Nonlinear Vibration

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Lecture - 14
Super and sub harmonic resonance conditions

Welcome to today class of Non-linear Vibration. So, we are continuing with the second lecture of the module 4. So, today class we will discuss regarding different responses what we got from the duffing equation. Also we will study regarding the stability of the obtained response further, we will study how to analyse the system when it is subjected to hard excitation.

So, last class we have studied regarding the weak non-linear forcing term. So, today class we will take the forcing term when it is not weak so; that means, when this forcing term is comparable to that of the linear term. So, in that case how the vibration behaviour will be there so, that part we will see.

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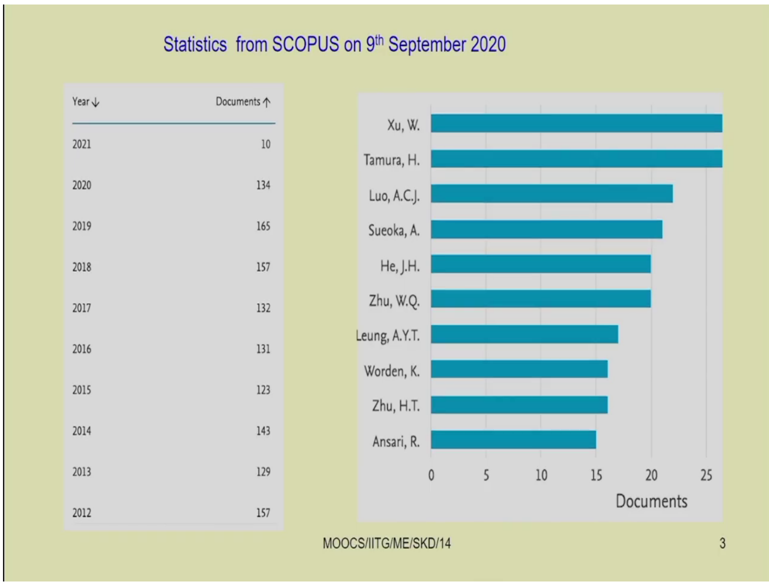

So, last class also we have discussed regarding the number of publications related to these duffing equation in the last 10 years.

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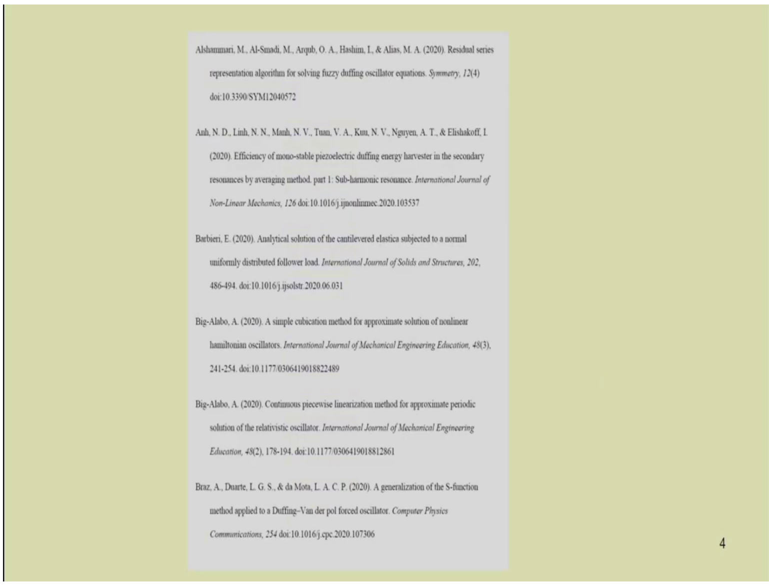

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Cândido, M. R., Llibre, J., & Valls, C. (2020). Non-existence, existence, and uniqueness of limit cycles for a generalization of the van der Pol–Duffing and the Rayleigh–Duffing oscillators. *Physica D: Nonlinear Phenomena, 407* doi:10.1016/j.physd.2020.132458

Cheng, Z., & Yuan, Q. (2020). Damped superlinear duffing equation with strong singularity of repulsive type. *Journal of Fixed Point Theory and Applications, 22*(2) doi:10.1007/s11784-020-0774-z

Demina, M. V., & Sinelshchikov, D. I. (2020). On the integrability of some forced nonlinear oscillators. *International Journal of Non-Linear Mechanics, 121* doi:10.1016/j.ijnonlinmec.2020.103439

El-Borhamy, M. (2020). Chaos transition of the generalized fractional duffing oscillator with a generalized time delayed position feedback. *Nonlinear Dynamics, 101*(4), 2471-2487. doi:10.1007/s11071-020-05840-y

El-Dib, Y. O. (2020). Modified multiple scale technique for the stability of the fractional delayed nonlinear oscillator. *Pramana - Journal of Physics, 94*(1) doi:10.1007/s12043-020-1930-0

Eze, S. C. (2020). Analysis of fractional duffing oscillator. *Revista Mexicana De Fisica, 66*(2), 187-191. doi:10.31349/RevMexFis.66.187

Fu, Y., Hu, M., & Li, Y. (2020). FPGA implementation for a chaotic digital receiver using duffing oscillators array. Paper presented at the *ACM International Conference Proceeding Series*, 341-346. doi:10.1145/3408127.3408163 Retrieved from www.scopus.com

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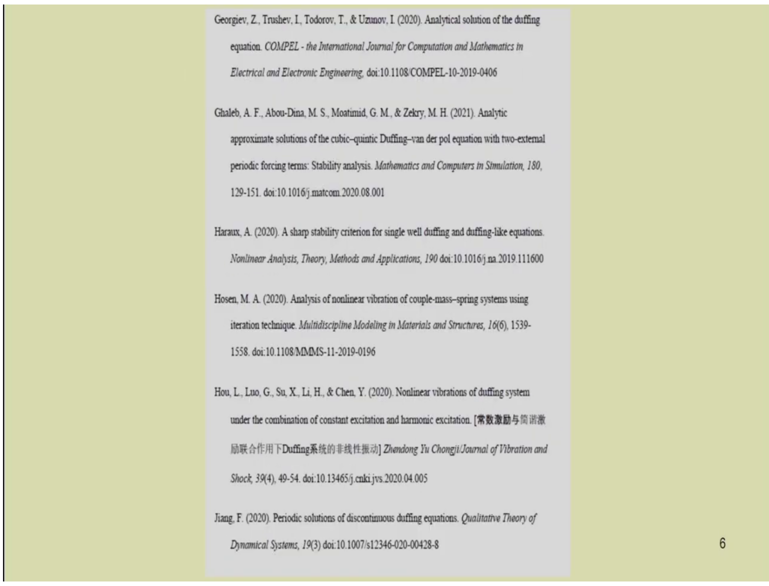

And we have seen so, many different ways. So, these analysis can be done for the duffing equation and the last 2 years publications.

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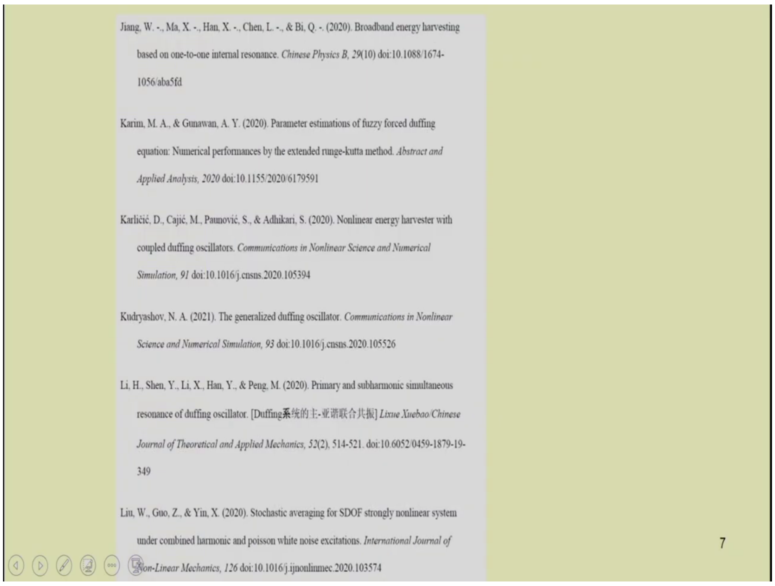

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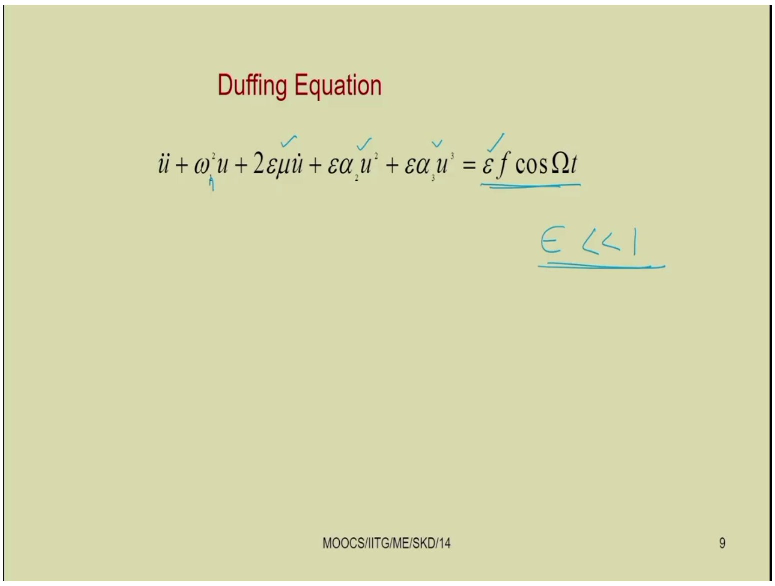

So, the duffing equation which can be written in this form that is u double dot plus omega 1 square u or omega n square u also you can write omega n is the natural frequency or in some paper you can find that is omega 0 square also plus 2 epsilon mu u dot plus epsilon alpha 2 u square plus epsilon alpha 3 u q equal to epsilon f cos omega t. So, if the forcing term is 0.

So, then it is equivalent to free vibration and in case we are taking the forcing term. And if the coefficient is written in terms of epsilon then this is a weak forcing. So, here epsilon is the bookkeeping parameter which is always considered to be less than very very less than 1. So, the damping term is a weak term these quadratic and cubic non-linear terms are also weak.

So, that is why we can use the method of multiple scale to obtain the response of the system. Also you can use directly these numerical techniques for example, this Runge Kutta method to obtain the response of the system. So, in that case so, you have to find the first order equations. So, two first order equation for the second order equation.

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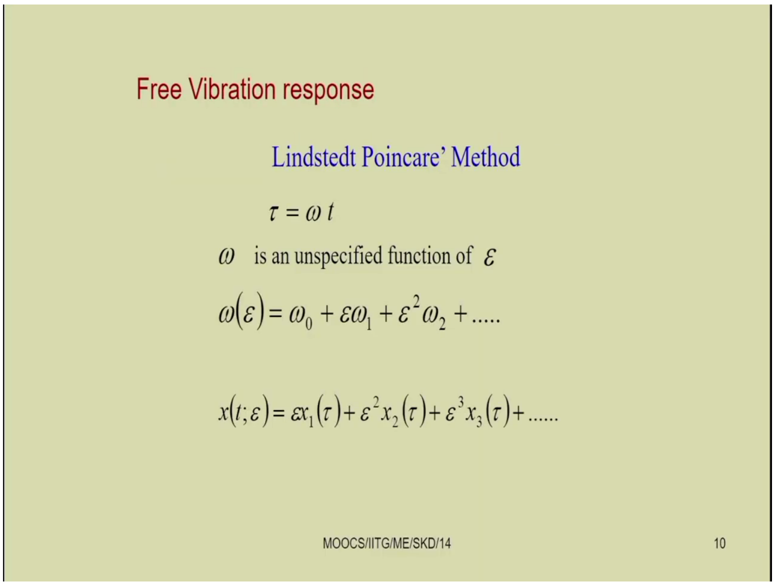

And by using those two first order equation and by using this by using this Runge Kutta method you can solve it.

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$$\frac{d^2x}{dt^2}+\sum_{n=1}^{N}\alpha_n x^n = 0 \qquad \alpha_1 = \omega_0^2$$

$$\left(\omega_0+\varepsilon\omega_1+\varepsilon^2\omega_2+......\right)^2\frac{d^2}{d\tau^2}\left(\varepsilon x_1+\varepsilon^2 x_2+\varepsilon^3 x_3+...\right)+$$
$$\sum_{n=1}^{N}\alpha_n\left(\varepsilon x_1+\varepsilon^2 x_2+\varepsilon^3 x_3+...\right)^n = 0$$

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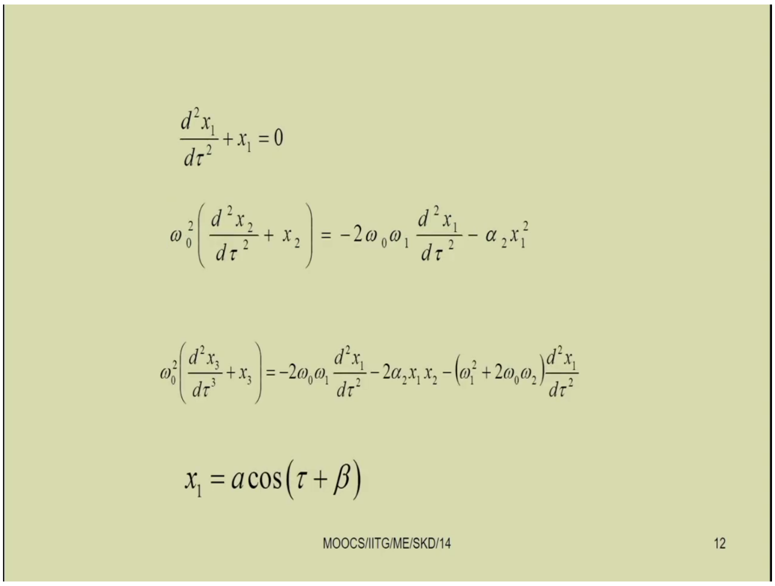

So, previously we have used this Lindstedt Poincare technique to find the free vibration response of the system.

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$$\omega_0^2\left(\frac{d^2x_2}{d\tau^2}+x_2\right)=2\omega_0\omega_1 a\cos(\tau+\beta)-\frac{1}{2}\alpha_2 a^2\left[1+\cos 2(\tau+\beta)\right]$$

To eliminate secular term $\omega_1 = 0$

$$x_2=-\frac{\alpha_2 a^2}{2\omega_0^2}\left[1-\frac{1}{3}\cos 2(\tau+\beta)\right]$$

$$\omega_0^2\left(\frac{d^2x_3}{d\tau^3}+x_3\right)=2\left(\omega_0\omega_2 a-\frac{3}{8}\alpha_3 a^3+\frac{5}{12}\frac{\alpha_2^2 a^3}{\omega_0^2}\right)\cos(\tau+\beta)$$

$$-\frac{1}{4}\left(\frac{2\alpha_2^2}{3\omega_0^2}+\alpha_3\right)a^3$$

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To eliminate the secular term from x_3 we must put

$$\omega_2 = \frac{\left(9\alpha_3\omega_0^2 - 10\alpha_2^2\right)a^2}{24\omega_0^3}$$

$$x = \varepsilon a\cos\left(\omega t + \beta\right) - \frac{\varepsilon^2 a^2 \alpha_2}{2\alpha_1}\left[1 - \frac{1}{3}\cos\left(2\omega t + 2\beta\right)\right] + O\left(\varepsilon^3\right)$$

$$\omega = \sqrt{\alpha_1}\left[1 + \frac{9\alpha_3\alpha_1 - 10\alpha_2^2}{24\alpha_1^2}\varepsilon^2 a^2\right] + O\left(\varepsilon^3\right)$$

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So, in that case we got the relation between these. So, you can see the relation between these forcing the relation between these omega and so, this root over alpha 1; alpha 1 is nothing, but omega 0 square. So, this omega equal to omega 0 into 1 plus 9 alpha 3 alpha 1 minus 10 alpha 2 square by 24 alpha 1 square epsilon square a square.

So, here these omega and a are related; that means, these frequency of the response depend on the amplitude of the response, frequency of the response depend on these amplitude of the response.

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$$\ddot{u}+u+0.1x^3=0 \qquad x=0.001\,\text{m and } \dot{x}=0.1\,\text{m/s.}$$

Solution: Here $\omega_0^2=1, \alpha_2=0, \alpha_3=1$ and ε=0.1

Substituting these parameters in equation (3.2.15),

$$\omega=\omega_0\left[1+\frac{9\alpha_3\omega_0^2-10\alpha_2^2}{24\omega_0^4}\varepsilon^2a^2\right]=1\left[1+\frac{9-10\times 0}{24}(0.1)^2a^2\right]=\left[1+\frac{3}{800}a^2\right]$$

Also, $x=\varepsilon a\cos(\omega t+\beta)-\dfrac{\varepsilon^2a^2\alpha_2}{2\omega_0^2}\left[1-\dfrac{1}{3}\cos(2\omega t+2\beta)\right]+O(\varepsilon^3)$

Now from initial condition

$$0.001=0.1a\cos\beta-\left(\frac{0.01a^2\times 0}{2}\right)\left[1-\frac{1}{3}\cos 2\beta\right]=0.1a\cos\beta$$

$$0.1=-0.1a\omega\sin\beta-\left(\frac{0.01a^2\omega\times 0}{3}\right)\sin 2\beta=-0.1a\omega\sin\beta$$

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$$a^2 = \frac{1}{0.01}\left(0.001^2 + \frac{0.001}{\omega^2}\right) = 0.0001 + \frac{0.1}{\omega^2}$$

$$\omega = \left[1 + \frac{3}{800}a^2\right] = \left[1 + \frac{3}{800}\left(0.0001 + \frac{0.1}{\omega^2}\right)\right] = 1 + 3e-7 + \frac{3}{8000\omega^2}$$

$$or, \omega - \frac{3}{8000\omega^2} = 1.0000003$$

$$or, 8000\omega^3 - 8000.0024\omega^2 - 3 = 0$$

ω=1.0004. The other two roots are complex numbers.

So, $a = 0.3266$

$$\tan\beta = -\frac{0.1}{0.01\omega} = -\frac{10}{\omega}$$

β = -1.4707.

So, $x = 0.03226\cos\left(1.004t - 1.4707\right)$.

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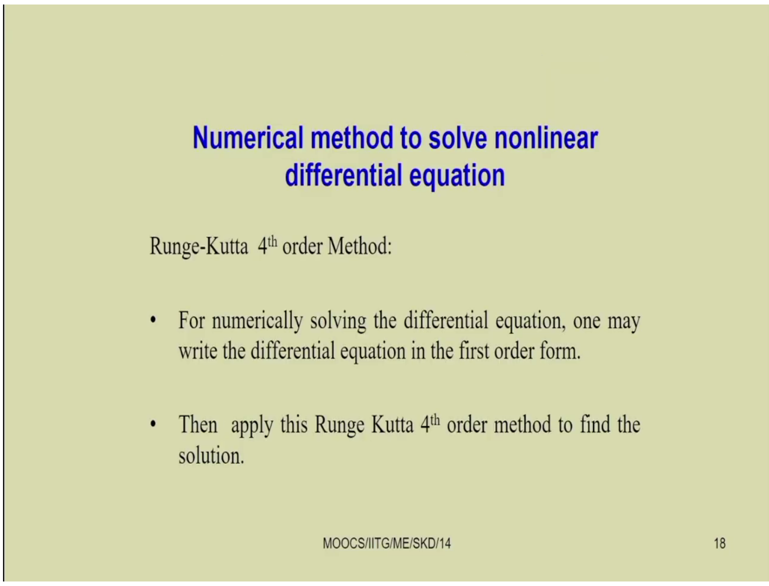

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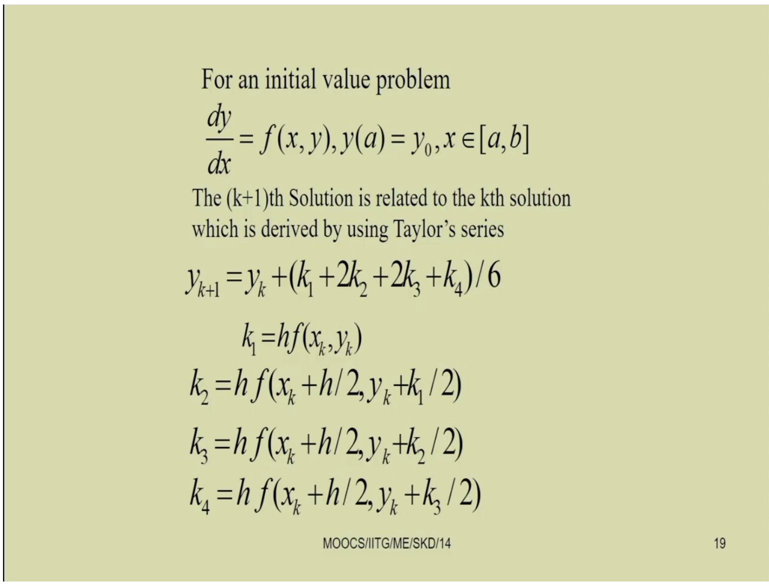

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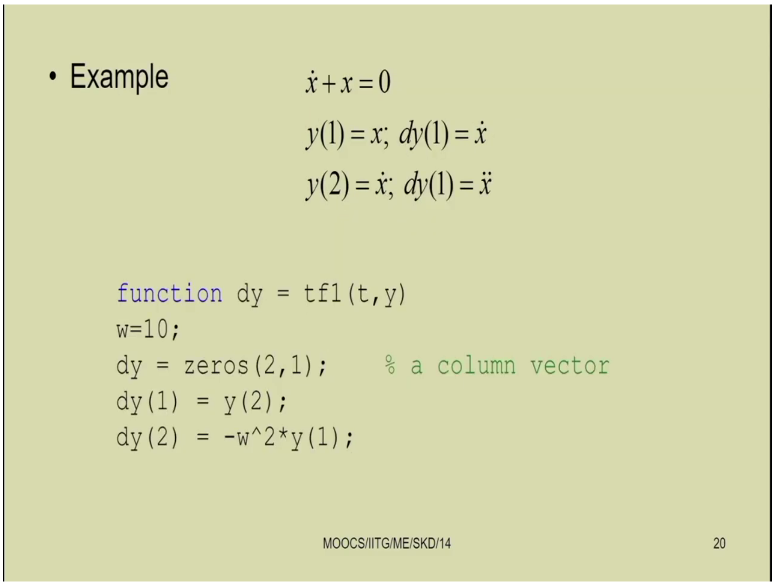

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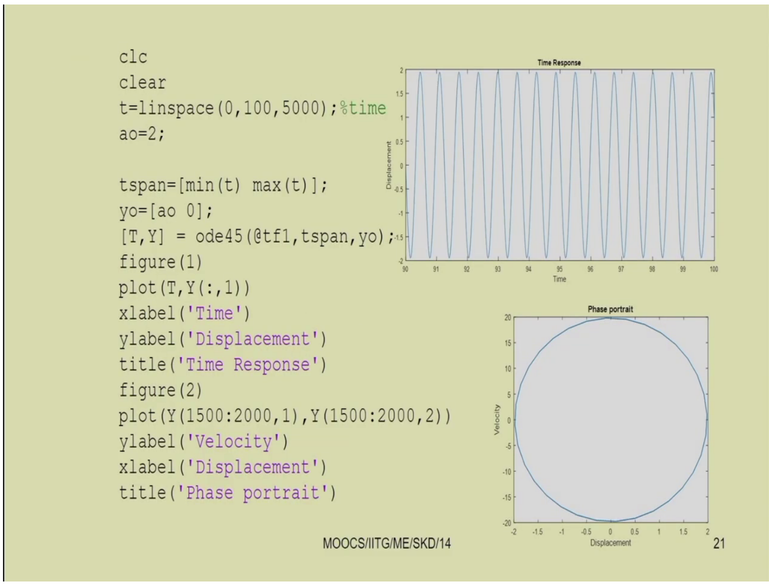

Similarly, so, already we have discussed regarding this Runge Kutta method also and we know how to write a programme to find the solution also regarding the stability briefly we have discussed and how to find these Jacobian also.

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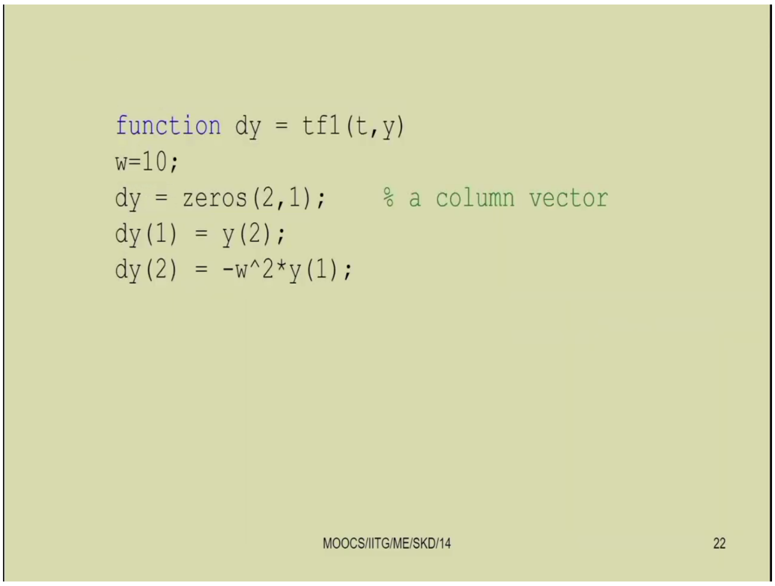

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$$\frac{d^2x}{dt} + f(x) = 0$$

$$\frac{dx}{dt} = y; \quad \frac{dy}{dt} = -f(x)$$

$$\frac{dx}{dt} = ax + b; \quad \frac{dy}{dt} = cx + d$$

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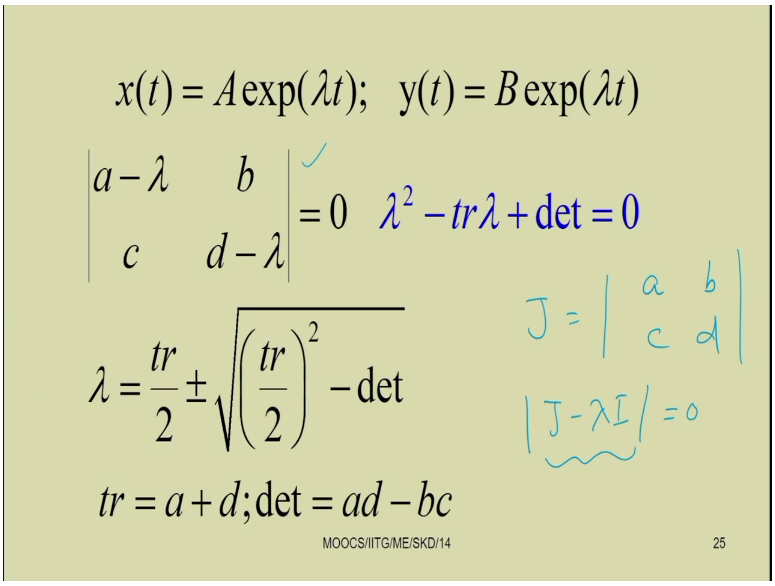

So, today I will tell you. So, what we have studied here. So, that is known as the Liapunov stability criteria. So, regarding that thing also we will see today. So, if you have a equation for example, d square x by dt square plus f x equal to 0. So, you can write this equation by using two first order equation.

So, that is dx by dt equal to y and the second equation equal to dy by dt equal to minus fx. Or these equations in linear form if you want to write you can write this way, that is dx by dt equal to ax plus b and dy by dt equal to cx plus d.

So, in matrix form if you can so by substituting these x t equal to A e to the power lambda t and yt equal to B e to the power lambda t.

So, you can write that thing in this form where you can find in this form that is Jacobian can be Jacobian matrix J can be equal to. So, in this form it can be written a b c and d and to find the eigenvalue of the Jacobian matrix J minus lambda I determinant of J minus lambda I equal to 0 determinant of J minus lambda I equal to 0.

So, this determinant of J minus lambda I is can be written in this form that is a minus lambda b c d minus lambda equal to 0. So, that thing can be written. So, if you take the determinant it can be written equal to lambda square minus trace of lambda plus determinant equal to 0; where lambda is the eigenvalue of the system. So, as you are two you have two equations.

So, you can get two eigenvalues. So, these eigenvalues can be written lambda equal to trace by 2 plus minus root over trace by 2 whole square minus determinant or if you take this half out then it will be trace square minus 4 determinant. So, depending on the trace square minus 4 into determinant whether it is positive negative.

So, you can have these either the imaginary roots or the real root and getting these roots then only you can decide whether the system is stable or not. So, here trace equal to. So, trace equal to a plus d and so, the diagonal. So, u are the diagonal term. So, that will give you the trace and the determinant equal to ad minus bc.

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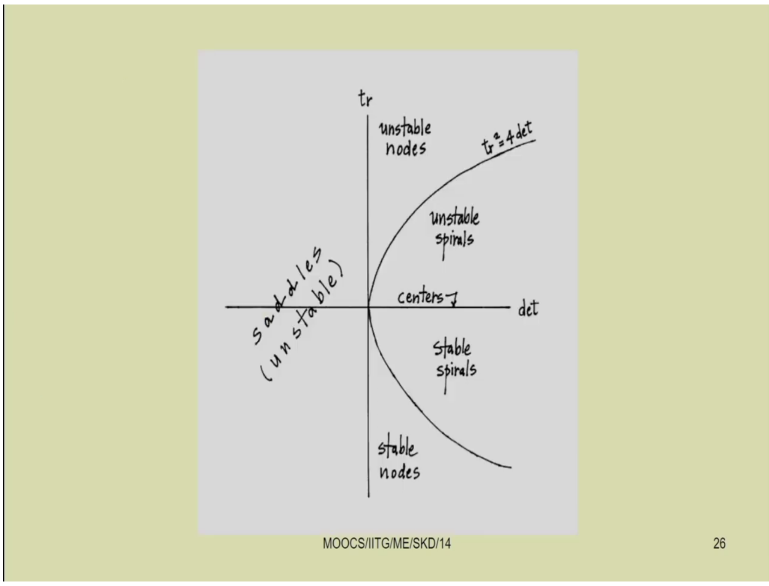

So, if you plot these trace the determinant versus trace. So, you can define the system to be stable or unstable and these stable unstable also can be defined in terms of. So, different type of responses for example so it can be. So, if it is in the left hand side in the that is in third and fourth quadrant. So, then it will be saddle that is unstable. So, already I told you.

So, it will be unstable when the real part of the eigenvalue becomes positive. So, if the real part of the eigenvalue becomes negative also. So, we can have this also we can see the unstable nodes, unstable spirals, center and stable spiral stable nodes. So, slowly we will understand all these terminology while we are discussing regarding different types of response of the system ok.

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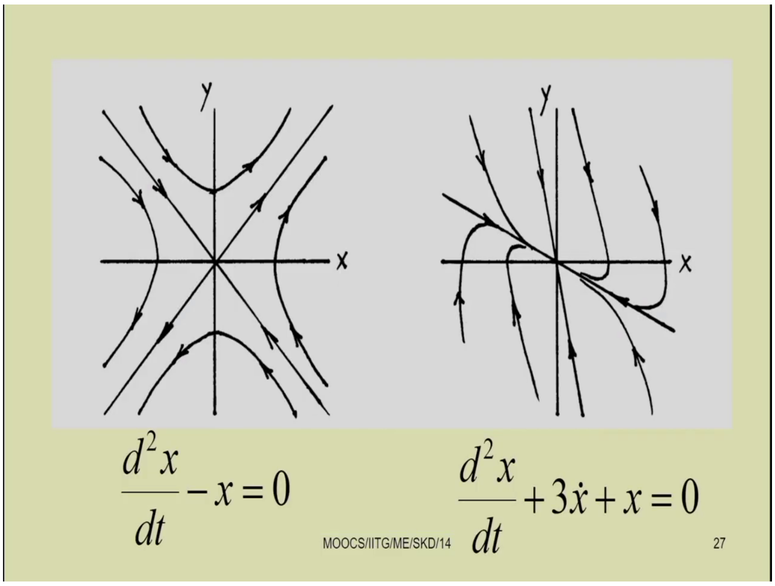

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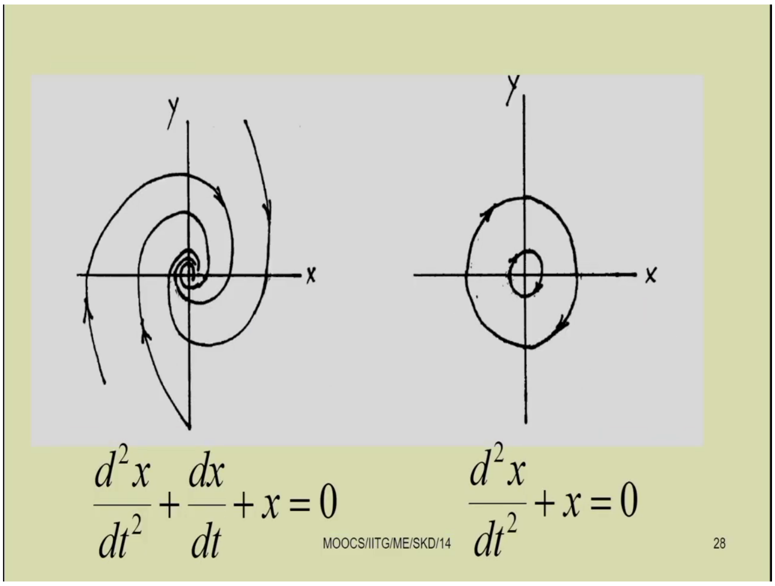

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```
%Duffing Equation : Straight forward expansion
% Duffing Equation
%x_tt+w^2x+alpha x^3=0;

syms x x0 ep x1 t wn alpha
x=x0+ep*x1

dEQ=diff(x,2)+wn^2*x+alpha*x^3
yy=expand(dEQ)
collect(yy,'ep')
```


So, we have discussed all these parts yesterday or in the last class.

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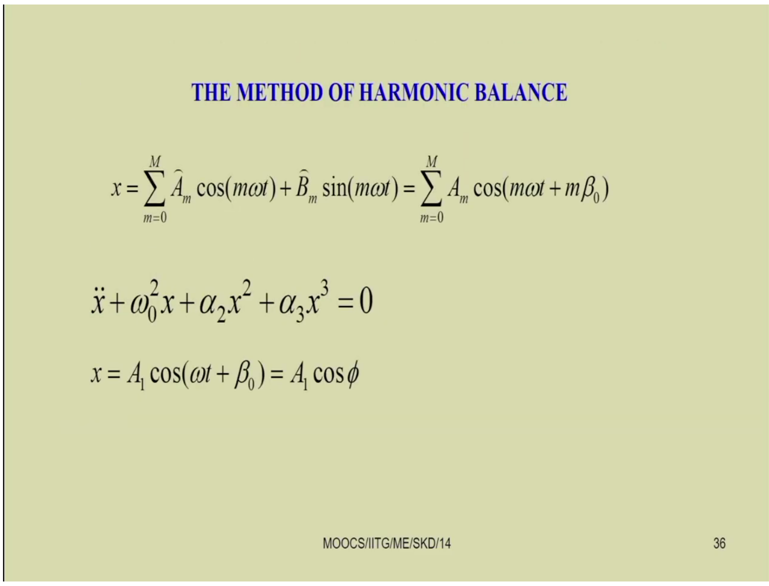

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$$-(\omega^2-\omega_0^2)A_1\cos\phi+\frac{1}{2}\alpha_2A_1^2\left[1+\cos2\phi\right]+\frac{1}{4}\alpha_3A_1^3\left[3\cos\phi+\cos3\phi\right]=0$$

$$\omega^2=\omega_0^2+\frac{3}{4}\alpha_3A_1^2$$

$$\omega=\left[\omega_0^2+\frac{3}{4}\alpha_3A_1^2\right]^{1/2}\simeq\omega_0\left[1+\frac{3\alpha_3}{8\omega_0^2}A_1^2\right]$$

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So, and by using method of multiple scales. So, we have seen the response.

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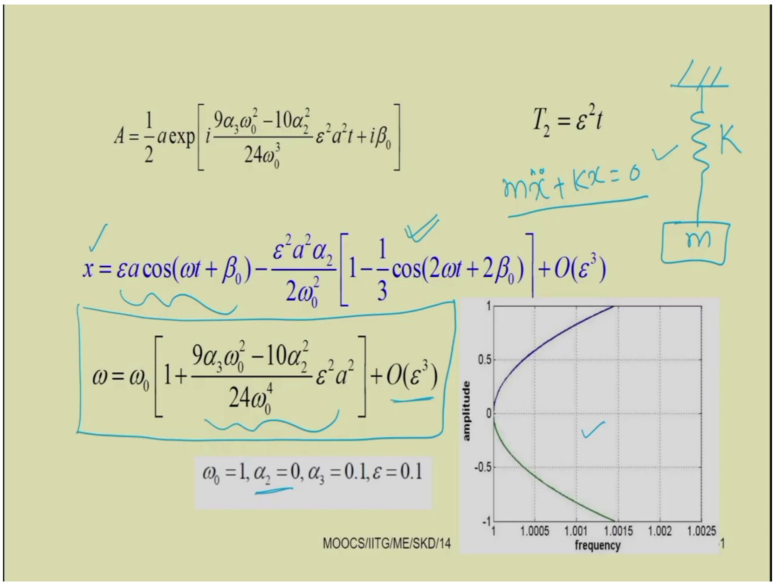

So, you can see. So, we got this relation that is either by using these Lindstedt Poincare technique or by using method of multiple scale. So, we got the relation between omega and a that is omega equal to omega 0. So, there will Lindstedt Poincare method we have written omega equal to root over that is alpha 1.

So, that root over alpha 1 is nothing, but this omega 0. So, the relation between. So, in case of free vibration. So, the relation between these 2 equal to omega equal to omega 0 into 1 plus 9 alpha 3 omega 0 square minus 10 alpha 2 square epsilon square a square by 24 omega 0 fourth. So, we have neglected the order of epsilon cube.

So, by taking some numerical value for example, by taking omega 0 equal to 1 alpha 2 equal to 0 that is there is no cube quadratic non-linearity, then alpha 3 equal to 0.1 and epsilon equal to 0.1. So, if you plot this amplitude frequency versus amplitude. So, you can see this relation. So, this shows the frequency versus amplitude; that means, by for a different frequency for a different value of frequency we can get different value of amplitude.

So, this way one can study free vibration response also one can find the response x can be written in this form that is epsilon a cos omega t plus beta 0 minus epsilon square a square alpha 2 by 2 omega 0 square into 1 minus one third cos 2 omega t plus 2 beta 0. So, this higher order terms we have neglected.

So, if you recall the free vibration of a spring mass simple spring mass system. So, if you recall the response of a simple spring mass system. See this is mass this is constant K where the equation is written in this form that is mx double dot plus Kx equal to 0. So, if you are adding this quadratic non-linearity then it becomes plus K 1 x cube because this quadratic non-linearity we have taken alpha 2 equal to 0.

So, in that case it will be this quadratic non-linearity 0. So, the response of this system can be only the first term. So, that is a cos omega t plus beta 0 either a cos omega t or a sin omega t plus beta 0 where a and beta 0 are constants which can be obtained from the initial conditions. So, this is the additional term we are getting.

So, when we are considering the non-linearity in the system for example, you have taken only quadratic cubic nonlinearity in this case. So, in that case this alpha 2 will be equal to 0. So, this term will not be there also, but this a is a function this omega is a function of a. So, no longer it is constant, but in case of the simple spring mass system where mk.

So, we have for constant value of a and k. So, we got this omega does not depend on a, but in this case. So, for a different value of omega we have different value of a and we can find this response using this expression.

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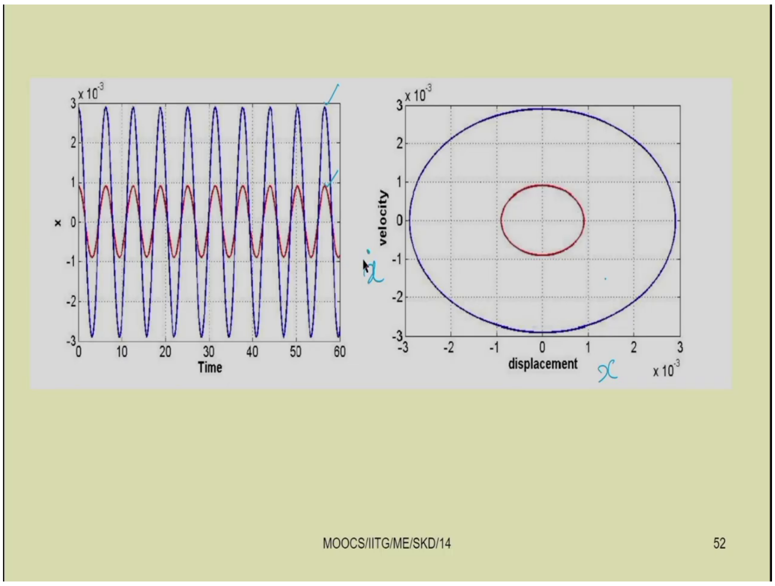

So, in that case. So, for different initial conditions. So, by taking different initial conditions. So, if we plot these this is known as time response. So, time versus this x is the time response. So, this is for one initial condition and this is for another initial condition. So, by starting a with some initial condition. So, we can plot this curves.

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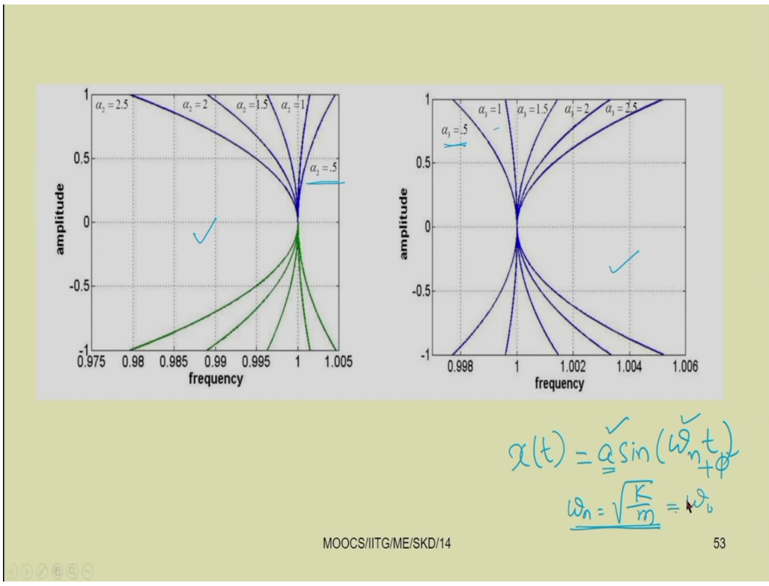

So, this is known as the phase portrait. So, in this phase portrait you just see both are the phase portraits. So, here you just see by changing. So, the previous curve what you have seen in the previous curve. So, we have now taken. So, your alpha 2 equal to 0. So, if you take alpha 2 non zeros. So, this is the phase portrait. So, now, by taking different value of alpha 2.

So, that is for example, these quadratic non-linearity if you are taking. So, then you can see by taking different value of alpha 2 the response this amplitude the frequency amplitude relation becomes like this. So, here you can see for small value of alpha 2. So, that is alpha 2 equal to 4 and 5 and alpha by increasing these alpha 2. So, you can see the response. So, different value of amplitude. So, for a particular value of frequency you are always finding 2 values of this amplitude.

So, depending on these initial conditions. So, we can get the response amplitude similarly by taking this alpha 3 equal to. So, alpha 3 by changing this alpha 3. So, for example, we have taken alpha 3 equal to 0.5 here, by taking alpha 3 equal to 0.5 and increasing this value of alpha 3. So, you can see the frequency response curve. So, here different value of amplitude you can get by taking different value of alpha 3.

So, you can see these response amplitude not only depend on this value of omega also it depends on different value of alpha. So, with different value of alpha 2 alpha 3, so we can get different value of amplitude by taking those amplitudes. So, we can plot the response of the system. So, that is x versus time or we can plot the phase portrait that is x versus x dot x. So, this is x versus x dot. So, this way you can study the pre vibration response of the of a non-linear system with quadratic and cubic nonlinearity.

So, in case of linear system the response amplitude can be for example, this x t in case of linear system x t equal to a cos or sin omega n t where these omega n t plus phi also you can write. So, omega n t plus phi where a and omega are constants. So, they are not function of time or they are not function of these omega response amplitude.

But in case of non-linearity. So, they are function of response amplitude. This response amplitude is a function of omega also. So, here in linear case these omega n is a constant term that is it is equal to root over K by m, but in non-linear case it is no longer root over K by m or it is omega 0; what do we have written in this case. So, it is multiplied by. So, this you have seen the relation that is omega equal to omega 0.

So, this is the additional term we have to take in case of the non-linear. So, you just see the first term it is only omega 0. So, in linear case this is only the first term, but in case of the non-linear equation. So, you have omega is the additional term is omega 0 into 9 alpha 3 omega 0 square minus 10 alpha 2 square by 24 omega 0 fourth into epsilon square a square.

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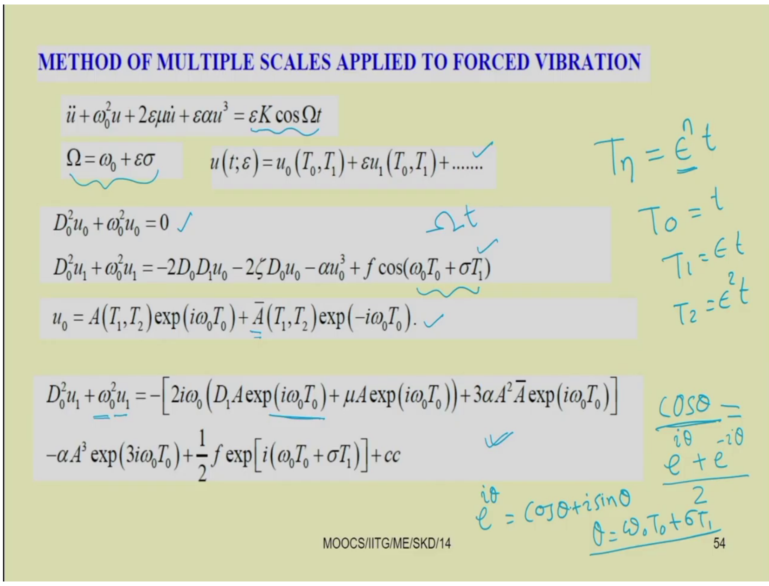

So, this way you can analyse the non-linear three vibration response of the system. So, you have several systems. So, in several systems, so depending on the system parameters so, you can analyse the response amplitude and frequency also. So, now by taking the force vibration.

So, last class also we have discussed how to obtain the response of the system by using this method of multiple scale. Today we will analyse the response of the system the method of multiple scale. Yesterday we have discussed where we are taking the resonance condition omega equal to omega 0 plus epsilon sigma, where sigma is the detuning parameter.

So, we have taken this u that is the response equal to u 0, T 0; T 1 plus epsilon 1, T 0, T 1 already you know in case of method of multiple scales. So, these T n is nothing, but epsilon n t. So, n for n equal to 0. So, you have this T 0 equal to t, T 1 equal to epsilon t and T 2 equal to epsilon square t. So, this way so, these are different time scale similar to these scales we are using in our watch that is the second hand, minute hand and hour hand.

So, these T 0, T 1, T 2 these are the different time scales. So, by taking different time scales.

So, we can write the response using these u 0 and u 1. So, now separating or substituting these equation in the original equation and separating different order of epsilon terms with different order epsilon. So, we can get this D 0 square u 0 plus omega 0 square u 0 equal to 0 and D 0 square u 1 plus omega 0 square u 1 equal to minus 2 D 0 D 1 u u 0 minus 2 zeta d 0 u 0 minus alpha u cube plus f cos omega 0 T 0 plus sigma T 1.

So, you can note that this external frequency here is written in this form the external frequency this is omega. So, already so, this omega t can be written as omega 0; omega 0 plus epsilon sigma T.

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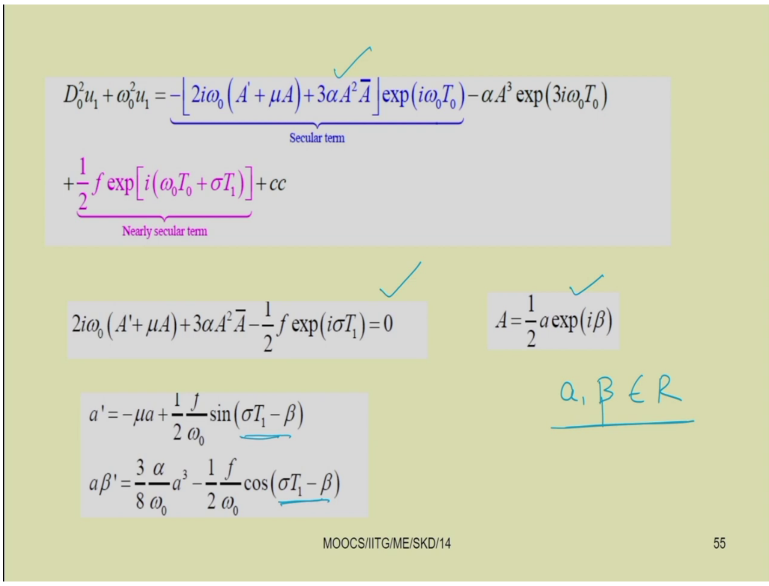

So, as epsilon T equal to T 1. So, that is why it is written in terms of. So, the first term is written in terms of T 0, but the second term is written in terms of T 1. So, here we are putting two different timescales. So, the solution of these one is known. So, that is D 0 square u 0 plus omega 0 square u 0 equal to 0. So, the solution equal to A T 1 T 2. So, a should not be a function of T 0 that is why it is written a function of T 1 T 2.

So, u 0 equal to A T 1 T 2 e to the power i omega 0 T 0 plus A bar T 1 T 2 e to the power minus i omega 0 T 0 here A bar is the complex conjugate of A. So, this A is assumed to be a complex number now substituting this u 0 in the second equation. So, we can get these terms and as we know that system response is bounded we can see some of the terms which will lead to or which will lead the response to infinite.

For example, the terms containing e to the power i omega 0 T 0 as the coefficient of u 1 equal to omega 0 square. So, the term containing i omega 0 T 0 will tends the response to be infinite. So, those are the secular term and we should eliminate those secular terms to find the solution. So, here you can see. So, this is the secular term which is coefficient of e to the power i omega 0 T 0.

So, in this case while deriving these equations. So, either you may derive it using manually you may derive or you may go for these symbolic software to derive these equations.

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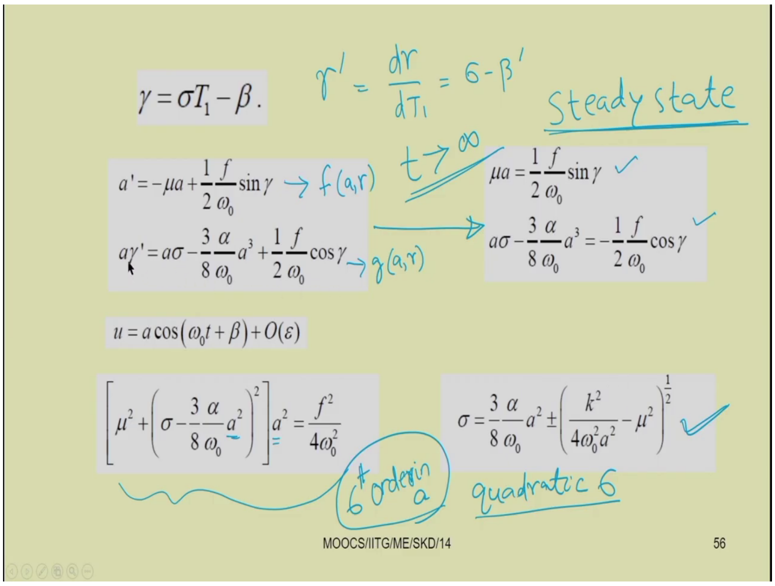

So, in symbolic software or like this you can use Mathematica, Maxima or you may use this MATLAB also to derive these equations or you may use maple also so to write down these equations. So, now by eliminating the secular term. So, you can get these condition that is 2 i omega 0 A dash plus mu A, A dash is nothing, but D a by dt 1.

So, plus mu A plus 3 alpha A square A bar minus half f cos f e to the power i sigma T 1 equal to 0. So, you may note that. So, here in this original equation in the foreseeing. So, we have a cos term. So, this cos term you can write it in this way. So, if you know this cos theta cos. So, how you can write these cos theta or cos omega t?

So, you are familiar with this thing. So, e to the power i theta plus e to the power minus i theta by 2 because e to the power i theta, because e to the power i theta equal to cos theta plus i sin theta. So, plus e to the power minus i theta equal to cos theta minus i sin theta. So, if you add e to the power i theta plus e to the power minus i theta. So, this becomes 2 cos theta. So, to write cos theta. So, cos theta will be equal to e to the power i theta plus e to the power minus i theta by 2.

So, this f cos omega 0 T 0 plus sigma T 1. So, you can write equal to half f e to the power omega 0 T 0 plus sigma T 1 plus half. So, you can this cos term you can write in this form. So, by using e to the power i for theta. So, here theta is nothing, but so theta is omega 0 T 0 plus sigma T 1. So, by substituting this thing so, this forcing term you can write.

So, if your forcing term is written in terms of sin similarly you can modify this equation. So, sin theta from these things sin theta you can get. So, e to the power i theta minus e to the power or e to the power ok e to the power i theta minus e to the power minus i theta by 2 i if you do so then you can get the sin theta term ok.

So, depending on whether you have a sin term or a cosine term in the forcing. So, you can accordingly modify your equation and you can find the corresponding secular term. So, after getting the secular term so you have to substitute as already you know A is a function A is a complex number. So, you can write A equal to half small a e to the power i beta where a and beta both a and beta are function are real number a and beta are real number.

So, A capital A is a complex number. So, the small a and beta are real number. So, by taking these or by substituting A equal to half a e to the power i beta in the term which are secular you can and separating it by real and imaginary terms. So, you can find A dash equal to minus mu A plus half f by omega 0 sin sigma T 1 minus beta ab dash equal to 3 by 8 alpha by omega 0 a cube minus half f by omega 0 cos sigma T 1 minus beta.

Here you just see. So, the right hand side these terms contain the time. So, these are known autonomous systems. So, first we should make it autonomous for simplifying our analysis. So, in that case so by taking the sigma T 1 minus beta as gamma. So, we can take gamma equal to sigma T 1 minus beta. So, you can find these gamma dash equal to. So, gamma dash is d gamma y. So, this is nothing, but a d gamma by d T 1. So, this is equal to sigma minus beta dash where beta dash is nothing, but d beta by d T 1.

So, it is differentiated with the timescale T 1. So, by substituting it in the previous equation. So, you can write down this equation in this form that is a dash equal to minus mu a plus half f by omega 0 sin gamma a gamma dash equal to a sigma minus 3 by 8 alpha by omega 0 a cube plus half f by omega 0 cos gamma.

So, for the steady state for steady state. So, what do you mean by steady state? So, when at steady state. So, the response will not depends on time; that means, time tends to for steady state time tends to infinite. So, when time tends to infinite t tends to infinite. So, these a dash and a gamma dash becomes 0. So, by substituting a dash equal to 0 and gamma dash equal to 0.

So, this equation reduces to mu a equal to half f by omega 0 sin gamma a sigma minus 3 by 8 alpha by omega 0 a cube equal to minus half f by omega 0 cos gamma. Now, by squaring and adding these two terms. So, we can get this equation that is mu square. So, mu is nothing, but your damping parameter. So, mu square plus sigma minus 3 by 8 alpha by omega 0 a square whole square into a square equal to f square by 4 omega square.

So, you can see these equation is sixth order in. So, this is A to the power 6 term you will get. So, a square whole square a to the power 4 outside we have one a square. So, a to the power 6. So, these equation is 6th order in 6th order in a. So, this is 6th order in a, but quadratic in, but quadratic so, but quadratic in alpha that is the detuning parameter or quadratic in sigma this is not alpha this is sigma quadratic in sigma, sigma is the detuning parameter.

So, we can write the quadratic equation and we can find the response or frequency response by using these equations. So, this is the equation for frequency plotting the frequency response plot. So, here so, you can see sigma equal to 3 by 8 alpha by omega 0 a square plus minus k square by 4 omega 0 square a square minus mu square to the power half.

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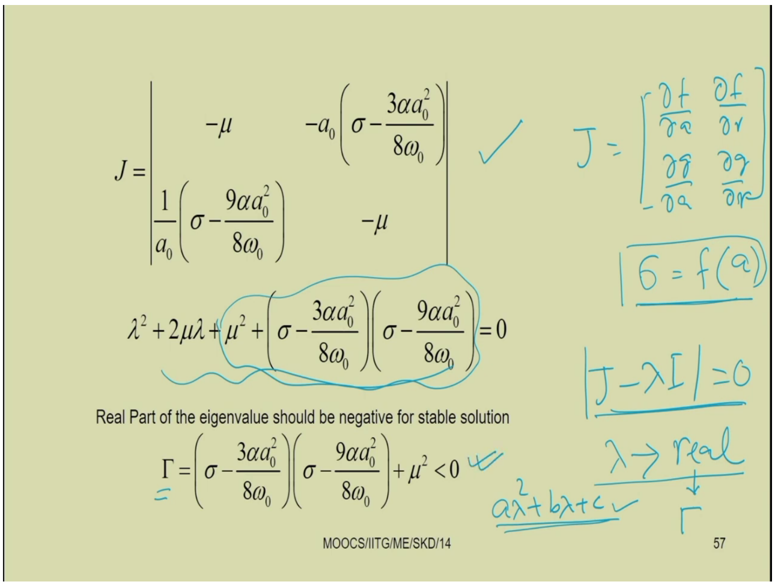

Now, to find the stability. So, we have to find the Jacobian matrix and find the real part of the eigenvalue to find the solution to be to study the solution. So, before that thing. So, let us see some other things.

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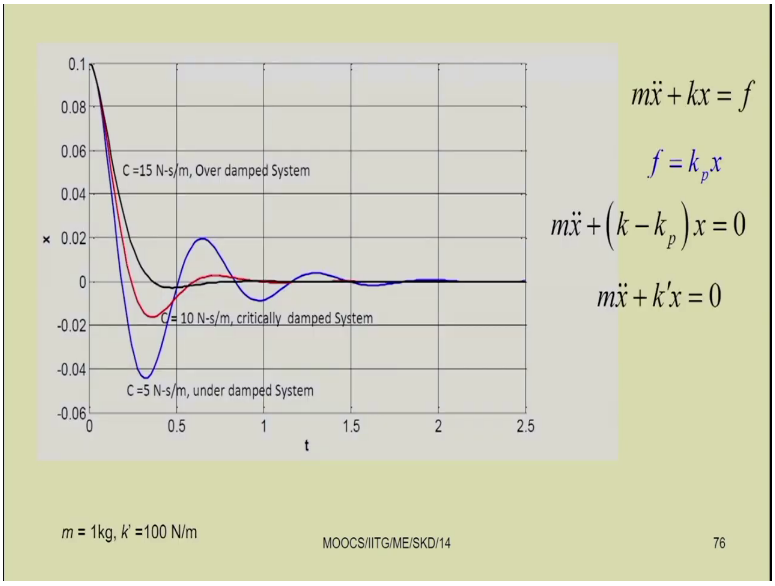

So, let us see what we mean by a stable and unstable system. So, we should first know what a stable system is or what is an unstable system. So, let us take the example of the same simple spring mass system.

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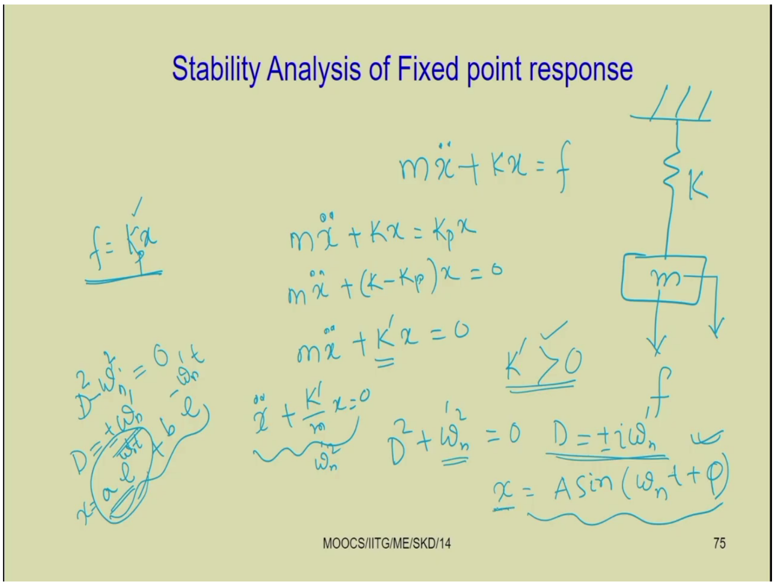

So, this is the spring mass system this is K and this is m and ok the equation becomes mx double dot plus Kx. So, if some force is acting on this thing. So, it is it can be written as f. So, in this case so let us take by taking simply f equal to let us take a f equal to Kx or K P x proportional. So, let us take a proportional controller. So, f equal to K P x.

So, this equation can be written mx double dot plus Kx minus or equal to K P x. So, or it can be written mx double dot plus K minus K minus K P into x equal to 0. So, or it can be written as mx double dot plus K dash x equal to 0. So, here depending on the value of K dash that is if K dash is K dash greater than 0.

So, if K dash greater than 0 the response can be. So, you know already that the auxiliary equation in this case that is or this equation can be written in this form x double dot plus K dash by mx equal to 0. So, here the auxiliary equation is the D square plus. So, this is omega n dash square.

So, omega n dash square. So, this is the auxiliary equation equal to 0. So, the roots of this equation that is D will be equal to plus minus i omega n dash. So, the solution will be equal to. So, the solution x can be written equal to A sin. So, in this case it can be written A sin omega n. So, A sin omega n t plus phi.

So, if so, this is the case if this omega n square is greater than or K dash is greater than 0. So, then omega n dash square. So, this is a positive term so, that we can get the response in this form, but if this K dash is negative. So, if K dash is negative then. So, this equation can be written D square minus. So, in that case the equation becomes D square minus omega n square omega n dash square equal to 0; auxiliary equation can be written this form or D equal to plus minus plus minus omega n dash. So, D equal to plus minus omega n dash.

So, here the solution will be x equal to a e to the power omega n dash t plus b e to the power minus omega n dash t. So, one term; so, this is the first term this is a e to the power omega n dash t as t tends to infinite. So, this term will tends to infinite and this term will second term will tends to 0.

So, due to the presence of this term. So, due to the presence of this term the response amplitude will grow. So, the system becomes unstable, but in the first case the system will have a bounded solution. So, that is as this sin omega n t plus phi will vary between plus minus 1. So, then this x is bounded.

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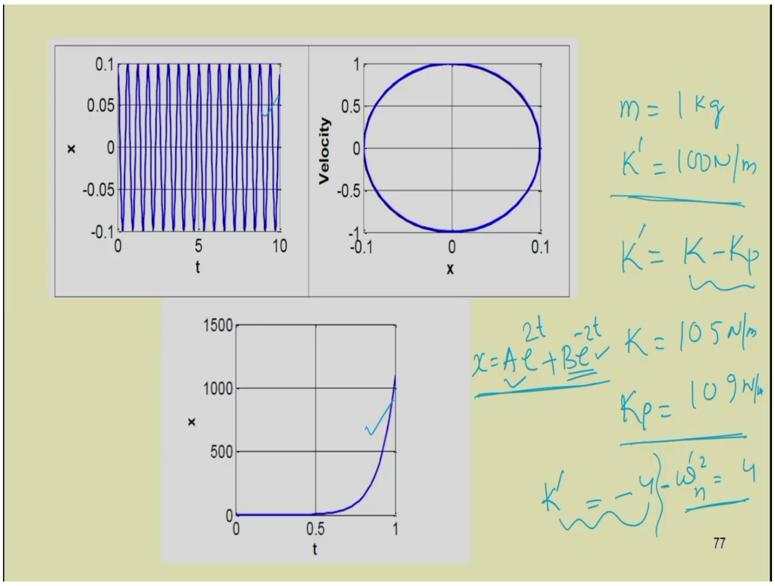

So, in the first case. So, for example, so, if we will take this m for example, let us take m equal to 1 1 kg K dash equal to 100. So, 100 Newton per metre. So, in this case. So, you can have a bounded solution and the response will be periodic. So, the solution is a sin omega t plus phi. So, this is the solution.

But if you take for example, this K dash is nothing, but K minus K P K minus K P. For example, let us take K equal to let K equal to we are taking 105 Newton per metre and K P let us take more than that thing. So, let us take it is equal to 109 Newton per metre. So, in this case K minus K P becomes minus 4 if we are taking this negative so; that means, this K dash equal to negative.

So, in this case K dash second case K dash equal to 105 minus 109 that is equal to minus 4. So, in this case K dash equal to minus 4 so, that these K dash by m that is omega m dash square becomes minus 4 or minus it can be written that minus omega n square equal to 4. So, the roots of the equation becomes omega n equal to plus minus 2. So, one part that is x equal to A e to the power.

So, you can have A e to the power 2 t plus B e to the power minus 2 t. So, due to the presence as one part. So, this part as t tends to infinite. So, this part becomes it will tends to 0, but this part will grow exponentially. So, you just see this part is growing exponentially. So, the response amplitude becomes unstable. So, if the response amplitude becomes unstable then the system is unstable, but if we are getting this bounded solution then the system responses.

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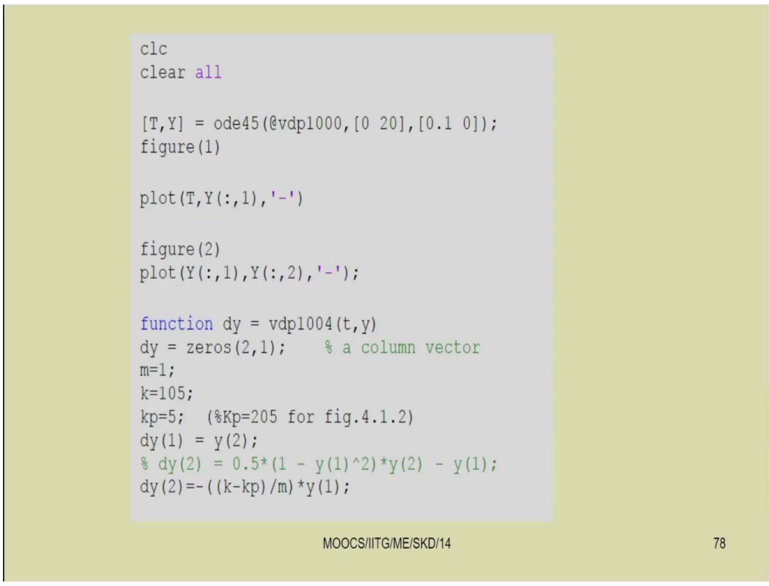

So, we are having a stable response. So, you can write a small MATLAB code for finding the solution.

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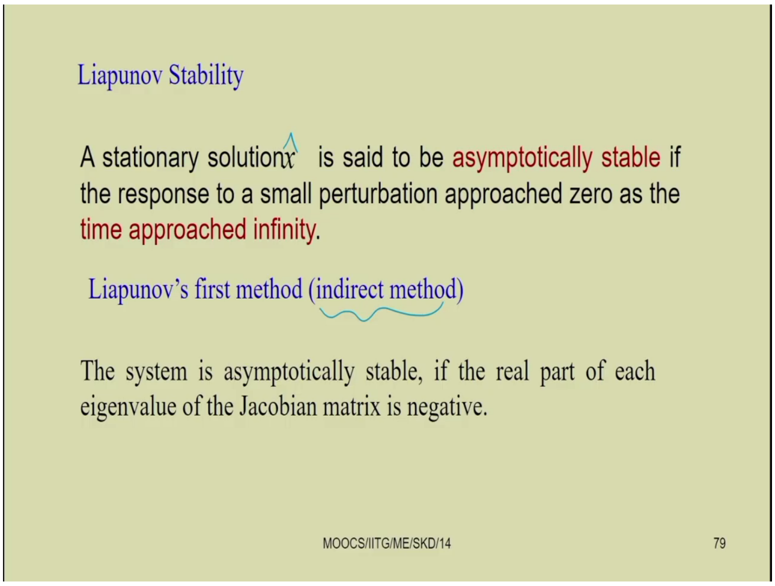

So, let us see what we mean by Liapunov stability. So, already you know.

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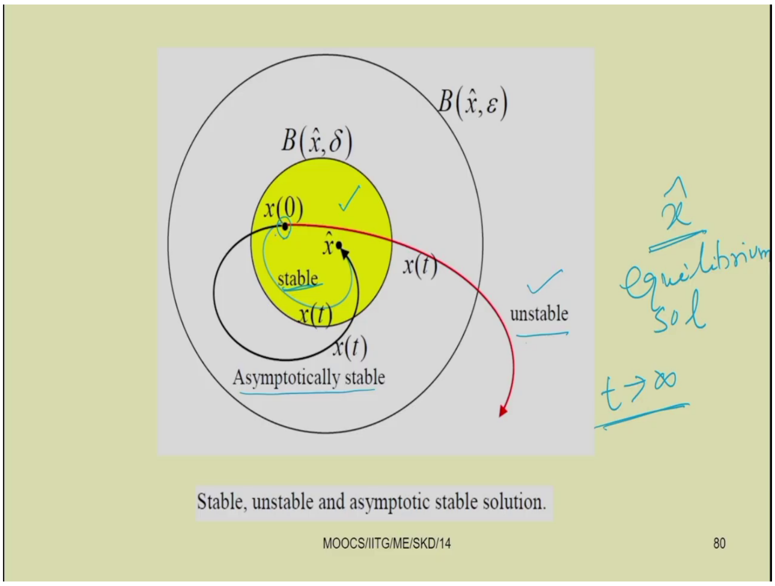

So, three type of things where we are going to three different type of things we are going to study. So, one is a stable system.

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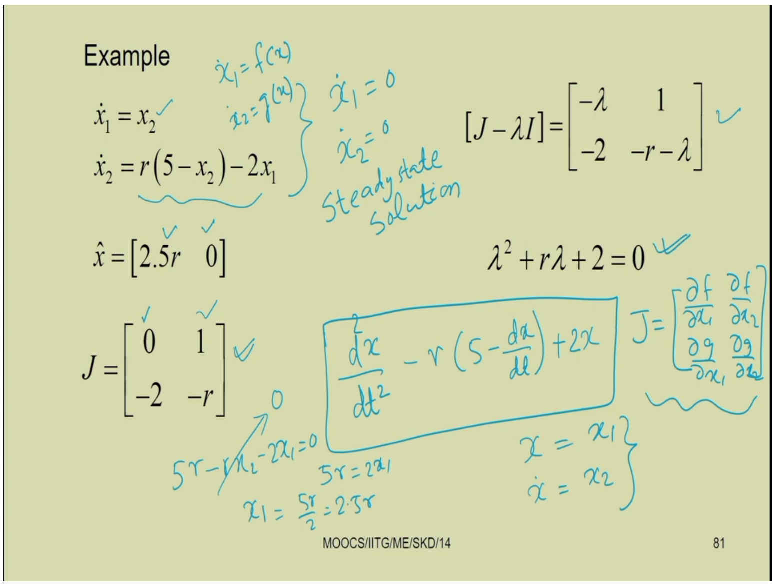

So, if the response; so, if the response becomes bounded. So, if the response becomes bounded. So, then we can tell the system to be stable. So, it is bounded. So, if it is inside always inside a small value. So, if we are perturbing the system; that means, let x dash be the stationary value or equilibrium position. So, for a given equation let x gap is the equilibrium or stationary equilibrium solution or the fixed point solution or the stationary solution. So, the solution is stable.

So, for a slide perturbing the solutions so, if always it remains inside a bounded value. So, then this becomes stable. But if it grows like you have seen in case of when that K dash equal to negative. So, if it grows. So, this is the starting point. So, let we have taken this x 0 as the starting point.

So, if the response grows, then it is known as unstable solution and the third one. So, we have another term. So, that is known as asymptotically stable; that means, if the response come back to the original state or within this bounded solution as t tends to infinite.

So, as t tends to infinite time tends to infinite. So, if the response remain within if the response remains within certain value of delta. So, then it is known as asymptotically stable. So, we know three different type of solution one is stable solution, second one is unstable solution. So, for all time t if we perturb the solution. So, if it goes to infinite or it grows then it is unstable, but if it is remain always within certain boundary.

So, then it is stable, but for certain case as t tends to infinity if its come backs to the or it is bounded then it is known as asymptotically stable. Now, we can see the Liapunov stability a stationary solution x cap. So, a stationary solution x cap is said to be asymptotically stable, if the response to a small perturbation approach to 0 as the time approach infinite. So, the Liapunov first method or indirect method of stability study. So, how we can do it?

So, the system is asymptotically stable if the real part of each eigenvalue of the Jacobian matrix is negative given the dynamic equation. So, we can find the Jacobian of that system. So, if the eigenvalue of the Jacobian matrix real part of the eigenvalue of the Jacobian matrix is negative then we can find the system to be asymptotically stable.

So, let us use this method for example, let us take this example this x dot equal to let us take this x dot equal to x 2 and x 2 dot equal to r into 5 minus x 2 minus 2 x 1. So, let us take this example. So, this is a second order equation and we have written these in terms of first order equation.

For example, the corresponding second order equation is nothing, but d square x by dt square. So, this is d square x by dt square. So, minus r into 5 minus. So, x 2 is nothing, but this dx by dt dx by dt minus 2 x 1 x 1 is x. So, this is the second order equation. So, which is written in terms of two first order differential equation that is x 1 dot equal to x 2 and x 2 dot equal to.

So, we have taken this thing to right hand side and from that thing we can see. So, this will be plus. So, this is plus 2 x.

So, when it is taken to right hand side this becomes r into 5 minus dx by dt minus 2 x. So, here x is you have taken x equal to x 1 and x dot equal to x 2 by substituting x equal to x 1 and x dot equal to x 2 that is displacement equal to x 1 and velocity equal to x 2. So, you can write this equation x 1 dot equal to x 2 and x 2 dot equal to r into 5 minus x 2 minus 2 x 1.

To find the equilibrium solution that is for the equilibrium solution. So, or steady state solution this x 1 dot. So, it will not be a function of time x 1 dot equal to 0 and x 2 dot equal to 0 by substituting x 1 dot equal to 0. So, we can get by substituting x 1 dot equal to 0 and x 2 dot equal to 0. So, that is for steady state solution. So, this r for steady state solution.

So, we can write. So, this is for steady state solution or equilibrium solutions. So, what you can get this x 1 dot equal to 0 and x 2 dot equal to 0. So, our equation becomes equilibrium solution become x 2 equal to 0 and the second equation becomes r into pi minus x 2 minus 2 x 1 equal to 0 from this thing.

So, let us write this 5 r minus r x 2 minus 2 x 1 equal to 0. So, this is the thing. So, already we got this x 2 equal to 0. So, this term equal to 0. So, x 1 becomes or 5 r equal to or 5 r equal to 2 x 1. So, x 1 equal to 5 r by 2. So, this becomes 2.5 r. So, x 1 equal to 2.5 r and x 2 equal to 0. So, this is the set is the stationary solution or steady state solution or the fixed point response of the system.

So, now we have to find the Jacobian. So, how to find Jacobian? So, our equation is x 1 dot equal to x 2. So, Jacobian is found this way. So, let this we have one equation let first equation is x 1 dot equal to fx and x 2 dot equal to gx. So, the Jacobian can be found in this way. So, that is del f by del x 1 then del f by del x 2 and in second. So, it is del z by del x 1 and a del z by del x 2.

So, you can find these Jacobian by using this method that is Jacobian J equal to del f by del x 1 and del f by del x 2. So, in this case f equal to x 2. So, if you differentiate with respect to x

1. So, this becomes 0. So, the first term becomes 0 now by differentiating with respect to x 2; x 2 when you differentiate with respect to x two. So, this becomes 1.

Similarly, now the second equation that is your g x; g x is nothing, but these r into 5 minus x 2 minus 2 x 1. So, here x 2 already ok. So, now, you differentiate with respect to x 1. So, when we differentiate with respect to x 1. So, then this becomes minus 2 and differentiating with respect to x 2. So, this becomes. So, this is r minus x 2 or minus r x 2 5 r is constant differentiation of the constant with respect to x 2 equal to 0 and differentiation of minus r x 2 with respect to x 2 equal to minus r.

So, this way by using this formula that is J equal to del f by del x 1 del f by del x 2 and del z by del x 1 and del z by del x 2. So, we got J equal to 0 1 and minus 2 r. We have to find according to this Liapunov this of the Liapunov stability. So, we can find this eigenvalue of these Jacobian matrix to find the eigenvalue. So, we can find J minus lambda I. So, J minus lambda I can be written. So, 0 minus lambda and 1 remain 1. So, this minus 2 minus r minus lambda.

So, J minus lambda equal to these. So, we have to find the determinant of this J minus lambda I and equal to 0. So, to find lambda then the determinant of this thing will be equal to minus lambda into minus r minus lambda minus 2 into 1. So, this becomes lambda square plus r lambda plus 2 equal to 0. So, we got this equation. So, now by solving these equations so we can get what is lambda. So, we got lambda square plus r lambda plus 2 equal to 0.

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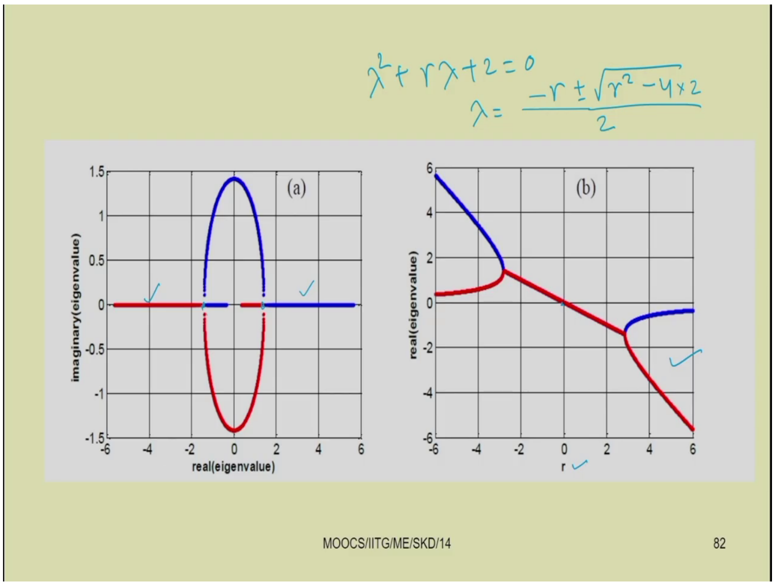

So, we got lambda square plus you just see lambda square plus r lambda plus 2 plus r lambda r lambda plus 2 equal to 0. So, we can find lambda equal to. So, minus b. So, it is equal to minus b that is minus r plus minus b square minus 4 ac by 2 a b square equal to r square minus 4 ac 4 a equal to 1 c equal to 2 minus 4 into c 1 into 2. So, that is into 2 minus by 2 a. So, this lambda equal to minus r plus minus r square minus 8 by 2.

So, here depending on the value of r. So, let us go on increasing this value of r and we can find the eigenvalue lambda. So, we have 2 eigenvalue that is lambda 1 and lambda 2. So, 4 plus 1 plus value 1 value of lambda will get and for negative. So, by putting this minus sign we can have the second value. So, by taking this lambda 1 and lambda 2 or by plotting the real and imaginary parts of the eigenvalue for a different value of r.

For example, let us take r equal to minus 6 2 plus 6. So, in that case if we are plotting the real value and imaginary value. So, up to certain. So, here up to these. So, we have the real part equal to 0 and up from this point. So, from this point so we can have this complex number. So, we have both real and imaginary part, this is up to this point and after that thing. So, then we can have. So, then we can have only this real part that is 0 real part we can having ok.

So, this way we can see we can analyse the system. So, whether it is having real part or imaginary part or it is a complex number. So, if you can see if we are plotting only the real part of the eigenvalue with different value of r. So, you can observe. So, this is r equal to 0. So, for r greater than 0. So, for r greater than 0.

So, you can see the real part of the eigenvalue is negative the real part of the eigenvalue is negative. So, for r greater than 0 the real part of the eigenvalue are negative, but for r less than 0 for r less than 0. So, we can see the real part is positive. So, real part. So, from these things. So, you can see the real part is positive.

So, in this case for the system with r greater than 0, the system to be stable and for r less than 0 the systems become unstable. So, this way you can do the stability analysis of the system ok. Now, coming back to the duffing equation again.

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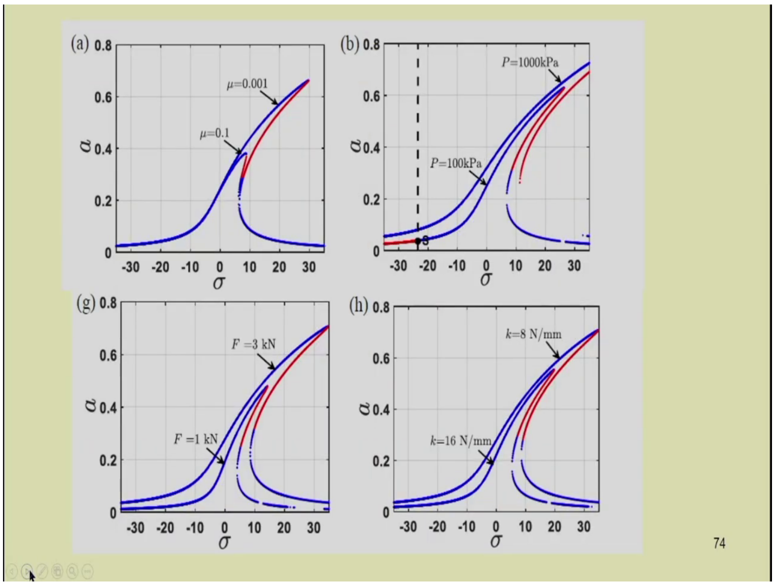

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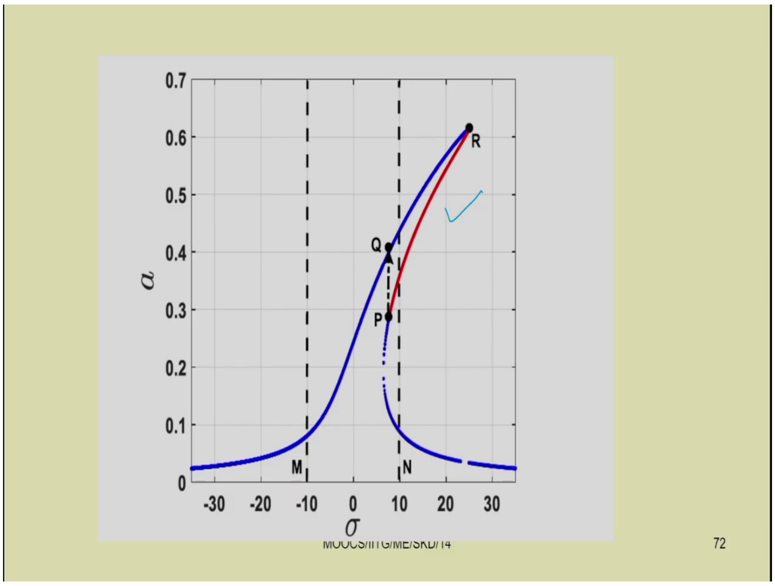

So, let us go back to the duffing equation or let us see. So, coming back to the duffing equation. So, we can see so if we can plot this a versus sigma. So, this is a typical frequency response in case of the duffing equation by changing different value of alpha. So, we can get actually the response with hardening effect or softening effect. So, that part also we will see. So, let me show you previously we have seems to see seen some other equations also.

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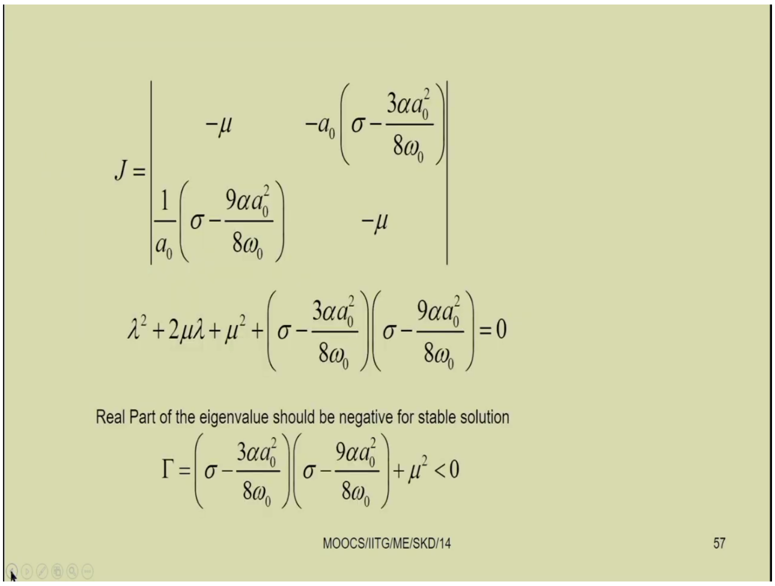

So, this is the equation actually now coming back to the duffing equation. So, already we got the frequency response between a and sigma. Now we can find the Jacobian from this equation that is a dash equal to minus mu a plus half f by omega 0 sin gamma, a gamma dash equal to a sigma minus 3 by 8 alpha by omega 0 a cube plus half f by omega 0 cos gamma.

So, this part. So, right hand side we can write these as f, a, gamma. So, the second. So, this is equal to our g a gamma. So, the Jacobian matrix can be written. So, we have two equations. So, one equation is a dash is equal to f a gamma second equation equal to gamma dash equal to a gamma dash equal to g a gamma.

So, these gamma dash we can write. So, these a we can divide it here. So, if we are dividing these a. So, then this equation becomes sigma minus 3 by 8 alpha by omega 0 a square plus f plus half f by omega 0 a cos gamma. So, now we can differentiate these equation and write down these Jacobian J equal to this.

So, Jacobian J equal to minus mu minus a 0 into sigma minus 3 alpha 0 a 0 square by 8 omega 0. So, you just see this a 0 is written. So, what is a 0? a 0 is the stationary solution or the fix point solution corresponding to the steady state solution. So, we have got this sigma and a. So, that a itself is a 0.

So, J equal to minus mu minus a 0 into sigma minus 3 alpha a 0 square by 8 omega 0. So, the second term. So, this is you got by del g by. So, del so you can write this J in this way that is del f by del a and del f by del gamma and the second equation you can get from this del g by del a and del g by del gamma.

So, by doing that way I am substituting actually the solution what you got in place of sigma. So, you got the from this equation between sigma is a function of the system parameter and a. So, from this thing or I can write this is a function of a. So, from that thing you got this one. So, now the. So, now, you can find J minus lambda I. So, for the system to be stable.

So, this J minus lambda I determinant of J minus lambda I should be equal to 0. So, from that thing you can get lambda and from the lambda for the system to be stable. So, we can write the real part real part must be real part of the eigenvalue must be negative real part of the eigenvalue must be negative.

So, the real part if we are writing that thing as lambda capital lambda. So, this lambda. So, we can get from this thing. So, you just see this is a quadratic equation in lambda, you can write this lambda equal to. So, lambda square plus 2 mu lambda plus. So, this is a constant part yes or no? So, this part is a constant part. So, this equation is can be written in this form a lambda square plus b lambda plus c.

So, lambda equal to minus b plus minus root over b square minus 4 ac by 2 a. So, by using that equation. So, you can write this is the condition that is lambdas this is the real part equal to sigma minus 3 alpha into a 0 square by 8 omega 0 into sigma minus 9 alpha a 0 square by 8 omega 0 plus mu square less than equal to 0. So, when this condition is satisfied the response is stable. So, if this condition if this real part is greater than 0. So, then the response is unstable.

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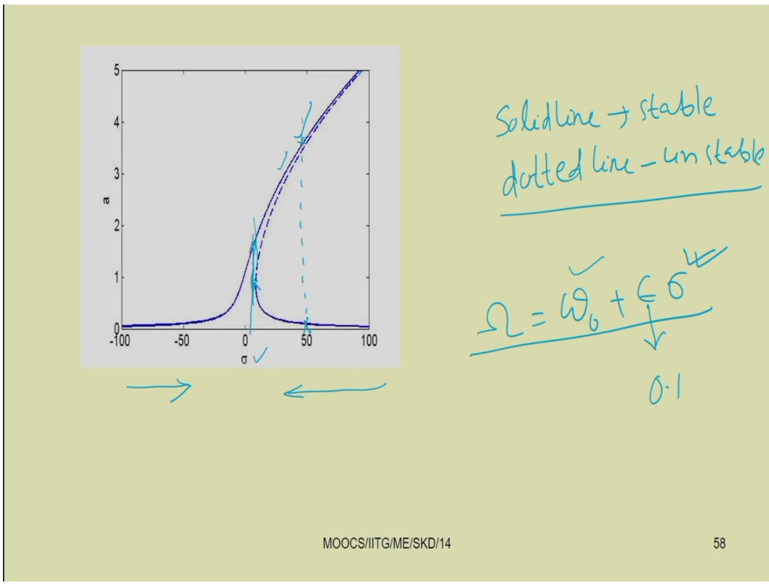

So, by plotting the stable part with solid line. So, solid line is stable. So, you can plot in colour also and dotted line unstable solution dotted line unstable solution. So, you can plot. So, you can plot the frequency response. So, a what is a sigma if you plot. So, we can see this thing. So, what is omega? So, omega already we know equal to omega 0 plus epsilon sigma.

So, here sigma is taking plus minus value; that means. So, our epsilon for example, let us take epsilon equal to 0.1 or 0.01. So, for a different value of omega 0 by taking different value of omega 0. So, we can plot by taking different value of sigma. So, we can plot the frequency response. So, you just see the frequency response. So, this is the response amplitude for a different value of sigma.

So, with increase in sigma. So, you can see with increase in sigma. So, this response amplitude increases. So, this is known as sweeping up. So, you sweep up the frequency. So, if you sweep up the frequency the frequency response amplitude a is a goes on increasing and after sometimes. So, the stable parts. So, the stable solution and the unstable solution meet at certain point.

So, later will know that these type that point is known as a bifurcation point and that bifurcation is known as saddle node bifurcation point. So, later we will study regarding different type of bifurcations and that time will know. So, that is the saddle node bifurcation. Similarly, if we sweep down the frequency for example, we are starting. So, you can see after this point the system has multi solutions multiple solutions multiple equilibrium points.

So, up to this thing the system has its single equilibrium point, but after this value after this line the system has multiple equilibrium point. So, here for example, for sigma equal to 50. So, we have a solution. So, this is a response you can draw a line here. So, you can see.

So, we have these and these two stable solution and one unstable solution and if we sweep down the frequency at this point the system. So, if you further sweep down as there is no solution towards left along this line. So, it will jump up from these to this. So, the response will jump from these points. So, these point to this point. So, there is a jump of phenomena takings place here.

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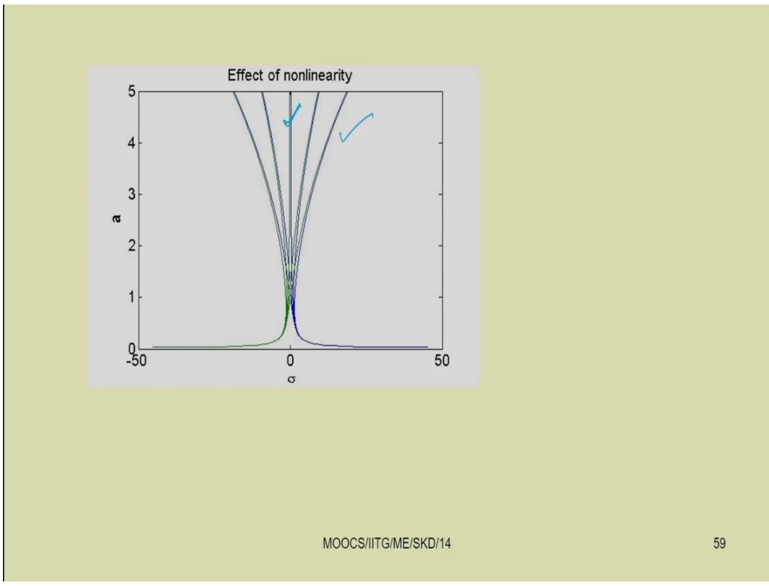

So, by changing the system parameter. So, we can see for example, if we are taking different value of non-linearity. So, with increasing non-linearity. So, we can have this hardening effect and by taking minus alpha we can have softening effect. So, we have this hardening and softening effect. So, this alpha. So, this non-linearity equal to 0 the coefficient of non-linear term equal to 0.

So, you can get this curve and by taking alpha positive. So, you can get this hardening type of effect and by taking alpha negative. So, you can get the softening type of effect in this response.

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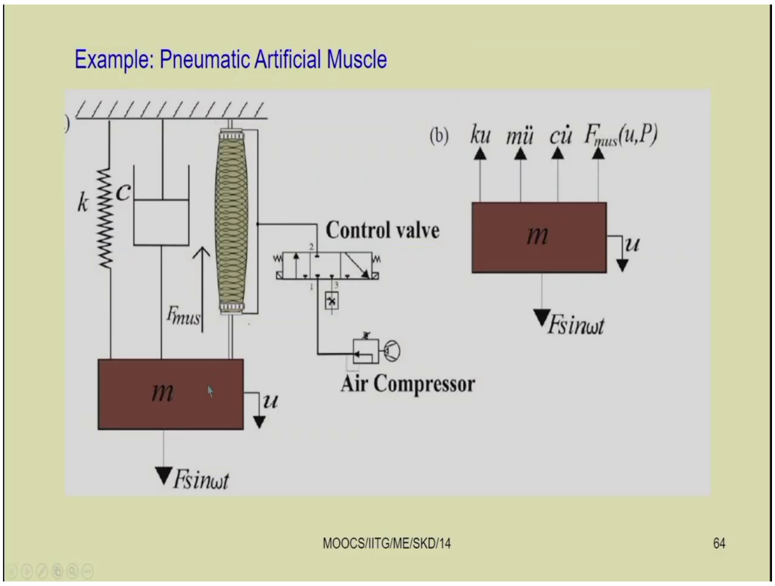

If we are taking so, for example, previously last time. So, we have taken a system with artificial muscle. So, by taking this artificial muscle which reduced to that of a duffing equation. So, we have taken these resonance conditions and we have seen. So, for example, in this case. So, we have 3. So, this is a bi stable solution this is known as bi stable solution. So, in case of bi stable. So, we have two stable solution and one stable solution.

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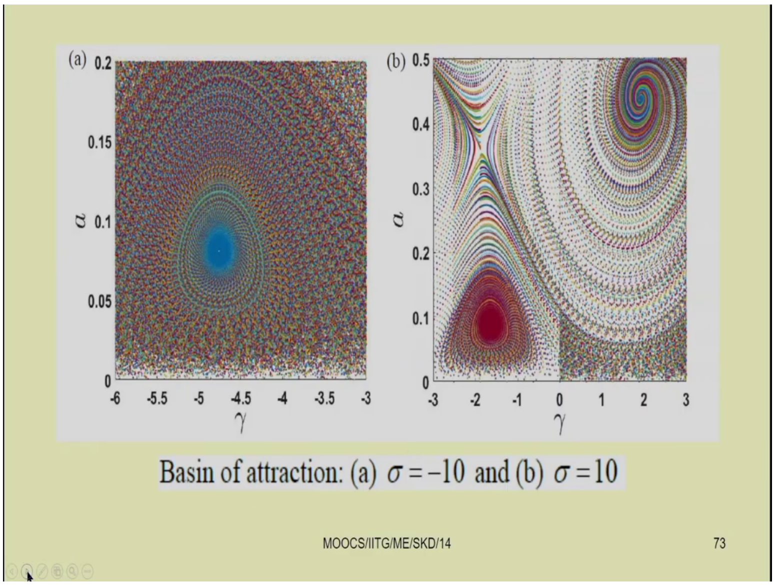

So, we can study the. So, here the basin of attraction by taking different initial conditions. So, we can see the clear basin. So, it is going. So, this is a saddle point, this unstable point is the saddle point and these and these points. So, these and these are stable point and this is the unstable saddle point.

So, this way one can study the duffing equation. So, when it is weak non-linear. So, or effect also one can study of the damping. So, by increasing the damping the response amplitude will decrease, by taking different value of forcing parameter you can see the response amplitude can be a response amplitude can be increased. So, this way one can study. So, different responses of the system.

So, today class again we have discussed regarding these duffing equation and we have studied the stability Liapunov stability the Liapunov stability of first type. So, that is by finding how we can find the stability by using a Liapunov first method that is indirect method for studying the stability and in this method we have we know how to find the Jacobian matrix and by finding the eigenvalue of the Jacobian matrix. So, we can study the stability of the system.

And next class we are going to take this hard excitation and also we will take the systems different systems with different type of damping. So, we will study with for example, by coulomb damping, viscous damping, negative damping, hysteretic damping. So, that is different type of damping. So, the response with the different type of damping and also with hard excitation.

Thank you very much.