is no closed orbit laying entirely in R.

3. A system for which a Liapunov function can be constructed will have no closed orbits.

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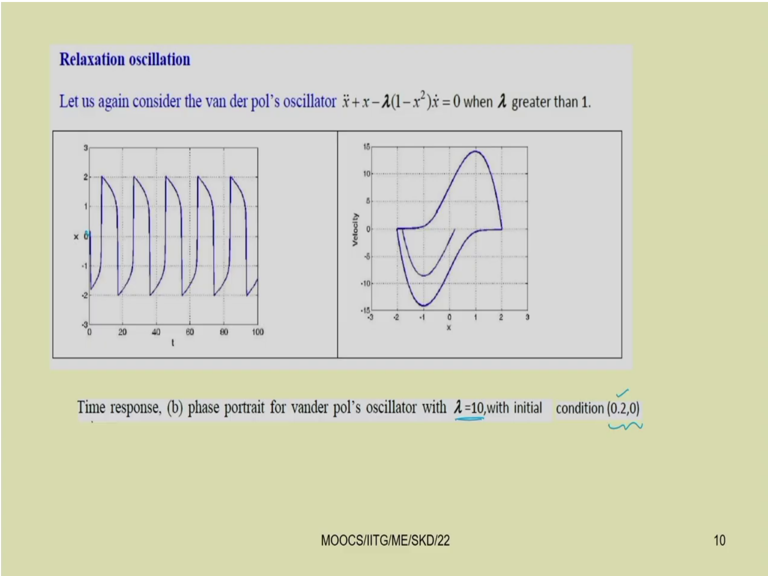

So, we will see another type of response that is known as this relaxation oscillation. So, let us take the same equation same equation of this Van der Pol, but here let us take this lambda greater than 1. So, previous case we have taken lambda equal to 1. Now, let us take lambda greater than 1. Let us take lambda equal to 10. So, if you take lambda equal to 10, we have plotted it with different two different initial conditions.

So, in one initial condition, so it is started with this point; in this case, it is starting with 0.2 and 0. So, you will just see it is started with 0.2, and displacement is 0.2, and velocity is 0. You can see the time response plot. So, it is different. So, this plot you can see physically it is different from the plot what you have seen in the previous case. The time response here it is in the form of sine and cosine, sine and cosine you have seen.

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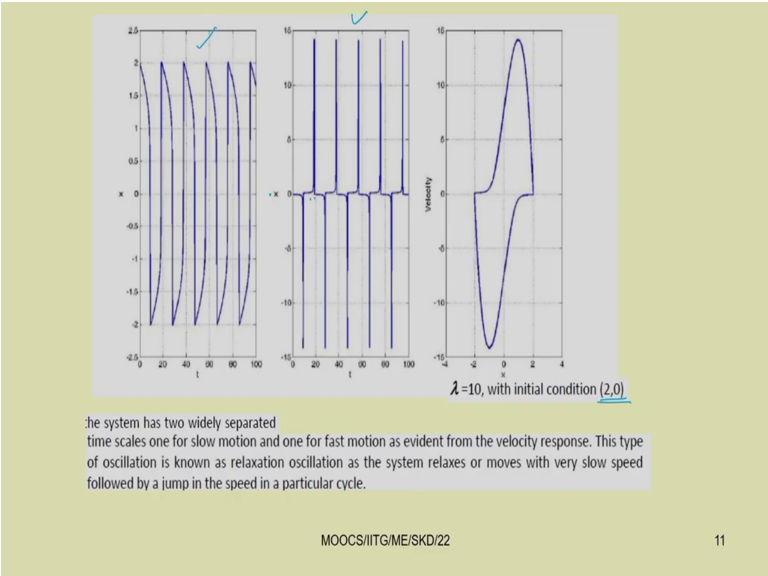

But in this case you can see the response is slightly different. So, let us take another initial condition, for example, it is 2 and 0.

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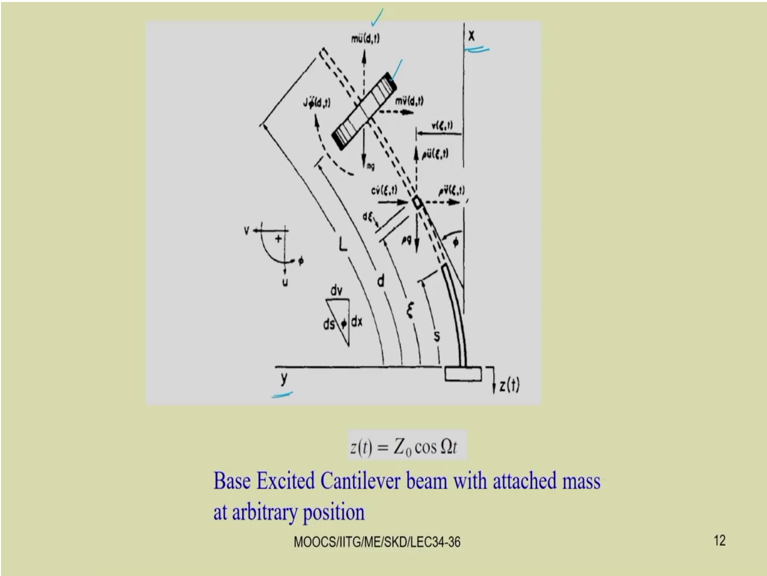

So, if you take another condition that is 2 and 0, the this is the time response plot and this is the velocity time versus velocity x dot. So, this is time versus velocity. And this part is the x versus velocity that is displacement versus velocity. So, from the velocity curve you can see, so it has two timescale. So, two widely separated timescales you have. So, one for the slow motion and one for the fast motion.

From this thing you can see one for the slow motion. Here it is slowly varying. So, this portion is slowly varying with time, and then with speed. So, this to this, you just see quickly it is moving. We have two response, two time responses are there. So, one part we have a slowly varying time. And from the velocity, you can see it is quickly moving.

One portion is slowly varying, and the other portion is quickly varying with time. That means so the system has two widely separated timescales one for slow motion and one for fast motion as evident from the velocity response. So, these type of oscillation is known as relaxation oscillation as the system relaxes or moves with very slow speed followed by a jump in the speed in particular cycle. So, this is a cycle.

So, in this cycle some part of this curve it is moving very slowly, and then suddenly it jumps and moves with very high speed that is why it is known as relaxation oscillations. In this Van der Pol oscillation, so by taking this parameter lambda away from or greater than 1, so you can see this relaxation type of oscillation.

And by taking it is equal to 1, so you have seen this limit cycle where the limit cycle has a form or it has a time response similar to that of a sine or cosine term. So, here you have seen periodic responses of the systems. So, already we have discussed regarding this base excited cantilever beam attached with a mass at arbitrary position.

So, again we will see this one. And here we will observe that the system the simple system has many different type of response, particularly all the four different type of responses you can find. So, here you can find the fixed point response, periodic response, this quasi-periodic response, and chaotic response for different system parameters.

So, let us examine this system again. So, here we have a base excited cantilever beam. So, this cantilever beam is moving up and down with a periodic motion that is z equal to z 0 cos omega t. At any time t, so you can take this beam is moving or it is taking a shape either to the left or to the right of the initial position that is the trivial state.

Let us now take this x. So, it is moving x in this direction. This is the x-direction along the length of the beam, and this is the y-direction that is transverse to the beam. And at a particular time t, so let the beam is making an angle phi with the x-axis. So, we have taken a small element here. In the small element, if you draw a tangent at the small element, so it is making an angle phi with this x-axis.

So, if we write what are the inertia force, what are the forces acting on that element, so you can note that forces acting are rho u double dot, then rho g, then this is rho v double dot and c v dot. So, v is the displacement in transverse direction. So, v dot is the velocity multiplying with c that is the term due to this damping. Similarly, rho v double dot is the inertia force in transverse direction, and inertia force in the longitudinal direction equal to rho u double dot. Rho is the mass per unit length is taken in this case.

Similarly, so in this beam, so there is an attached mass. So, this is the attached mass here. This attached mass has a mass of m. So, it will also be subjected to similar type of forcing. For example, m g is the weight of that mass. Then m v double dot, then m u double dot. So, these are the inertia force.

So, you just see body is moving towards left. As the body is moving towards left, the inertia force is acting towards right. The body is moving towards left and also downward, as the body is moving downward in longitudinal direction, so the inertia force is in upward direction m u double dot is in upward direction, and m v double dot towards the right.

Similarly, here, so we have applied this rotary inertia as j phi double dot can be written. We have taken the small element at a distance zeta, and this arbitrary mass is put at a distance d from the fixed end. If we want to write the equation of motion at a distance s from the base, we can write that equation of motion. Now, we can take the movement of all these forces. So, about this point s and we can write down this equation of motion.

So, here one point you can note, so if along this x-direction that is dx and transverse direction this is dv if we are taking a small element ds. So, we can write this or you can use this triangle and we can write the sine phi equal to dv by ds. And from that thing, we can see that if this phi is not small, then we can expand this thing and we can have a non-linear equation of motion.

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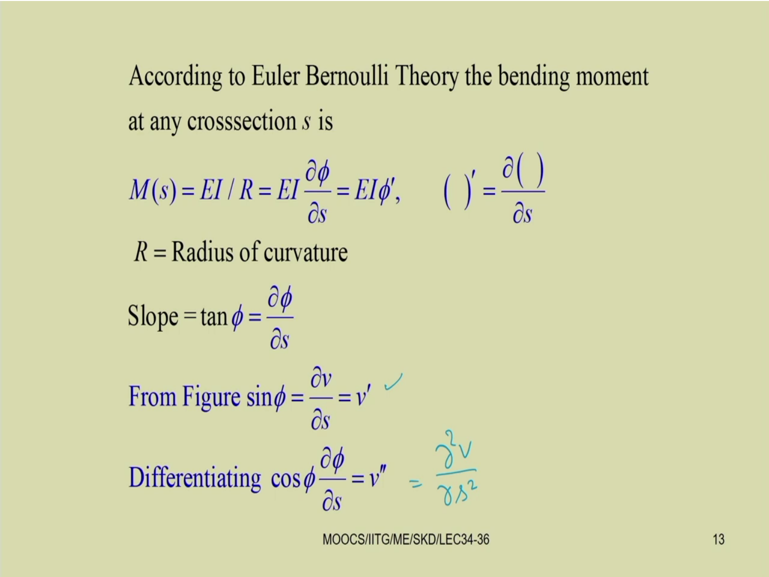

According to Euler Bernoulli theory, the bending moment at any cross section s can be written as m s equal to EI by R. So, this 1 by R can be written as del phi by del s. Here we are using this partial derivative because this phi is a function of both space and time.

So, that is why you are using this partial derivative. This 1 by R equal to del phi by del s. So, your M s equal to EI by R equal to EI del phi by del s or E phi dash we can write. R is the radius of curvature. So, slope we can find that is tan phi. So, tan phi will be equal to del phi by del s.

So, from this figure already I have shown you that sin phi equal to del v by del s or v dash. So, we can differentiate this equation, and you can write cos phi will be equal to, so differentiating sin phi it is cos phi into del phi by del s equal to v double dash. v double dash is nothing but so this is del square v by del s square del square v by del s square equal to cos phi into del phi by del s.

So, already we know the sin phi equal to v dash. So, cos phi will be equal to root over 1 minus sin square phi, so it will be equal to root over 1 minus v dash square.

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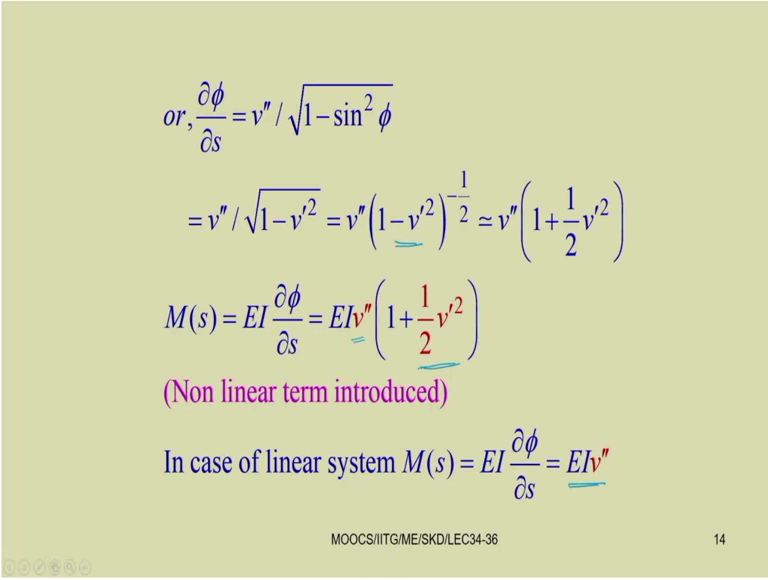

So, that way we can write this del phi by del s equal to v double dash divided by 1 minus sin square phi that is equal to v double dash divided by root over 1 minus v dash square.

Now, it can be further written as v double dash into 1 minus v square to the power minus half. And by expanding it binomially, so we can write this is nearly equal to v double dash into this to the power minus half will go inside this thing. So, this becomes minus minus, plus. So, this becomes v double dash into 1 plus half v dash square.

We can write this M s equal to EI del phi by del s equal to EIv double dash into 1 plus half v dash square. So, you just see. So, this non-linear term we have introduced this non-linear term as it is a product of this v dash square into v double dash. The second term is the non-linear term.

So, in simple Euler Bernoulli beam equation, so we used to write M s equal to EI del square v by del s square or EIv double dash. But here we are using this additional term that is v double dash into v dash square divided by 2. So, in case of non-linear system, so already you know. So, we are writing it EIv double dash.

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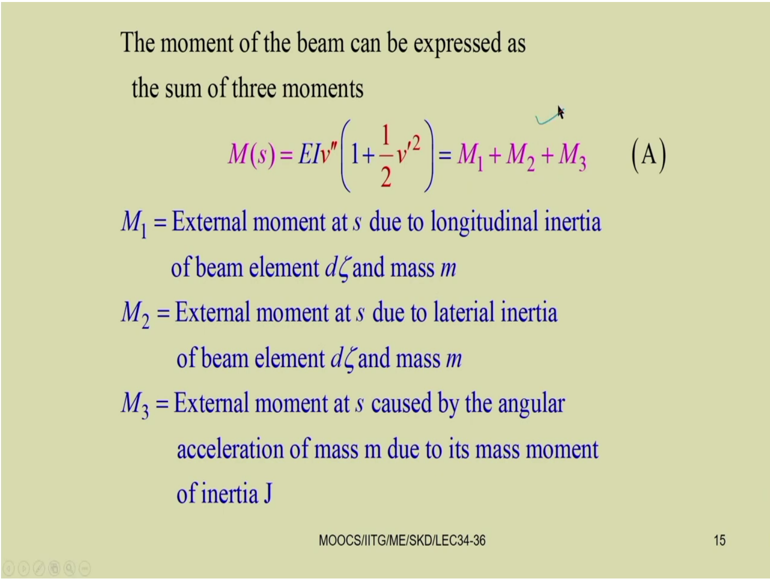

So, from this equation, so you can see the moment of the beam can be expressed as the sum of three moments these equal to M 1 plus M 2 plus M 3, where M 1 is the external moment at s due to longitudinal inertia of the beam element d zeta and mass M. Second will be M 2 that is external moment at s due to laterial inertia of the beam element d zeta and mass M.

And third one M 3 that is external moment at s caused by the angular acceleration of mass M due to its mass moment of inertia J. So, that additional mass what we have put at the arbitrary position; so here we are putting it at some arbitrary position. In many literature, you can find, so it is put at the tip of the beam.

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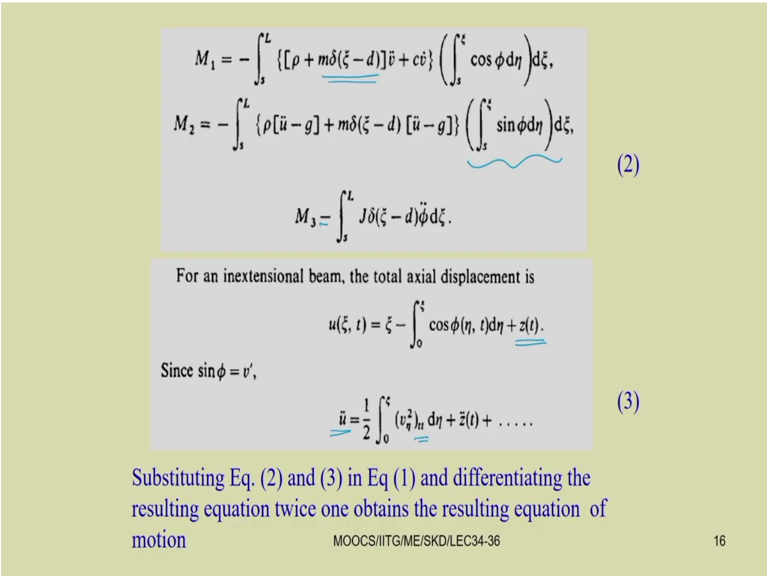

So, that way, so now let us see this M 1, so we will take the moment about c about that point s, here it will be so M 1 will be when you are taking this moment of the small element, then we will integrate it over the length that is from s to L. So, this M 1 equal to minus integration s to L rho plus M delta zeta minus d. So, these thing you just note this is delta is the direct delta function, we are writing this direct delta function to show the arbitrary position of the attached mass.

So, we can put it at any location and this is point mass. So, we are considering this as a point mass that is why this direct delta function is used. Then this M 1 equal to minus integration s to L rho plus m delta zeta minus d into v double dot plus c v dot into this force into you have to find the distance force into distance will give the moment. So, for the small element d zeta, that the distance will be d zeta into integration s to zeta cos phi d eta, so here eta is the dummy variable we are using.

Similarly, M 2 equal to minus s to L integration minus s to L rho u double dot, so rho u double dot the in the longitudinal direction the inertia force. So, the weight also we are considering that is why this is minus g plus m delta zeta minus d u double dot minus g into also the distance part can be written.

So, this is the distance part. So, the first part is the force acting on that small element and integration will give us length of the portion from s to L for which we are finding the moment. So, this distance equal to d zeta into integration s to zeta sin phi d eta.

So, M 3, so that is due to the inertia part rotary inertia. So, M 3 equal to s to L J delta zeta minus d phi double dot d eta for an extensional. So, considering in extension condition, that means, there is no laterial extension of the beam in extensional condition so if you are considering, then the total axial displacement can be written equal to u zeta t equal to zeta minus 0 to zeta cos phi eta t d eta plus z t; z t is the base motion. So, u t will contain this z t plus this part.

So, since sin phi equal to v dash we have taken before. So, this u double dot differentiating it twice. So, you can write this u double dot equal to half integration 0 to zeta v eta square t t; t t means so it is double differentiation with respect to time into d eta plus z double dot t. So, now, you just see so we have eliminated.

So, by writing this u double dot term, so we have eliminated or we have reduced or we have written this u term this axial displacement in terms of the transverse displacement. So, this is due to this in extensional property of the beam. So, we are assuming the beam does not extend in the direction.

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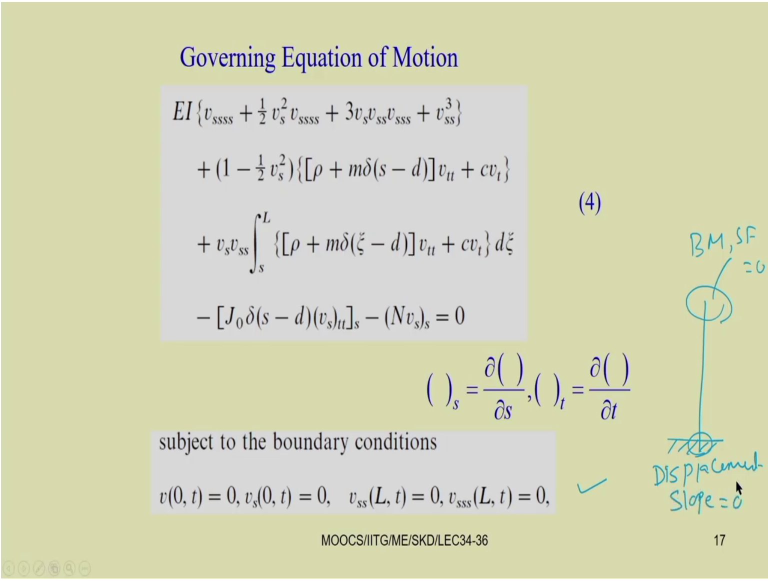

So, now by substituting this equation in this original equation, the governing equation of motion can be written in this form that is EI v 4 s, 4 s means this is del 4th v by del s 4th plus half del v by del s whole square into del 4th v by del s 4th plus 3 del v by del s into del square v by del s square into del q v by del s q plus del square v by del s del square v by del s square to the power 3 plus 1 minus half v s square into rho. So, actually you will get this equation by twice differentiating the equation, this equation.

So, by twice differentiating after substituting this M 1, M 2, M 3, so by twice differentiating this equation. So, you know moment, so if you differentiate once moment, you will get this shear force; and further differentiation, we will give the loading condition load. So, that is why you have to differentiate it twice. So, by differentiating twice, you can get this equation. And with boundary condition, so these are the boundary condition v 0 t equal to 0 that is at the base there is no displacement.

Similarly, there is no slope at the base. As it is fixed at the base, then v s s L, t equal to 0 at as we have a free end, mass is not attached at the end rather mass is attached at any arbitrary position. So, if we are putting this mass at the end, then the boundary condition should have been this equal to the bending moment there.

And this shear force would have been equal to this shear force this v s s L, t should have been proportionate to this shear force. But as it is a free end, so that is why the bending moment and shear force are equal to 0. So, that is why v s s L, t equal to 0, and del q v by del s q L, t equal to 0. So, this end in the fixed end that is in the fixed end. So, this is the cantilever.

So, at the fixed end so this is the fixed end displacement and slope are 0. And at the free end, bending moment and shear force are 0. So, bending moment and shear force are 0 here; and here displacement and slope are 0.

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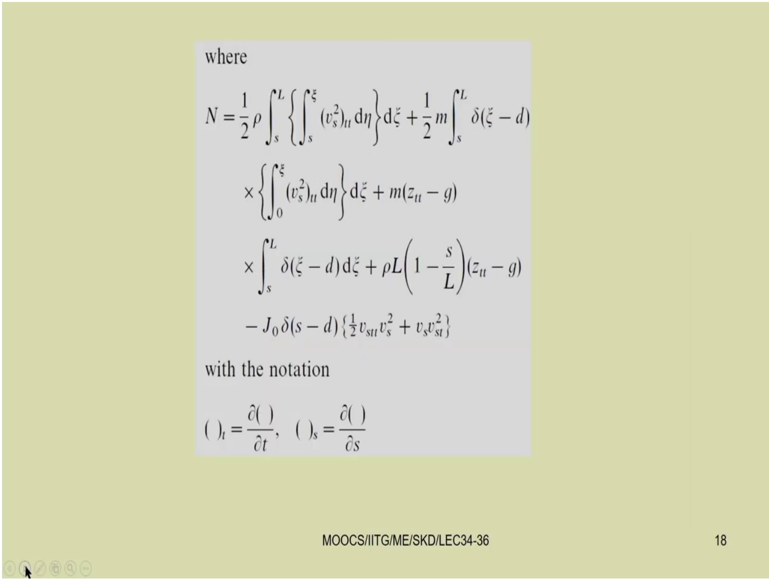

In this way, we got the equation of motion. So, here that N term is also this non-linear term is here.

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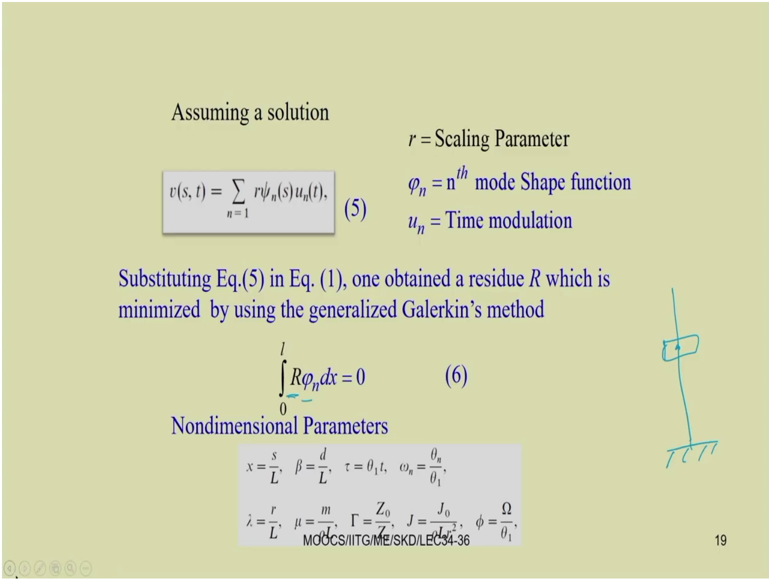

So, now, we can assume a solution v s, t equal to. So, multimode we can take r psi n s and u n t. So, where r is the scaling factor, psi n is the mode shapes, and u n is the time modulation. So, mode shape function psi n and u n is the time modulation. So, by taking this Euler Bernoulli beam equation, so we can find the mode shape.

So, here while finding the mode shape, so we may take one can take the simple Euler Bernoulli beam Euler Bernoulli equation for a cantilever beam and write down the psi, or you can use so one can divide this beam into two parts; so one of two where the mass is attached, and another one after the mass is attached. And taking this continuity conditions at this position, so one can find the psi n. So, that derivations can be found in the paper by (Refer Time: 40:37).

And now assuming this solution, this is assumed mode solution and applying this Galerkin's method, so we are taking a weight function R so this residue. So, we can take the by substituting this v s, t in this original equation. So, as these the assumed mode shapes we are taking which is not satisfying this governing equation.

So, we will have some residue. So, this residue is written in terms of R because equation will not be equal to 0. So, we will have this residue. So, in this residue, we can multiply this weight function. So, here psi n is taken as the weight function also, and integrate it over the length of the beam, so and equate to it to 0. So, we want to minimize this residue.

So, from that thing, so we know so this R psi n d x equal to 0. And using these parameter, so for example, where we can use this non-dimensional parameter x equal to s by L, beta equal to d by L, tau equal to theta 1 by t, omega n equal to theta n by theta 1, lambda equal to r by L, and this mu equal to m by L, mu is the mass ratio you have taken here, and gamma that is the z 0 by z.

So, we have taken this amplitude non-dimensional amplitude of base motion, and J equal to j 0 by j 0 by L r square. So, phi equal to omega by omega by theta 1, so external frequency by this theta 1.

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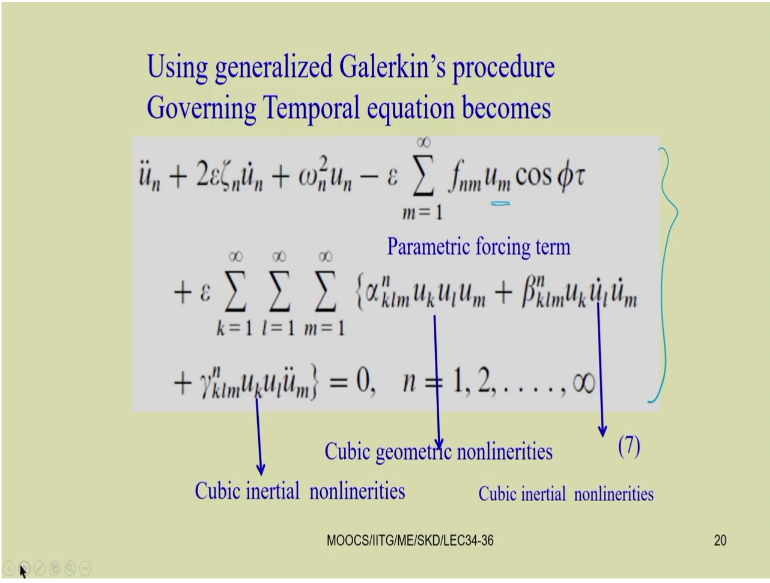

So, this way we can find this using generalized Galerkin procedure, the governing temporal equation of motions we can find. So, this is the governing temporal equation of motion. So, here you just see the equation is written in this form that is u n double dot plus 2 epsilon zeta n u n dot plus omega n square u n minus epsilon m equal to 1 to infinite f n m u m cos phi tau.

So, here you just see these coefficient of u m that is the displacement is a time varying term that is why you have a parametric forcing term, and here we have cubic geometric nonlinearity. And this and this, these two are cubic; inertia non-linear terms are also present in the system.

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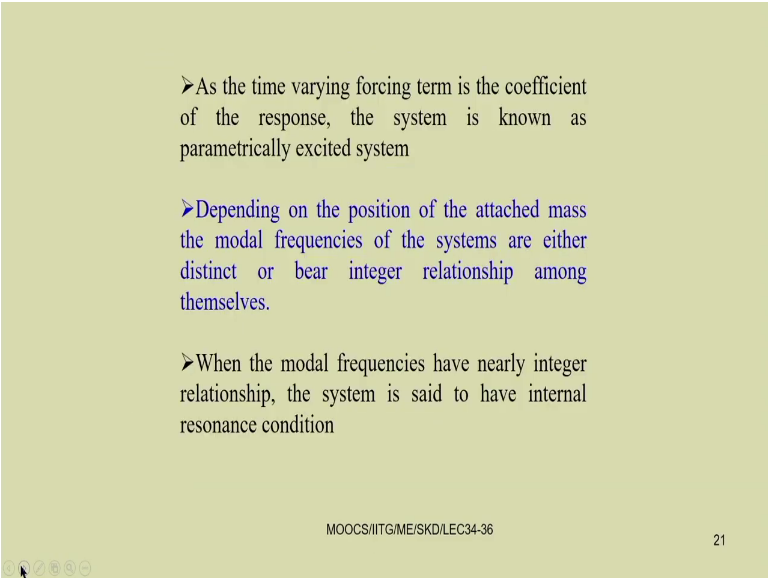

As the time varying, as the time varying forcing term is the coefficient of the response, the system is known as parametrically excited system. So, depending on the position of the attached mass the modal frequencies of the systems are either distinct or bear integer relationship among themselves. When the modal frequencies have nearly integer relationship, the system is said to have internal resonance condition.

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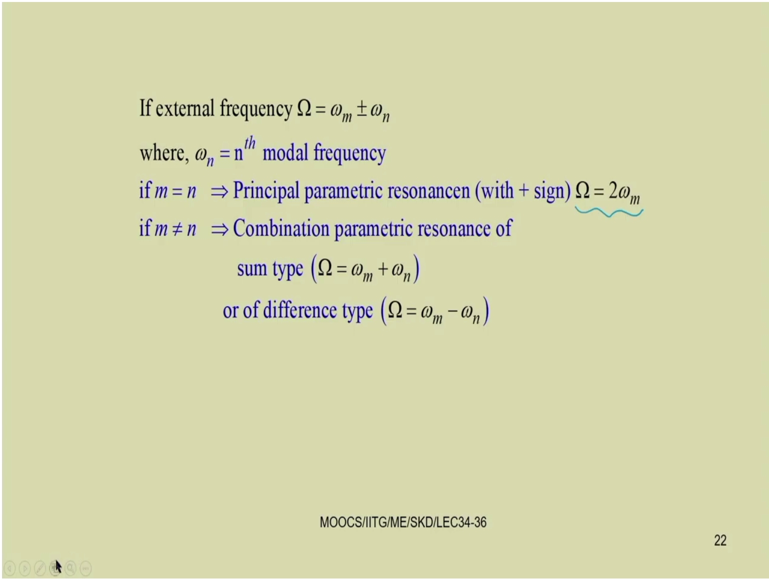

So, here if the external frequency omega is taken to be omega plus minus omega n where, omega n equal to nth model frequency. If m equal to n, so we call it as principal parametric resonance conditions, so that is equal to omega equal to 2 omega m. And if m not equal to n, so we have this combination parametric resonance of sum type if you are taking omega equal to omega plus omega n, and it will be of difference type if you are taking omega equal to omega m minus omega n.

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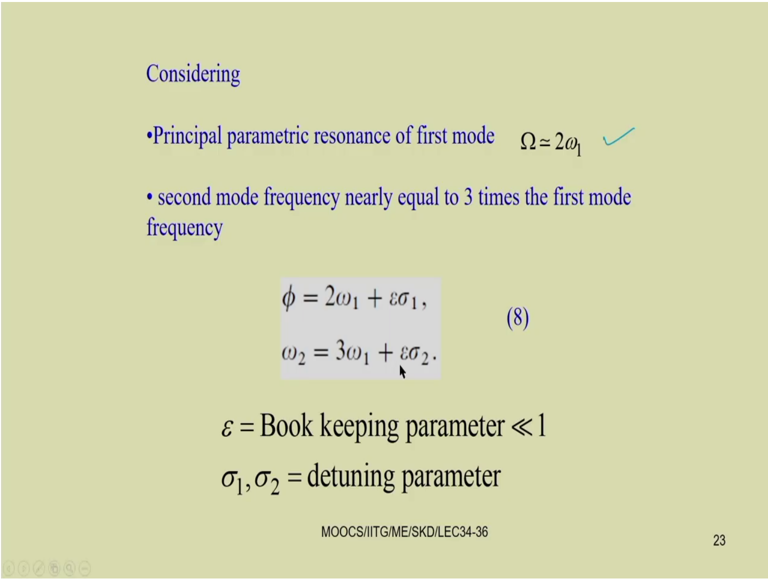

Considering, so in this case, I will show you different type of response by considering only principal parametric resonance of first mode that is omega equal to 2 omega 1. So, if you take omega equal to 2 omega 1, then we can use this detuning parameter phi equal to 2 omega 1 plus epsilon sigma 1, and omega 2 equal to 3 omega 1 plus epsilon sigma 2.

So, here we are assuming the second model frequency is 3 times the first mode natural frequency, and the sigma one sigma 2 are the detuning parameter. Epsilon is the bookkeeping parameter which is very very less than 1.

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Using Method of Multiple scales

$$u_n(\tau;\varepsilon) = u_{n0}(T_0, T_1) + \varepsilon u_{n1}(T_0, T_1) + \cdots,$$
$$T_0 = \tau, \quad T_1 = \varepsilon\tau, \quad n = 1, 2, \ldots, \infty. \qquad (9)$$

$$D_0^2 u_{n0} + \omega_n^2 u_{n0} = 0, \qquad (10)$$

where $D_0 = \partial/\partial T_0$ and $D_1 = \partial/\partial T_1$.

$$D_0^2 u_{n1} + \omega_n^2 u_{n1} = -\Bigg[2\zeta_n D_0 u_{n0} + 2D_0 D_1 u_{n0} - \sum_{n,m=1}^{\infty} f_{nm} u_{m0} \cos\phi\tau + \sum_{klm}(\alpha_{klm}^n u_{k0} u_{l0} u_{m0} + \beta_{klm}^n u_{k0} D_0 u_{l0} D_0 u_{m0} + \gamma_{klm}^n u_{k0} u_{l0} D_0^2 u_{m0})\Bigg] = 0, \qquad (11)$$

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Using method of multiple scales standard procedure or method of multiple scale, we can solve this problems.

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$$u_{n0} = A_n(T_1)\exp(\mathrm{i}\omega_n T_0) + \mathrm{cc}, \quad (12)$$

Substituting Eq. (12) in Eq. (11) and eliminating secular term for n =1

$$\begin{aligned} &2\mathrm{i}\omega_1(\zeta_1 A_1 + A'_1) - \tfrac{1}{2}[f_{11}\bar{A}_1\exp(\mathrm{i}\varepsilon\sigma_1 T_o) \\ &\quad + f_{12}A_2\exp\{\mathrm{i}\varepsilon(\sigma_2 - \sigma_1)T_0\}] \\ &\quad + \sum_{j=1}^{\infty}\alpha_{e1j}A_j\bar{A}_jA_1 + Q_{12}A_2\bar{A}_1^2\exp(\mathrm{i}\varepsilon\sigma_2 T_0) = 0, \end{aligned} \quad (13)$$

For $n = 2$,

$$\begin{aligned} &2\mathrm{i}\omega_2(\zeta_2 A_2 + A'_2) - \tfrac{1}{2}f_{21}A_1\exp\{\mathrm{i}\varepsilon(\sigma_1 - \sigma_2)T_0\} \\ &\quad + \sum_{j=1}^{\infty}\alpha_{e2j}A_j\bar{A}_jA_2 + Q_{21}A_1^3\exp(-\mathrm{i}\varepsilon\sigma_2 T_0) = 0. \end{aligned} \quad (14)$$

And we can get these reduced equation.

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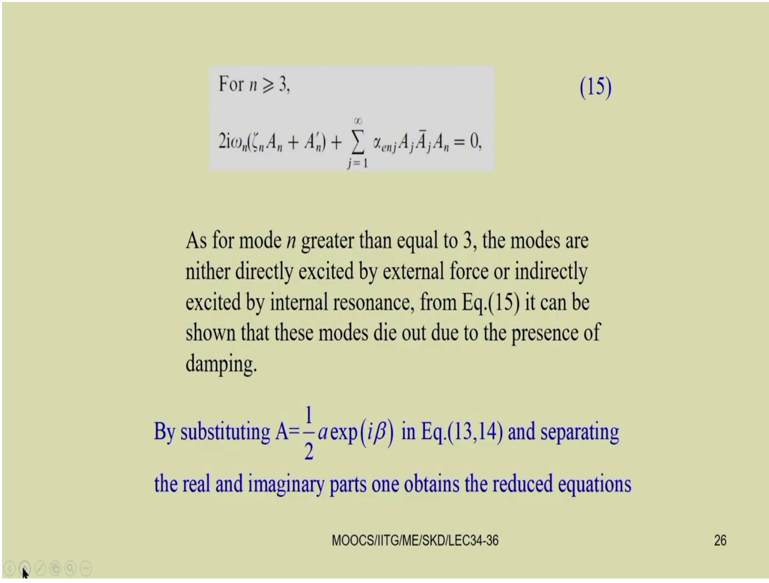

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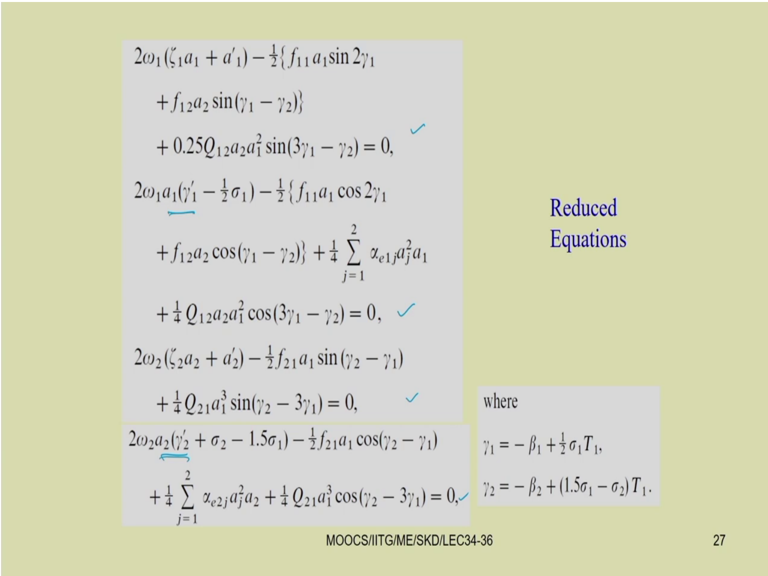

So, details are given here. And we can get the reduced equation. So, these are the set of reduced equations where we got. So, this is 1, 2, 3 and 4 as we have taken 2 modes see that is why we have 4 equations. So, a 1 and a 2 are the response amplitude of the first and second mode, gamma 1 and gamma 2 are the phase of the first and second mode. This way we can find the reduced equation.

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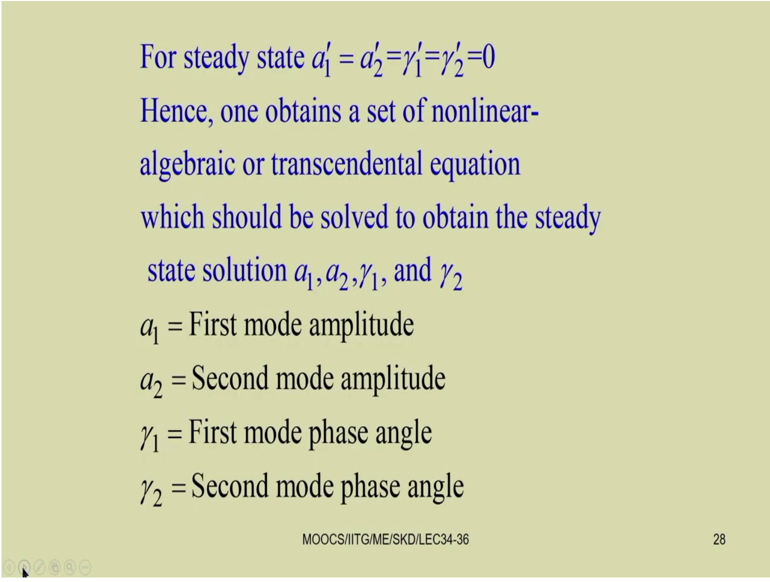

So, after getting the reduced equation for steady state response, this a dash, a 1 dash equal to a 2 dash gamma 1 dash equal to gamma 2 dash equal to 0. So, hence one obtain a set of non-linear algebraic or transcendental equation which would be solved to obtain the steady state solution that is a 1, a 2 gamma 1 gamma 2. a 1 is the first mode amplitude, a 2 is the second mode amplitude, gamma 1 – first mode phase angle, and gamma 2 – second mode phase angle.

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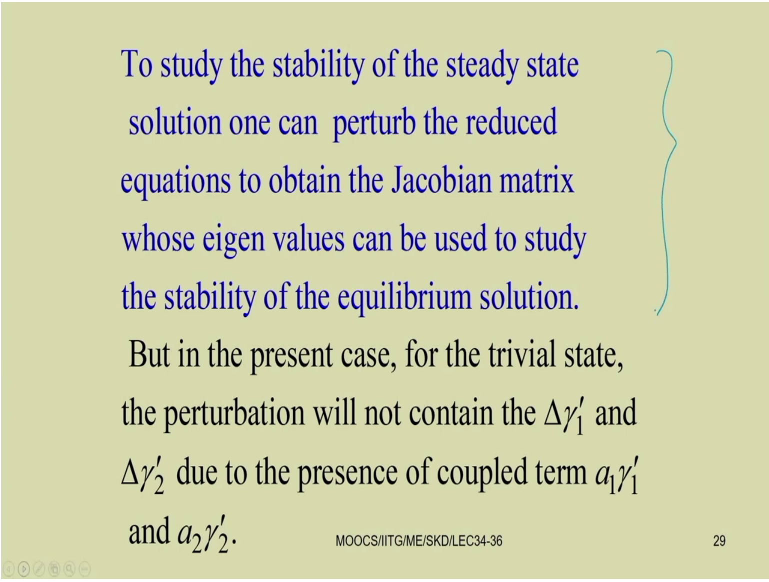

So, to study the stability of the steady state solution, one can perturb the reduced equation to obtain the Jacobian matrix whose eigen values can be used to study the stability of the equilibrium solution. So, that thing we have studied extensively last class or last few classes. But in this present case, you can see that this we have these terms that is a 2 gamma 2 dash and a 1 gamma 1 dash.

So, if you perturb these thing for trivial state, so this gamma delta gamma 1 dash and delta gamma 2 dash terms will not be available. So, that is why by just by perturbing these equation, so we cannot study the stability of the trivial state, but we can study easily the stability of the non trivial state.

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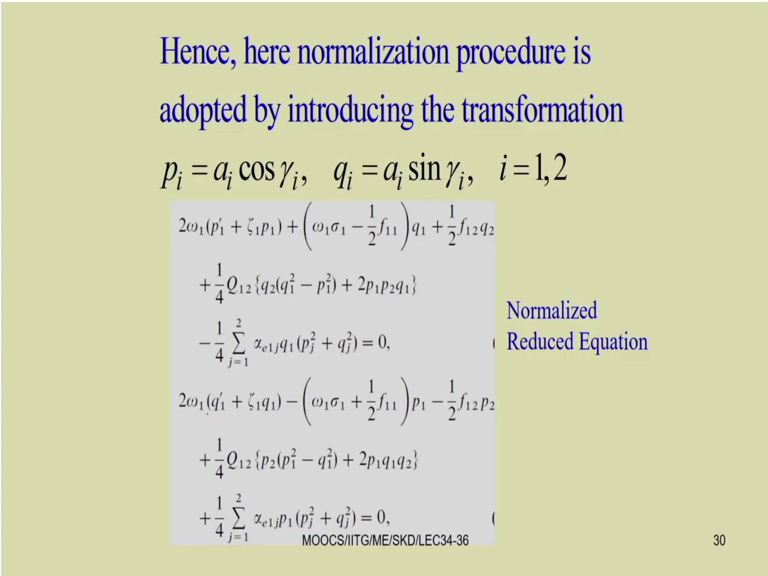

That is why one can do this transformation. So, for example, one can use this transformation that is p i equal to a i cos gamma i, and q i equal to a i sin gamma i. And one can get the normalized reduced equation in terms of p 1 dash, q 1 dash, p 2 dash, q 2 dash. So, you just see that amplitude term is not multiplied with this q dash term here, or p dash term here. So, that is why independently these equations can be perturbed around this p 1, q 1, and p 2, q 2.

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$$2\omega_2(p'_2 + \zeta_2 p_2) + \frac{1}{2} f_{21} q_1 + \omega_2(3\sigma_1 - 2\sigma_2) q_2$$

$$-\frac{1}{4} Q_{21} q_1 (3p_1^2 - q_1^2) - \frac{1}{4} \sum_{j=1}^{2} \alpha_{e2j} q_2 (p_j^2 + q_j^2) = 0,$$

$$2\omega_2(q'_2 + \zeta_2 q_2) - \frac{1}{2} f_{21} p_1$$

$$-\omega_2(3\sigma_1 - 2\sigma_2) p_2 + \frac{1}{4} Q_{21} p_1 (p_1^2 - 3q_1^2)$$

$$+\frac{1}{4} \sum_{j=1}^{2} \alpha_{e2j} p_2 (p_j^2 + q_j^2) = 0.$$

Normalized
Reduced Equation

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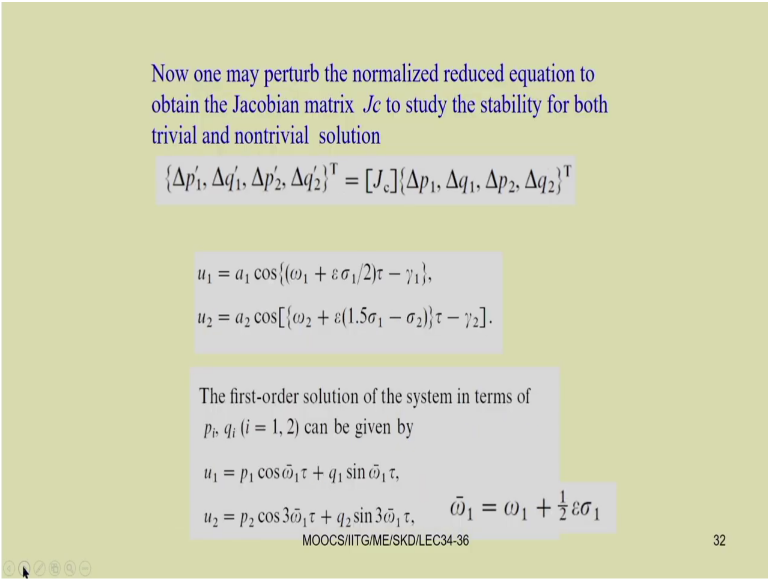

And one can get the Jacobin matrix J c. And by finding this eigenvalue of this Jacobin matrix, so one can find the stability and bifurcation of the fixed point response, already we have studied different type of bifurcations in case of the fixed point response. So, we have the static bifurcation and dynamic bifurcation.

So, in case of static bifurcations, we know the pitchfork bifurcation, saddle node bifurcation, and the transcritical bifurcation. Similarly, in case of dynamic bifurcation, so we study about the Hopf bifurcation. So, these are for the fixed point response. So, these are for the fixed point response.

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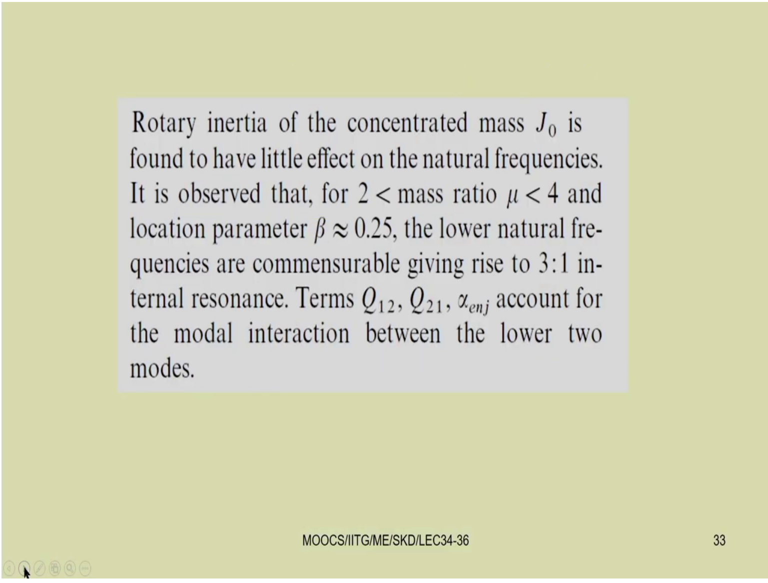

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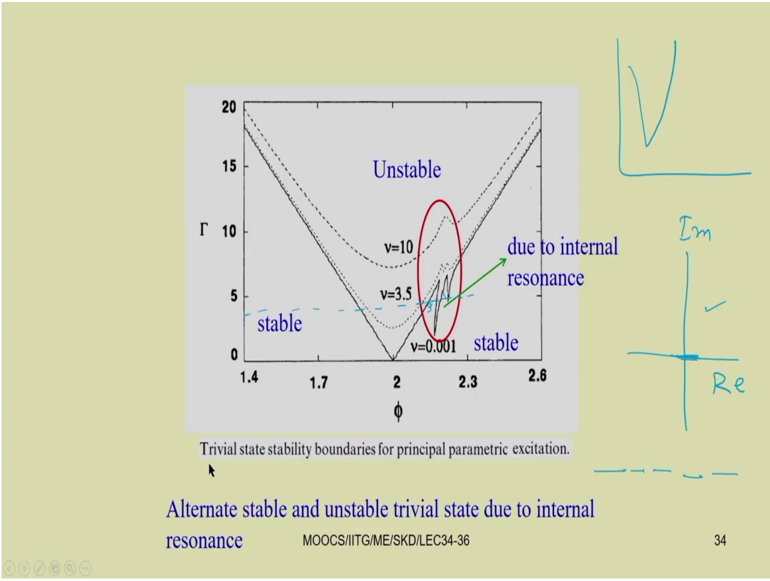

So, here you can see. So, you have plotted the parametric instability region particular a sample case has been shown here. So, for example, so in this case, so by varying these frequency and amplitude of the non-dimensional frequency and amplitude of the base excitation, so one can plot this instability region. In the previous class, we know we have used this procreate theory to plot this transition curve, also we have seen we have use this method of multiple scales also to find this thing.

So, here to plot this thing we can take these Jacobian matrix put these p, q, and p 1 q 1, and p 2 q 2 equal to 0 for the trivial state and then find the eigenvalue. So, we can find the value of phi and gamma for which so there is a change of stability, that means, the system change from stable to unstable region or from unstable to stable region. That means, so it crosses this the Jacobian.

So, the eigenvalue the real part of the eigenvalue crosses the imaginary axis. By changing the system parameter, we can see the real part of the eigenvalue crosses the imaginary axis. So, we can plot the real and imaginary part of the eigenvalue and we can see. So, it crosses the imaginary axis with 0 value. In that case, so we will have the static bifurcation. So, in this case, you can see as we are studying this fixed point response mostly you can find these things.

That means, so if it is going from the left side of the x plane. So, this is known as the x plane. So, if it is going from the left side of the x plane to the right hand side of the x plan, the system changes its stability from stable to unstable. Taking that point at which it is changing from stable to unstable, so one can plot this instability region. Unlike in case of this Mathieu Hill equation, so where we use to get a curve like this we got a curve like this thing, but in this case you can observe.

So, there are multiple, so if you take a line here. So, you can see there is a multiple stable and unstable region. So, up to this it is stable, then this becomes unstable, and then outside it has come outside of this curve so then this become stable, and then unstable, then stable. So, this is stable; this is stable. And in between it is unstable, and then finally, it becomes stable.

So, you can get. So, due to the presence of this internal resonance, you can observe multiple stable and unstable region. So that means, in the trivial state so you can find so multiple bifurcation points which will be the root of the non-trivial state. So, you can find, so there are multiple points which will give rise to nontrivial state.

After knowing this instability region, one can plot the so you just see. So, this part is due to this internal resonance condition. So, now, one can plots plot the frequency response.

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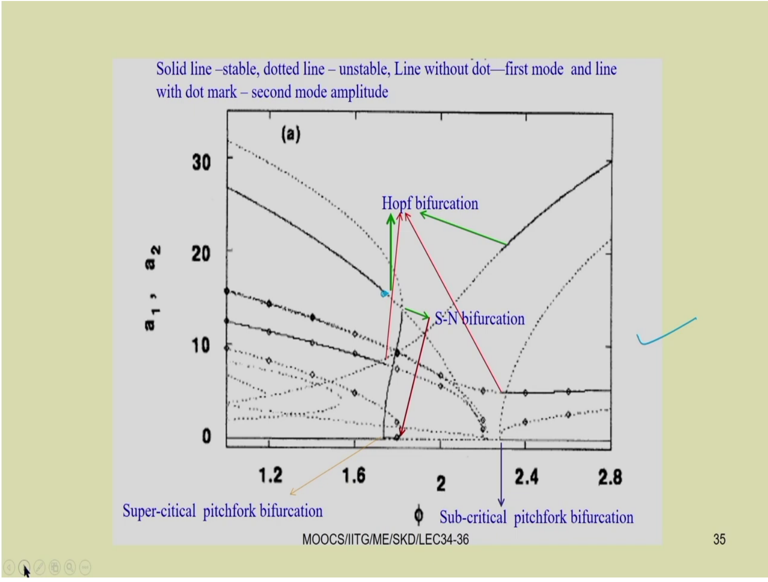

So, this is a typical frequency response plot. Here you can observed that there are several type of bifurcation points. So, clearly the bifurcation point. So, here is the saddle node bifurcation point, then we have a Hopf bifurcation point here.

So, this point is a Hopf bifurcation point. And here you can note that if you decrease this or if you increase this phi value, so here it will give rise to periodic response. There are another Hopf bifurcation points also you have seen.

So, here you have multiple Hopf bifurcation points in this case. And we have different saddle node bifurcation point, Hopf bifurcation point, then we have supercritical pitchfork bifurcation point,and subcritical pitchfork bifurcation point.

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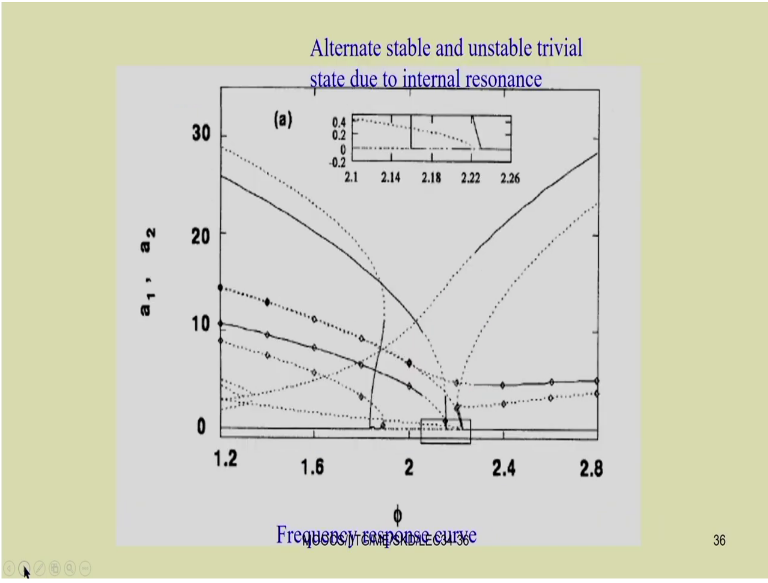

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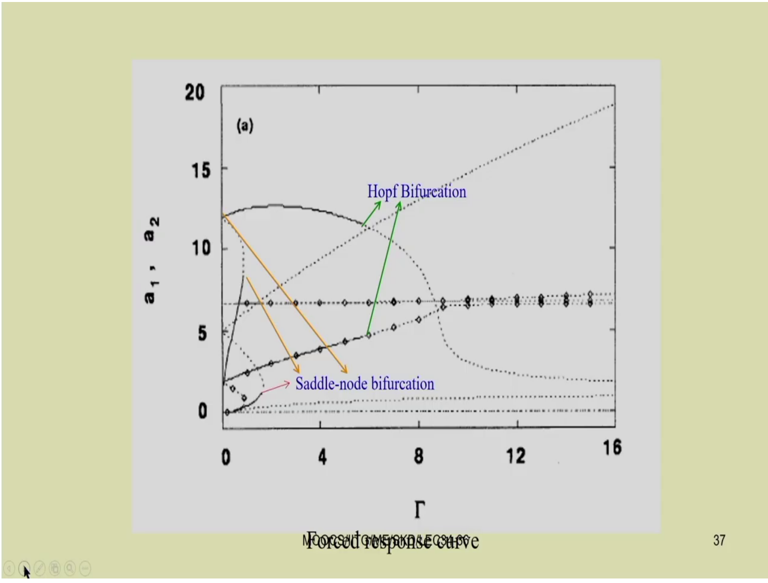

Similarly, here we can plot the force response plot. After plotting this frequency response plot, we can plot the force response plot also. So, here you just see in case of the force response, this gamma is the amplitude of the forcing. So, here you just see several type of bifurcation points.

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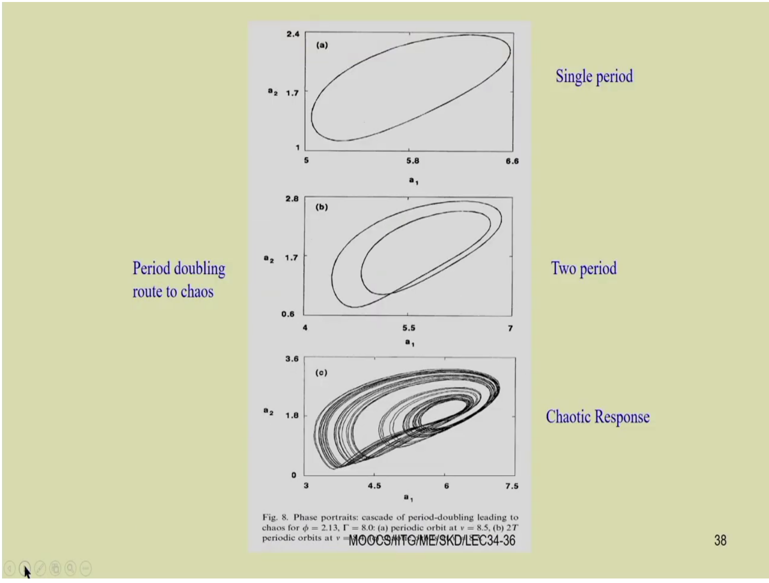

So, several bifurcation point. So, you have the saddle node bifurcation point, then this Hopf bifurcation point. Already we studied or we know so during this Hopf bifurcation point, so we will have the periodic response. So, you just see periodic response. So, this is a period doubling route to chaos. So, initially, we have a periodic response.

So, this period now it becomes two period by changing the system parameter, and finally, it goes to a chaotic response. So, we will define a chaotic response later. So, but you note that this chaotic response has a, so this is a deterministic response. So, it is not like this random response.

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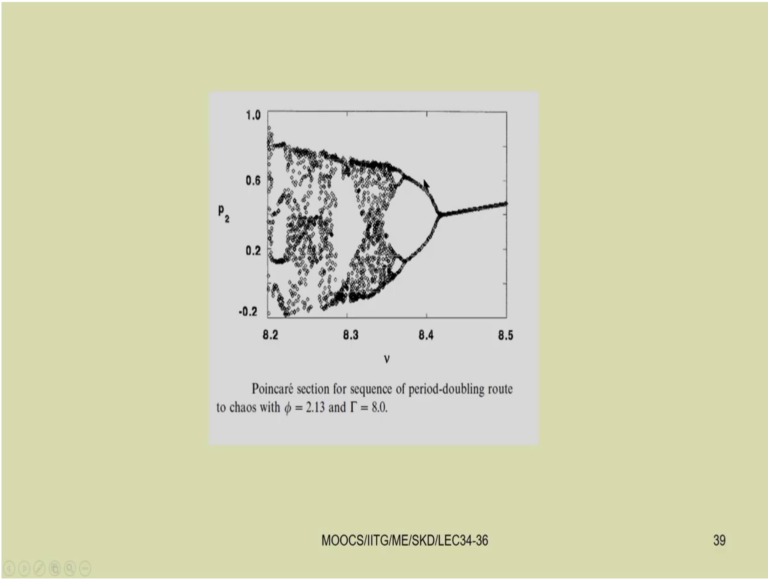

So, this is a deterministic response. So, here one can draw the Poincare section. So, later I will tell how you want to you can you can find the Poincare section. So, if you draw the Poincare section, so you can see, so initially you have one point. So, this gives rise to 2, then this gives rise to 4, and then 8, then 16, so that way it will go on increasing, and finally, the response will be chaotic.

But in between you can have some window so where the response is periodical. So, in between the chaotic response, so you have some windows where the response is periodic.

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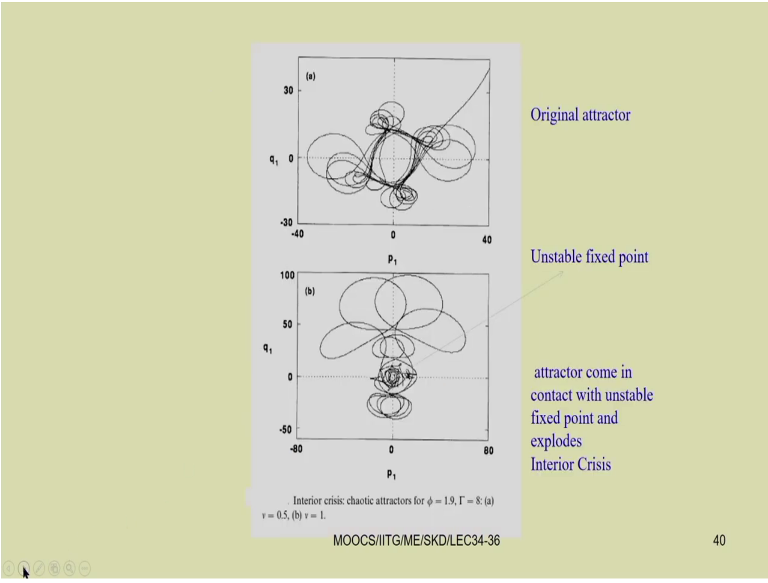

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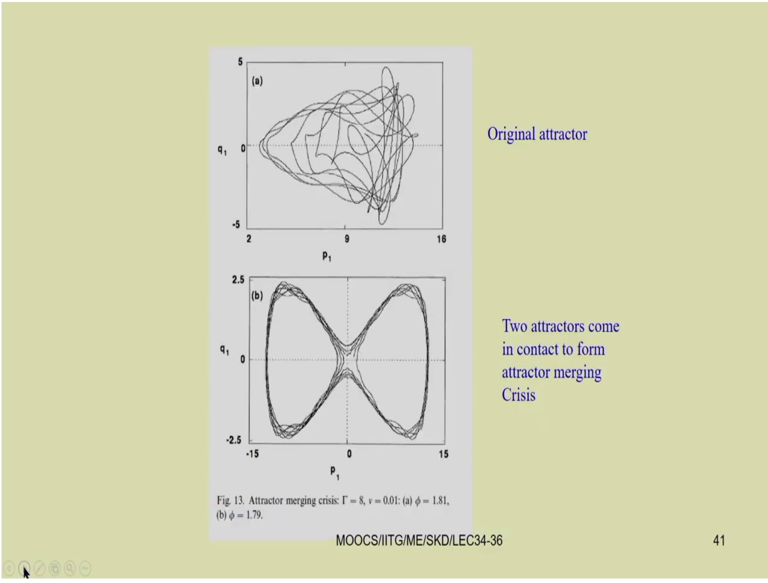

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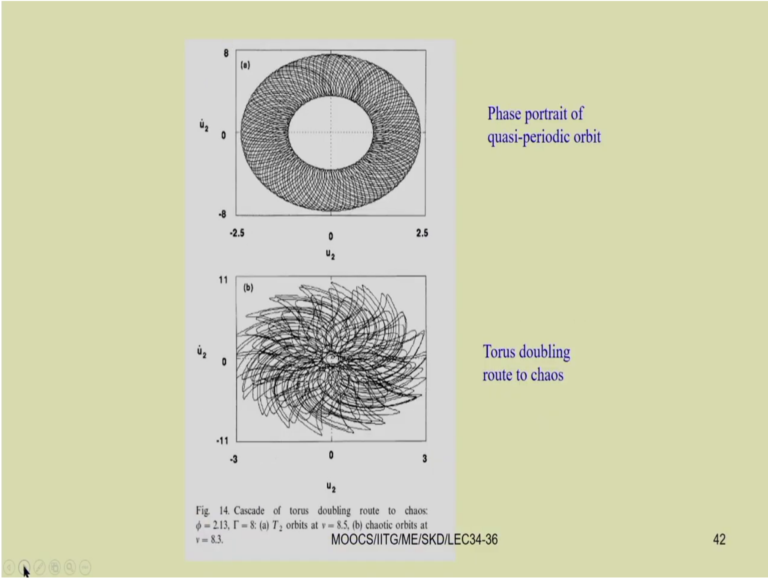

Several other chaotic responses are there also, and this is a quasi-periodic response this phase portrait of a quasi-periodic response. So, next class, we are going to study regarding this more regarding this quasi-periodic and periodic chaotic response also.

You have seen in case of the parametrically excited or base excited system, so you have several different type of responses that is fixed point response periodic response, quasi-periodic response and chaotic response.

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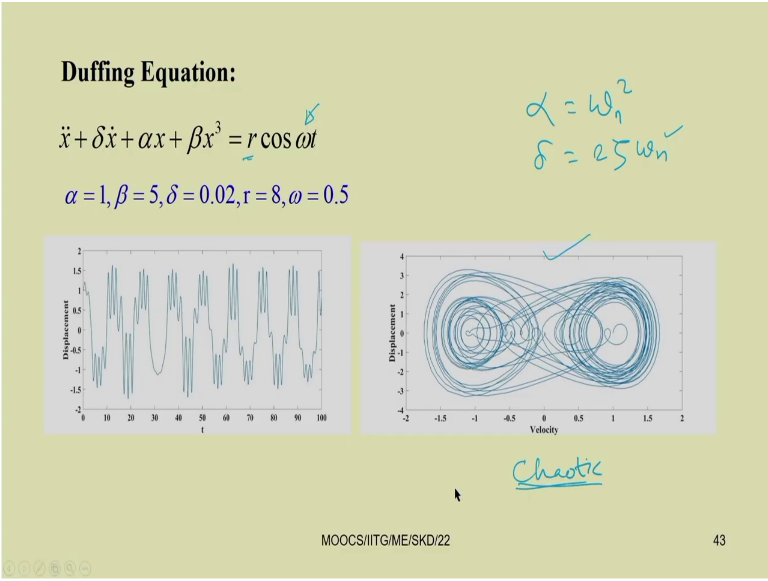

So, if we will take the duffing oscillator also, so you can observe the similar thing. So, we have written a simple code to see how the duffing equation will behave with different coefficient. So, the duffing equation can be written already you know by x double dot plus delta x dot plus alpha x plus beta x cube equal to R cos omega t.

So, here this delta is nothing but omega square generally we take, and then this know this we take this alpha equal to coefficient of x equal to omega n square. So, delta equal to, so this is the coefficient of damping 2 zeta omega n.

So, if you are writing using epsilon also you can write that way; otherwise you can write it equal to 2 zeta omega n where zeta is the damping ratio and omega n is the natural frequency of the system. Then beta x cube that is coefficient of the cubic non-linear term; and r cos omega t r is the this amplitude of the forcing and omega is the frequency of the forcing.

So, by taking different system parameter, one can plot these response for this displacement that is x versus t 1 can plot. So, you just see for alpha equal to 1 that is coefficient of x equal to 1, beta equal to 5, delta equal to 0.02. So, damping very small it is taken. And this r equal to 8, omega equal to 0.5. So, one can easily see the response to be chaotic.

One can observe a chaotic response here. So, this chaotic response is nothing but it contains a number of harmonics greater than 16 you can tell. So, it contains a number of harmonics. And it wonders between so to attractor. So, here it is wondering between this point and this point.

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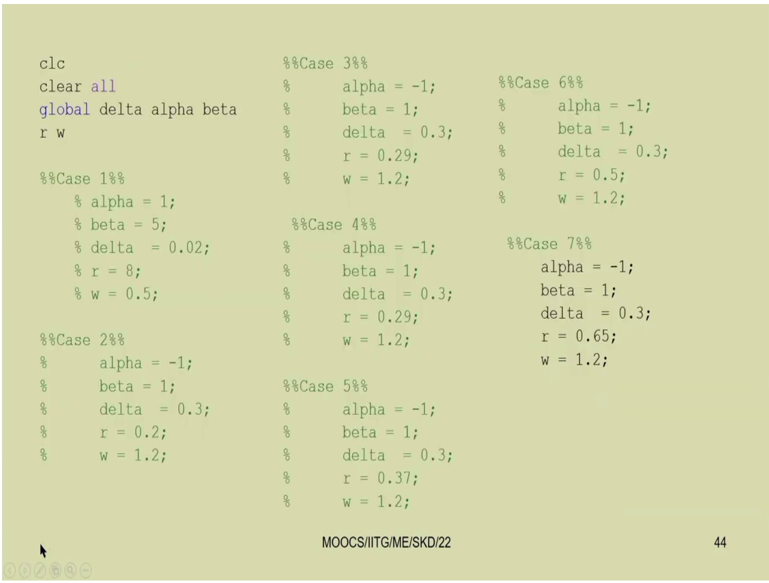

So, you can write a simple code to find these responses for different cases. For example, here, so we have taken 7 case. The first case what just now I have shown. So, here alpha is taken to be 1, beta 5, delta 0.02, r equal to 8, and omega equal to 0.5. Similarly, different cases, 7 different cases have been taken.

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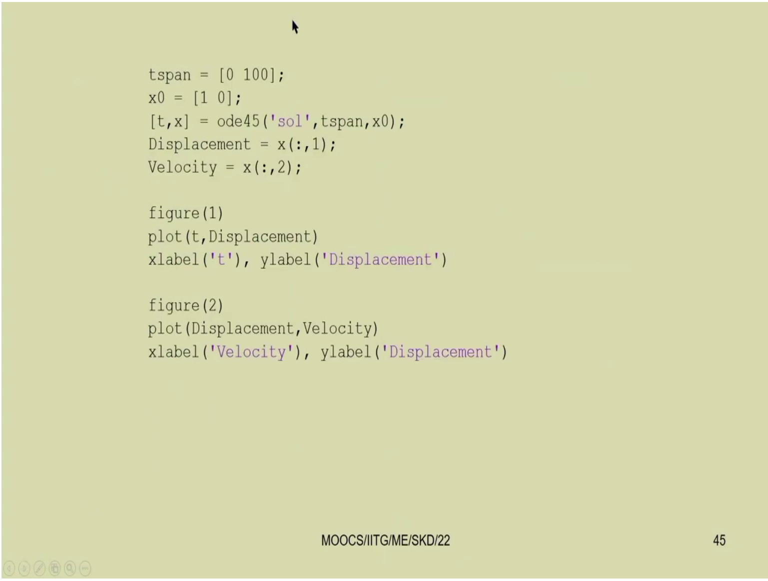

And this is the simple ode45 code you can write. So, you can write this taking initial condition. For example, let us take initial condition x 0 equal to 1 0. So, t, x equal to ode45. So, you can write this equation first order differential equation. So, that is saved in a file sol. So, then this tspan. So, 0 to 100, you have taken. So, and this is x 0 is initial condition.

So, then you can get t and x, then you can plot this displacement versus velocity. So, this x will contain two component that is x 1 and x 2. The first component is this displacement and second component is the velocity. So, you can plot the displacement versus time response that is time versus displacement. You can plot these phase portrait by plotting this velocity and displacement.

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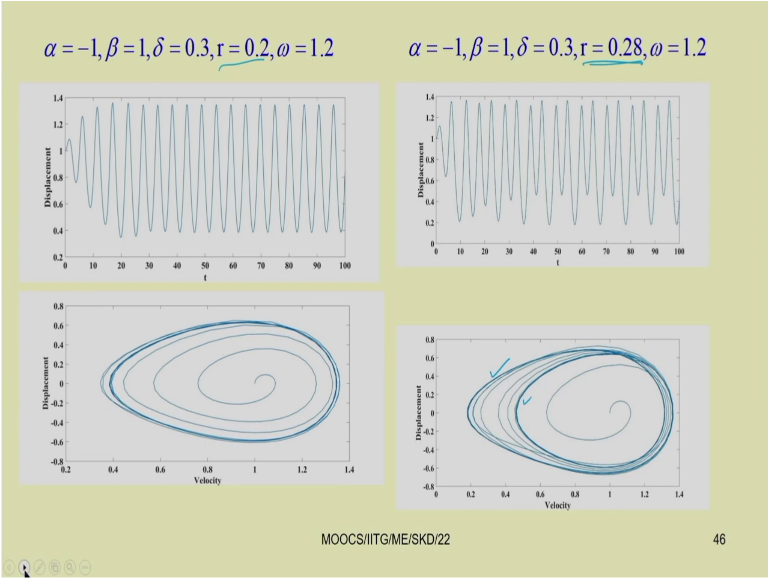

Displacement and velocity, or velocity and displacement it is up to you. So, you can plot you just see by taking this alpha equal to minus 1 that is we have taken what is alpha you just note it again. So, here in this equation, so we have taken alpha is negative, that means, the stiffness term we have taken negative.

If we are not taking this non-linear term, so generally it tends to infinite. So, this will be as the stiffness term is negative, generally the response should have been unstable, but due to the presence of this due to the presence of this cubic non-linear term, so you can see one can get a periodic response.

So, already I told you so this contain both transient and steady state that is why you are getting all these thing. But if you plot only the a study state part, you can have a only a closed curve and clearly you can visualize the response. So, by changing you just see here the parameter r is 0.2, now it is changed to 0.28.

By increasing into 0.28, so you can see so it will have a 2 periodic. So, this is one and this is the other one it is not clearly shown, but you can plot yourself and check verify the steady state part it is two periodic.

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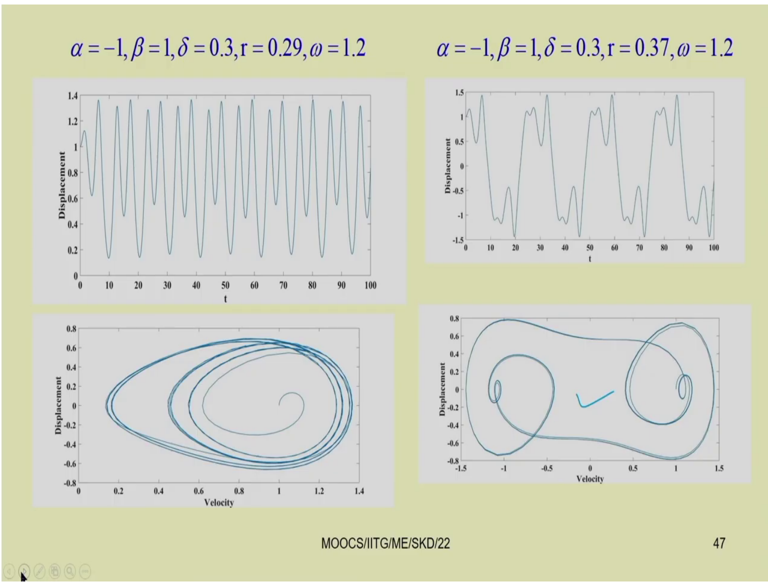

Similarly, by increasing this r equal to 0.29, so this becomes 4 periodic. And then finally, by changing for example, by taking it equal to 0.37, so you can see this is wondering between two attractor here. So, this becomes 8 periodic.

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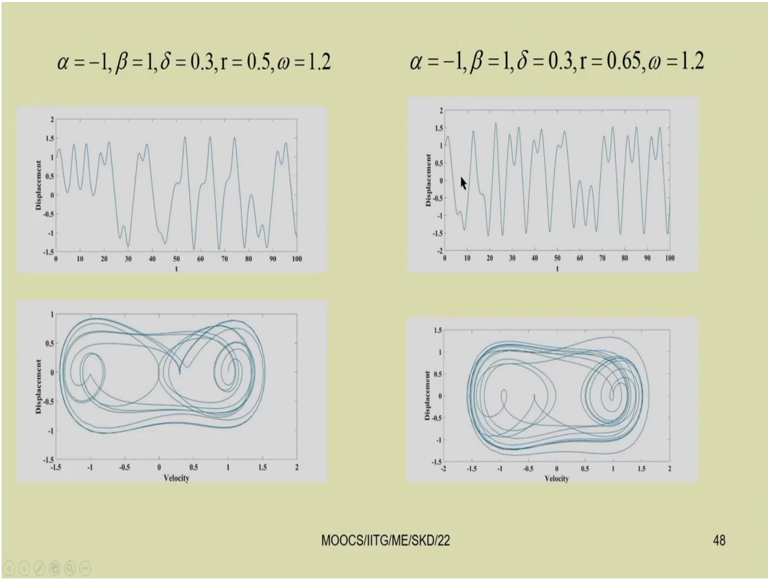

And then you can see finally it becomes chaotic by further changing r equal to 0.5, and r equal to 0.65, so the response becomes chaotic in this way. So, today class you have seen different type of response, so fixed point response, periodic response, quasi-periodic response, and chaotic response. And in the next class, we are going to study the stability of the periodic response, and also we will discuss something related to quasi-periodic and chaotic response.

Thank you.