Nonlinear Vibration
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Lecture – 12
Method of generalized Harmonic Balance method

Welcome to today module of Nonlinear Vibration. So, today class we are going to discuss regarding these method of averaging. So, before that thing let us revisit what we have studied in the last class. That is the application of method of multiple scales to nonlinear systems.

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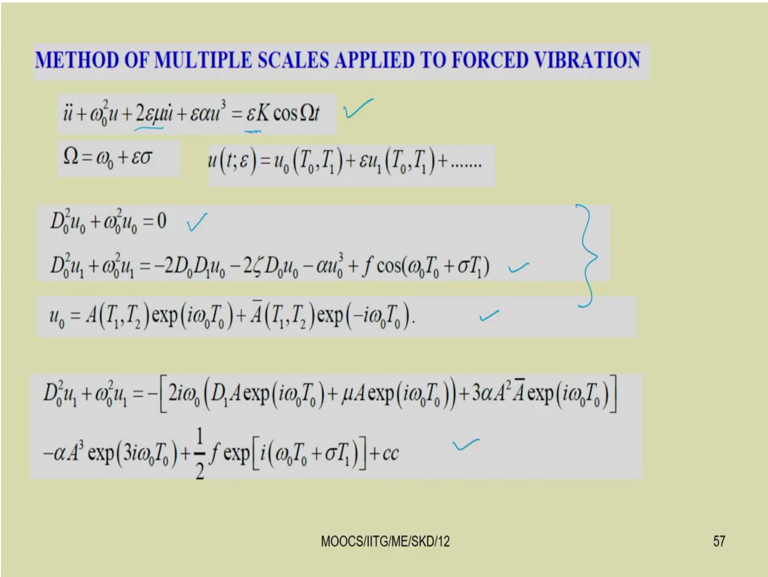

Particularly, we have studied this application of method of multiple scale to force vibration. So, here we have taken the governing equation in this form that is u double dot plus omega 0 square u plus 2 epsilon u u dot plus epsilon alpha u cube equal to epsilon K cos omega t.

So, we have discussed regarding weak form of forcing. So, as the forcing contains a term epsilon to or to represent weak form of forcing. So, we are using the term epsilon. So, similarly here also we have used this damping to be weak form. So, that is 1 order less than that of the linear term. So, here the damping and then this cubic nonlinearity and the forcing term are considered to be weak.

So, following the similar procedure as that in method of multiple scale that is by taking u t epsilon equal to u 0 plus epsilon u 1. So, and substituting this equation in this original equation and separating different order of epsilon one can obtain these equations. And then the solution of D this D 0 square u 0 plus omega 0 square u 0 equal to u 0 equal to A T 1 T 2 e to the power i omega 0 T 0 plus A bar e to the power minus i omega 0 T 0. So, substituting this equation in the higher order equation that is order of epsilon, then we can get this equation.

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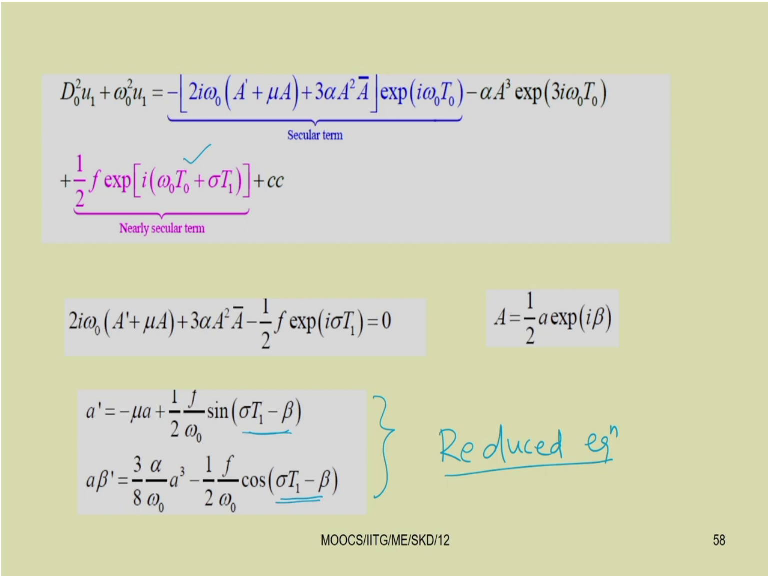

Now, by separating these or by killing the secular term, so secular terms are the terms having or with the coefficient of e to the power i omega 0 T 0 similarly we have some nearly secular term. For example, due to this forcing term this one when epsilon the sigma that is the detuning parameter tends to 0. So, we have a near secular term. So, by eliminating secular and near secular terms, so we can get some equations. So, these equations are known as reduced equation. So, these are known as reduced equation.

So, these reduced equation. So, we can see in this equation the time term explicitly appear in the right hand side. So, these are not autonomous system. So, to make it autonomous one can assume the sigma T 1 minus beta equal to gamma.

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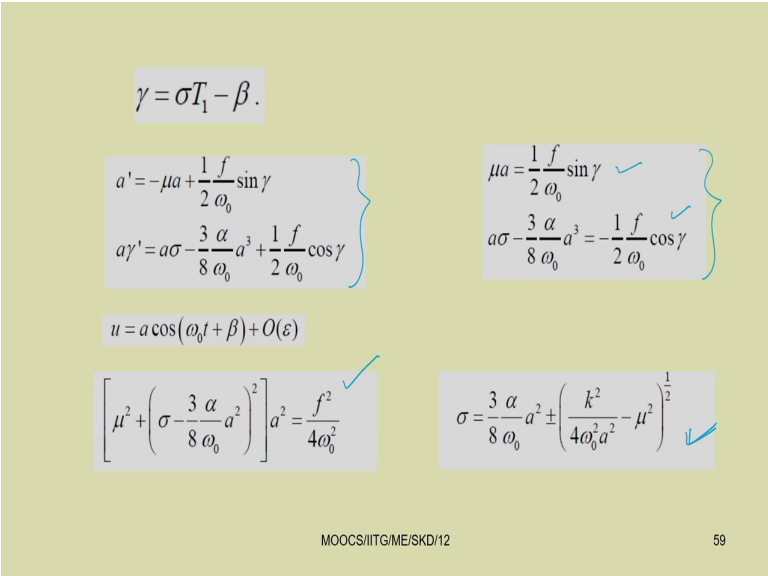

And then write down this equation in its autonomous form. So, these equations you can see. So, there is no time term explicitly appearing in the right hand side of the equation. So, these are autonomous equation.

So, now for steady state by putting at steady state this a and gamma are no longer function of time. So, this a dash and gamma dash must be equal to 0, so by substituting a dash and gamma dash equal to 0 in the reduced equation. So, one can get these equation, that is mu a equal to half f by omega 0 sin gamma a sigma minus 3 by 8 alpha by omega 0 a cube equal to minus half f by omega 0 cos gamma.

So, you can note that in the right hand side and in this both the equation. So, you have the term where you can square and add to get the sin square gamma plus cos square gamma.

So, this gamma term can be eliminated or it will not squaring and adding. So, this gamma term will not appear in this equation and we can get a close form equation or close form equation in this form. That is mu square plus sigma minus 3 by 8 alpha by omega 0 a square whole square a square equal to f square by 4 omega 0 square. Here this equation is in the form of a to the power 6 or square in sigma. So, one can conveniently use this quadratic equation in terms of sigma and solve it to get this equation.

So, that is sigma equal to 3 by 8 alpha by omega 0 a square plus minus k square by 4 omega 0 square a square minus mu square to the power half. So, this way one can solve or use this method of multiple scales to any equation.

Particularly, when they are written in its weak form and can solve this equations to find the resulting equation which can conveniently used to plot the force or free force by force response or frequency response of the system. So, later we will see during this application of these methods. So, we can see how the response curve looks like.

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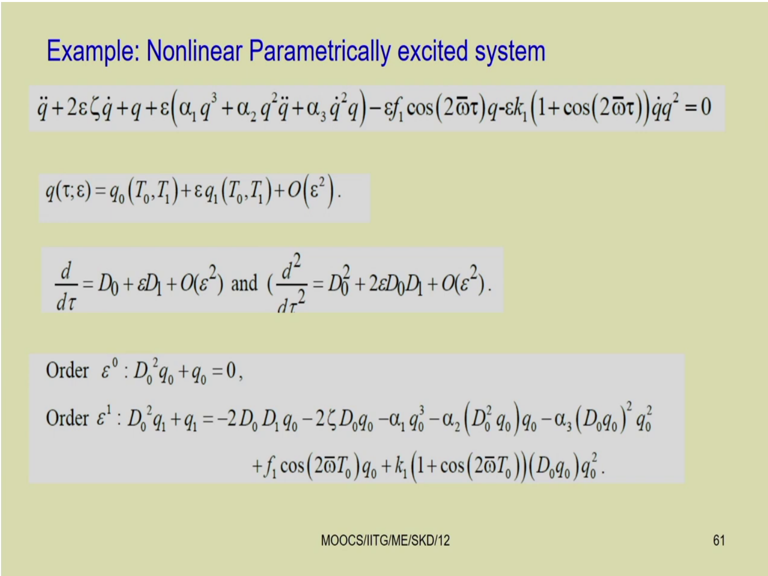

And similarly also last class we have taken some more examples of a parametrically excited system.

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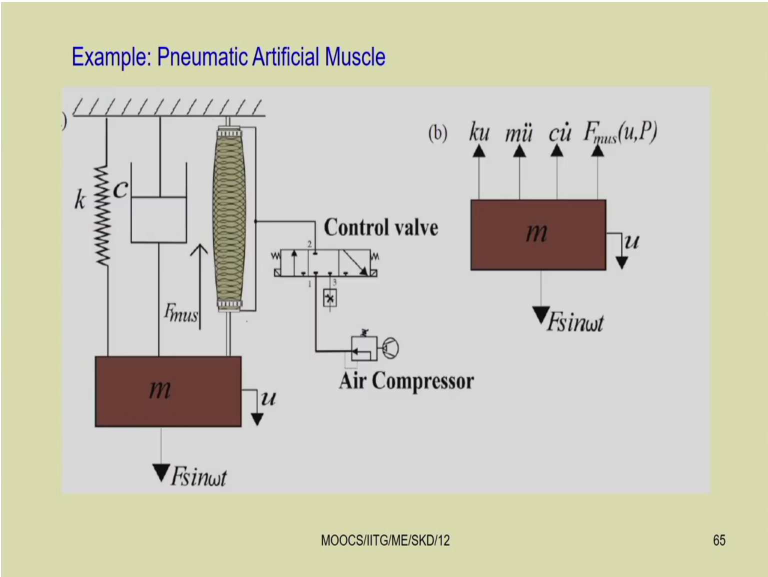

And also we have taken another example of pneumatic artificial muscle.

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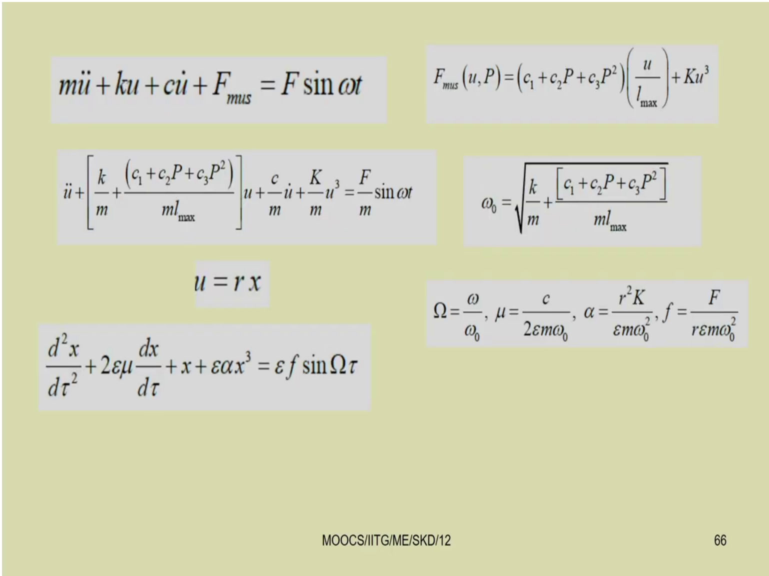

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$$x(\tau,\varepsilon)=x_0(T_0,T_1,T_2)+\varepsilon x_1(T_0,T_1,T_2)+\varepsilon^2 x_2(T_0,T_1,T_2)$$

$$D_0^2x_0+\varepsilon D_0^2x_1+\varepsilon^2 D_0^2x_2+2\varepsilon D_0D_1x_0+2\varepsilon^2 D_0D_1x_1+\varepsilon^2 x_0(D_1+2D_0D_2)+2\varepsilon\mu D_0x_0$$
$$+2\varepsilon^2\mu D_0x_1+2\varepsilon^2\mu D_1x_0+x_0+\varepsilon x_1+\varepsilon^2 x_2+\varepsilon\alpha x_0^3+\varepsilon^2 3\alpha x_1x_0^2=\varepsilon f\sin\Omega\tau.$$

Order of ε^0

$$D_0^2x_0+x_0=0$$

Order of ε^1

$$D_0^2x_1+x_1=f\sin\Omega T_0-2D_0D_1x_0-2\mu D_0x_0-\alpha x_0^3$$

Order of ε^2

$$D_0^2x_2+x_2=-2D_0D_1x_1-x_0(D_1+2D_0D_2)-2\mu D_0x_1-2\mu D_1x_0-3\alpha x_1x_0^2$$

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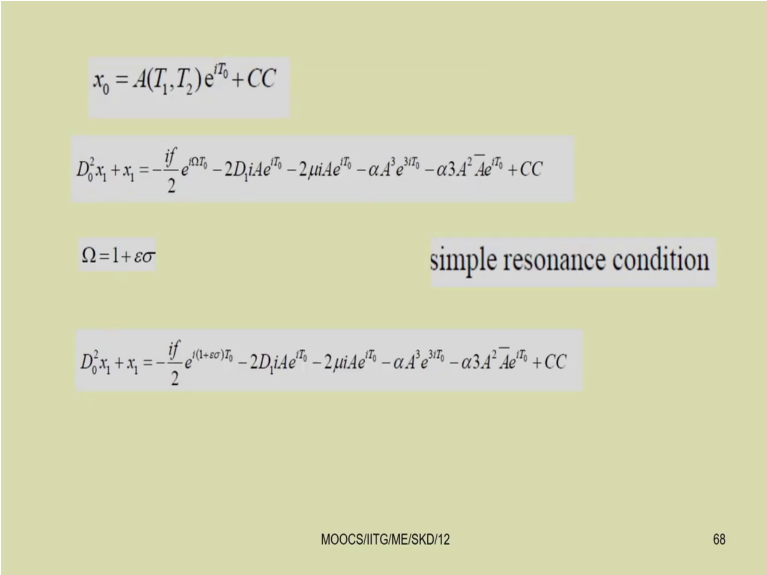

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$$-\frac{if}{2}e^{i\varepsilon\sigma T_0}-2D_1iA-2\mu iA-\alpha 3A^2\bar{A}=0$$

$$D_1A=-\frac{f}{4}e^{i\varepsilon\sigma T_0}-\mu A+\frac{3i}{2}\alpha A^2\bar{A}$$

$$D_0^2x_1+x_1=-\alpha A^3e^{3iT_0}+CC$$

$$x_1=\frac{\alpha A^3e^{3iT_0}}{8}+CC$$

$$D_2A=-(1+2\mu)\frac{D_1A}{2i}+\frac{i3\alpha^2A^3\bar{A}^2}{16}$$

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$$\frac{dA}{dt}=\frac{dA}{dT_1}\frac{dT_1}{dt}+\frac{dA}{dT_2}\frac{dT_2}{dt}=\varepsilon D_1A+\varepsilon^2D_2A$$

$$\frac{dA}{dt}=\left[\varepsilon-\frac{\varepsilon^2(1+2\mu)}{2i}\right]\left[-\frac{f}{4}e^{i\varepsilon\sigma T_0}-\mu A+\frac{3i}{2}\alpha A^2\bar{A}\right]+\varepsilon^2\frac{i3\alpha^2A^3\bar{A}^2}{16}$$

$$A=\frac{1}{2}ae^{i\beta}$$

$$a'=\frac{da}{dt}=\varepsilon\left(-\frac{f}{2}\cos\gamma-\mu a\right)+\frac{\varepsilon^2(1+2\mu)}{2}\left(\frac{f}{2}\sin\gamma-\frac{3}{8}\alpha a^3\right)$$

$$a\gamma'=a\frac{d\gamma}{dt}=a\varepsilon\sigma-\left[\varepsilon\left(-\frac{f}{2}\sin\gamma+\frac{3}{8}\alpha a^3\right)+\frac{\varepsilon^2(1+2\mu)}{2}\left(-\frac{f}{2}\cos\gamma-\mu a\right)+\varepsilon^2\frac{3\alpha^2a^5}{256}\right]$$

$$x_1=\frac{\alpha a^3}{32}\cos(6T_0)$$

$$x(\tau)=a\cos(\Omega T_0-\gamma)+\varepsilon\frac{\alpha a^3}{32}\cos(6T_0)$$

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$$\frac{da}{dt} = \varepsilon\left(-\frac{f}{2}\cos\gamma - \mu a\right)$$

$$a\frac{d\gamma}{dt} = a\varepsilon\sigma - \varepsilon\left(-\frac{f}{2}\sin\gamma + \frac{3}{8}\alpha a^3\right)$$

Table 2.1 System parameters used for numerical simulation.

Parameter	Numerical Value	Parameter	Numerical Value	Parameter	Numerical Value
l_{max}	74 mm	α	150	ε	0.1
c_1	-234.25 N	m	6 N	k	12 N/mm
c_2	1.96 N/kPa	μ	0.01	F	2 kN
c_3	-0.003 N/kPa^2	P	500 kPa	r	1

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So, by taking all these examples we have seen or in the last example particularly I have shown the frequency and force.

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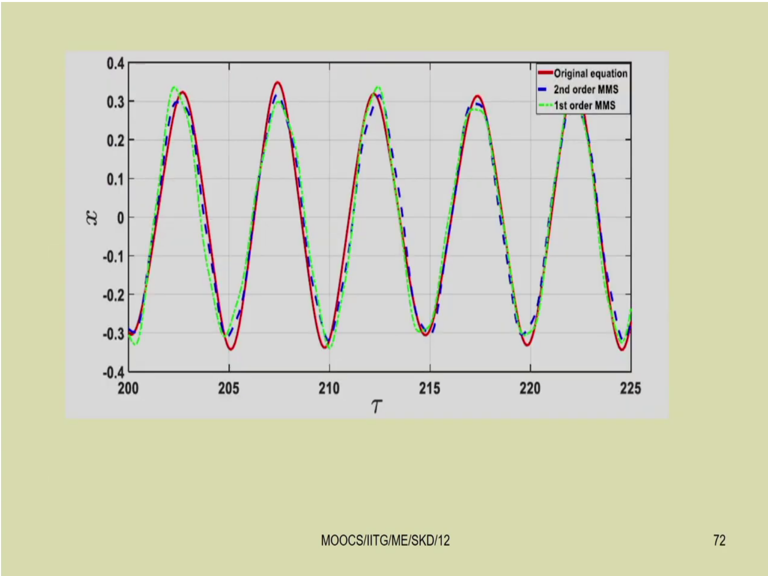

The this is the response curve, so this comparison of the original equation solving this original equation using this ode 4 5 then by using this first order method of multiple scale and second order method of multiple scales.

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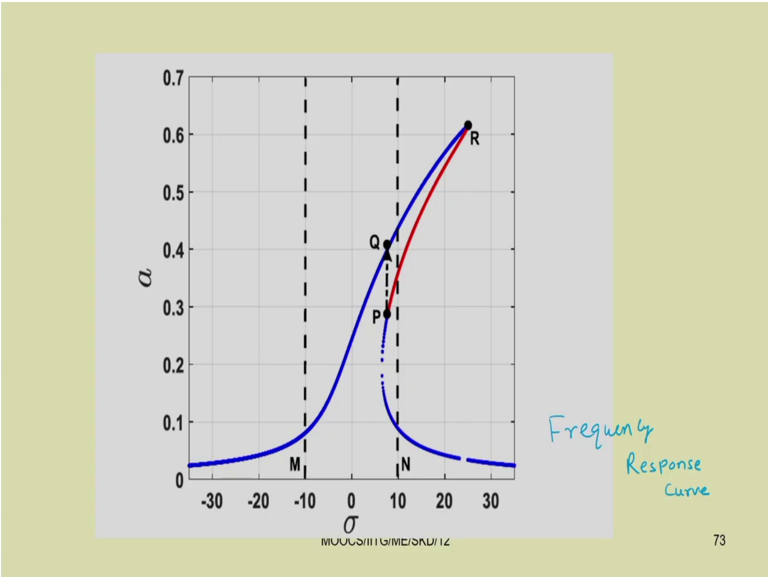

And then the, this is the, this is known as the frequency response curve. So, previous one is the force response curve and this is the frequency response curve. So, later we can see the jump up and jump down phenomena and also different bifurcation points in this forcing and frequency response plots.

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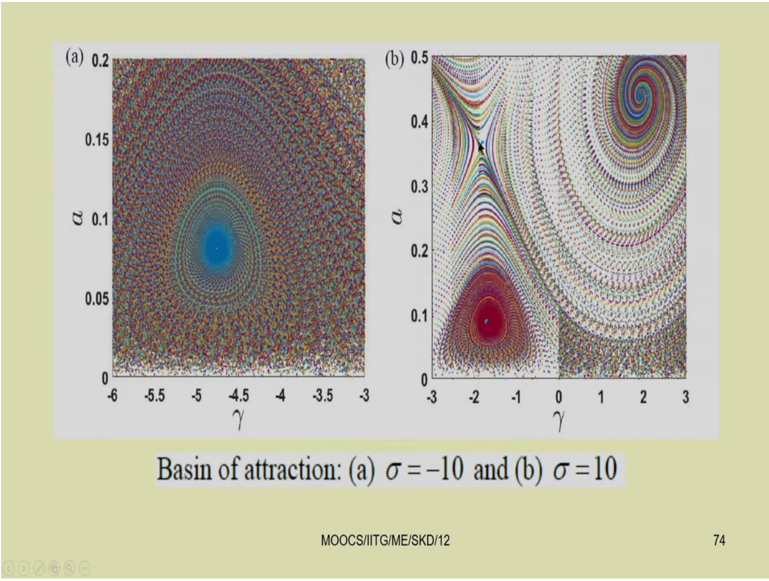

And also we will study regarding this basin of attraction. So, basin of attraction briefly I told. So, for example, at this position, so we can have 3 solutions. So, this is one solution, this is the second solution and this is the third solution.

So, these solutions can be obtained by using different initial conditions. So, if one plot all the initial conditions of a system at a particular value of sigma. Then we can obtain the basin of attraction. So, here with different value of initial conditions this basin of attraction has been plotted. So, one can see, so this is one solution, this is the other solution and this is the unstable solution which looks like a saddle point.

So, and at a previous point for example, at M, so one can see so as only one solution is there. So, that one can obtain this is the single solution whatever initial condition you just take. So, always it will come to this response or this point, so this point the stable point. So, here also we have two stable points and one unstable point.

So, these way these by using this method of multiple scales so, one can find the reduced equations, which can be solved conveniently to get a set of algebraic or transcendental equations, which can be solved to obtain the response of the system. So, now we will study one more method that is method of averaging. So, let us see one more method that is method of averaging also.

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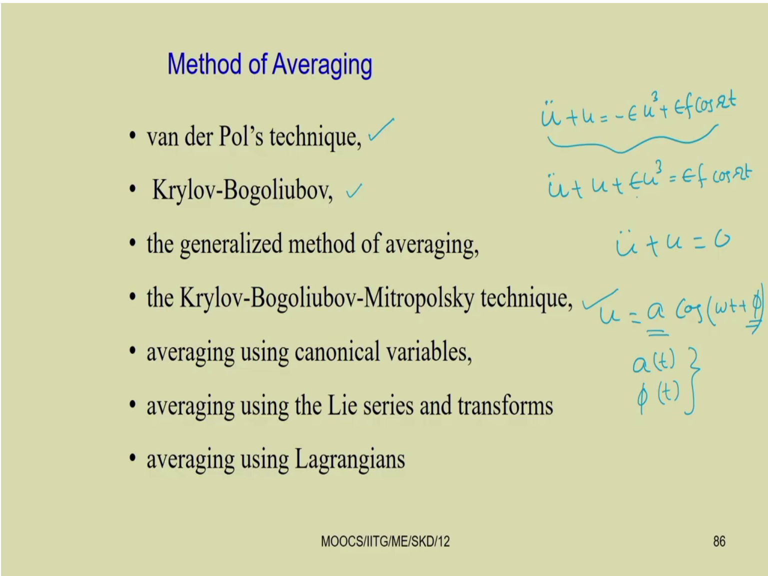

So, in method of averaging actually there are several methods are there. So, several methods includes for example, this van der Pol's technique, Krylov-Bogoliubov technique, the generalized method of averaging. The Krylov-Bogoliubov-Mitropolsky technique, averaging using this canonical variables, averaging using the lie series and transform averaging using Lagrangian. So, out of this three.

So, today class we will study this 3 methods that is van der Pol technique, Krylov-Bogoliubov method and the Krylov-Bogoliubov and Mitropolsky technique and other methods. You will be given assignments to study all other methods and also the recently available many other methods which you may study also as part of the assignments.

So, in averaging methods what we used to do. So, for example, so we are writing for example, you just write the equation of motion using a doping oscillator. So, that is u double dot plus u. So, I can write this thing equal to minus epsilon u cube minus or plus epsilon f cos omega t.

So, if you are taking doping equation with forcing and cubic nonlinearity. So, we can write this equation in this form. So, previously we have written this equation in this way u double dot plus u plus epsilon u cube equal to epsilon f cos omega t.

So, here what we did? So, we have taken all the terms with epsilon to the right hand side and written this equation. So, in this method of averaging it is assumed that the solution of this equation can be considered as that of a solution by putting this epsilon equal to 0. So, let us put epsilon equal to 0.

So, we have the resulting equation is that is u double dot plus u equal to 0. So, if u double dot plus u equal to 0 u equal to 0. So, its solution becomes so u can be written as a cos omega t so, a cos omega t plus phi. So, here in method of averaging, so it is assumed that this a and phi are, a and phi are not constant, but slowly varying function of time.

So, a is a slowly varying function of time and phi is also a slowly varying function of time. So, they are slowly varying function of time, that means, that is a dot and phi dot or a double dot and phi double dot that is the higher derivatives of a and phi are 0. So, they are not varying very much with respect to time only a and phi are slowly varying with time. So, here the assumption is in method of averaging the assumption is we take the solution of that of a system by substituting epsilon equal to 0.

After substituting epsilon equal to 0 find the solution of the resulting equation and here instead of taking the constant. So, these constants are considered to be slowly varying function of time. So, let us take the example of van der Pol equation.

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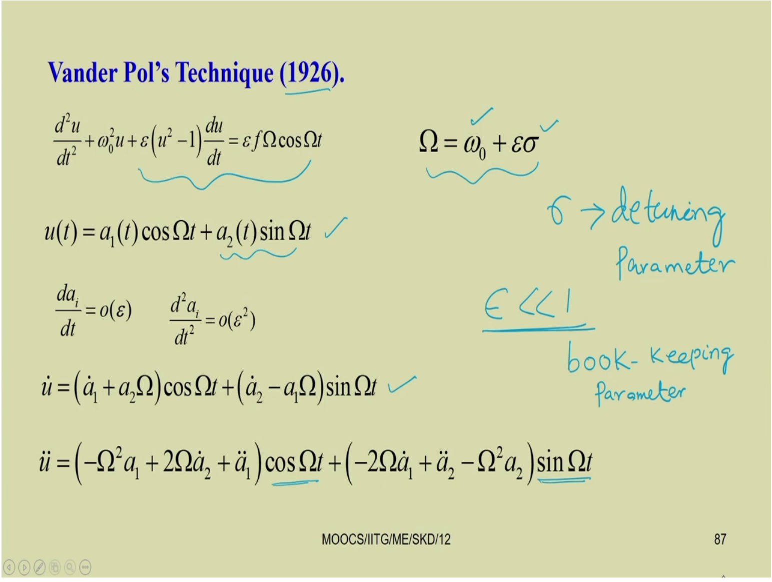

And see so how it can be solved. So, this Vander Pol technique, so solved in 1926. So, the equation van der Pol equation already I told you. So, it is it can be written in this form that is d square u by d t square plus omega 0 square u plus epsilon into u square minus 1 d u by d t equal to epsilon f omega cos omega t.

So, in this case so now, by substituting this epsilon equal to 0. So, these two terms will not be there. So, then the solution, so here substituting this omega equal to omega 0 plus epsilon sigma, that is prints that is taking the simple resonance condition.

That means, when the external frequency is nearly equal to the natural frequency of the system, that is omega 0 of the system and sigma is the detuning parameter sigma is the detuning parameter. So, it represent the nearness of these external frequency to that of the natural frequency of the system.

So, in many literature, so, a y sigma is taken to be 10 percent plus minus 10 percent of that of omega 0. So, u t so as by substituting this epsilon equal to 0. So, we can write down this solution of u equal to a t cos omega t plus a 2 t sin omega t. Either one can write this way or one can write this equal to a cos omega t plus phi also one can write. So, the solution of this equation can be written u 1 t equal to a 1 cos omega t plus a 2 t sin omega t.

So, here as already I told you the higher derivative the derivative of this a 1 a 2 as a 1 and a 2 are slowly varying function of time. So, the derivative of this thing is considered to be order of epsilon and epsilon d i by d t is considered to be order of epsilon and d square i by d t square is considered to be order of epsilon square. Where epsilon is a book keeping parameter, which is very, very less than 1 book keeping parameter so, which is considered to be very, very less than 1.

For example, if I take it 0.1 or 0.01, so like that so, now, as u t equal to a t cos omega t plus a 2 t sin omega t. So, u dot can be written as differentiating this equation it can be written, a derivative of a 1 equal to a 1 dot then derivative of a 2 t sin omega 2 t. So, now, this can be written a 1 dot cos omega t plus a minus a 1 sin omega t into omega.

So, taking the first term a 1 dot cos omega t and also from differentiating this thing. So, this is a 2 dot sin omega t plus a 2 into omega into cos omega t. So, by taking this cos term separately and sin term separately. So, you one can write this u dot equal to a 1 dot plus a 2 omega cos omega t plus a 2 dot minus a 1 omega sin omega t.

So, this way u 1 dot can be written similarly by differentiating once more one can write this u double dot, so differentiating and collecting the term of cos omega t and sin omega t. So, one can write this u double dot equal to minus omega square a 1 plus 2 omega a 2 dot plus a 1 dot a 1 double dot cos omega t plus minus 2 omega a 1 dot plus a 2 double dot minus omega square minus omega square sin omega t. So, next we can see now this now by getting this omega u double dot, now u dot and u.

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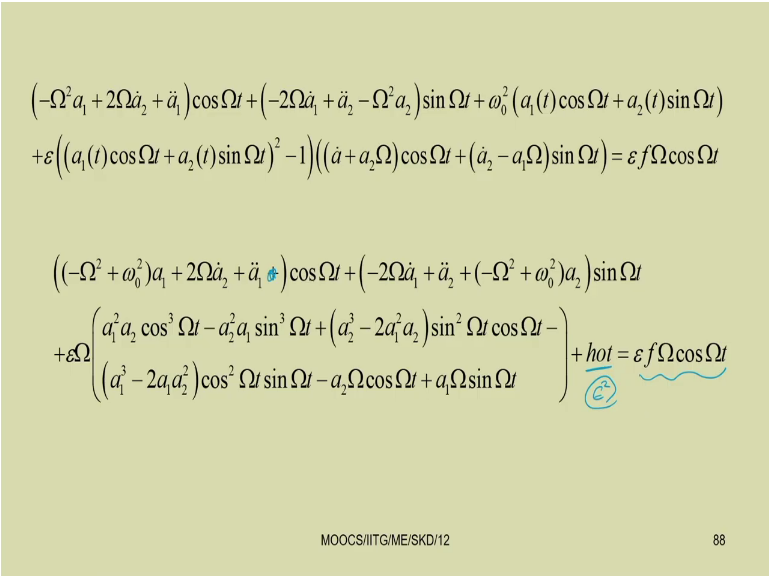

So, by substituting all these things in the original equation so, one can write. So, this is u double dot then omega 0 square u plus epsilon. So, this is u, so this is u square minus 1, so this is u square. So, this term becomes u square a 1 t cos omega t plus a 2 t sin square sin omega t whole square minus 1. So, u square minus 1 into u dot. So, that way, so this is equal to epsilon f omega cos omega t.

So, now again simplifying, so writing all the terms in terms of or simplifying this equation. So, one can write this is equal to minus omega square plus omega 0 square a 1 plus 2 omega a 2 dot plus a 1 double dot cos omega t.

Similarly this becomes plus minus 2 omega a 1 dot plus a 2 double dot plus minus omega square plus omega 0 square a 2 sin omega t, then epsilon omega into. So, all this terms will be there that is a 1 square a 2 cos cube omega t minus a 2 square a 1 sin cube omega t plus a 2 cube minus 2 a 1 square a 2 sin square omega t sin omega t.

Then minus a 1 cube minus 2 a 1 a 2 square cos square sin omega t cos square omega t into sin omega t minus a 2 omega cos omega t plus a 1 omega sin omega t. You just see plus the terms epsilon square or higher it is written h o t that is higher order terms. So, these are the higher order terms, which are not been written here.

So, we have written the terms with constant plus epsilon that is epsilon to the per 0 term and epsilon. And we have taken epsilon square and higher as the higher order terms equal to epsilon f omega cos omega t as right hand side we have a term with only epsilon order. So, in the left hand side also we have kept the term with these epsilon order of 2 epsilon.

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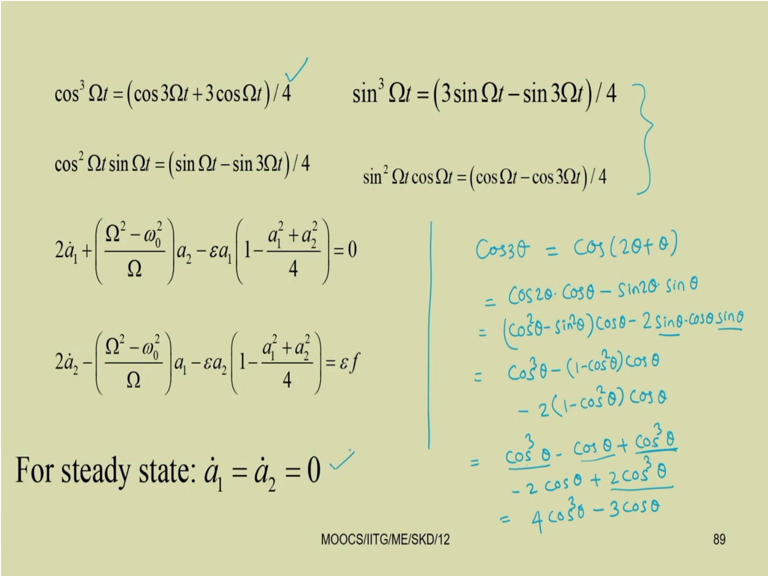

So, now we can write this cos cube omega t equal to cos 3 omega t plus 3 cos omega t by 4, similarly sin cube omega t can be written as 3 sin omega t minus sin 3 omega t by 4. Then this cos square omega t into sin omega t can be written as sin omega t minus sin 3 omega t by 4, then the sin square omega t cos omega t can be written as cos omega t minus cos 3 omega t by 4.

So, this trigonometric things can be easily derived and so for example, by writing this cos 3 omega t one can get this thing. So, this cos 3 omega t one case if we can do. So, that is cos 3 omega t, so 3 omega t can be written or simply cos 3 theta, so cos 3 theta can be written as cos 2 theta plus theta.

So, cos a plus b, so cos a plus b so this is equal to cos a into cos b that is cos 2 theta into cos a into cos b minus sin a, sin a into sin b. So, again this cos 2 theta can be written as cos square theta minus sin square theta into cos theta; minus the sin 2 theta can be written as 2 sin theta into cos theta into sin theta, so sin theta. So, this gives rise to.

So, we have to write down all the things in terms of cos theta. So, this thing can be written as cos cube theta cos cube theta and the sin square theta can be written as 1 minus cos square theta 1 minus cos square theta into cos theta minus. So, this sin theta into sin theta equal to sin square theta, so minus 2.

So, sin square theta can be written as 1 minus cos square theta into cos theta. So, this becomes so this is cos cube theta minus. So, this becomes cos theta again this minus minus plus. So, this becomes another cos cube theta so, minus 2 cos theta minus minus this plus. So, 2 cos square theta into cos theta this becomes cos cube theta.

So, now, so from this thing two cos cube theta here another cos cube theta. So, this become 3 cos cube theta. So, this becomes 3 cos cube theta 3 cos cube theta minus cos theta and minus 2 cos theta so this is. So, we can have you can see cos cube theta another cos cube theta here and 2 cos cube theta. So, this becomes 4 cos cube theta minus 3 cos theta.

So, cos 3 theta equal to 4 cos cube theta minus 3 cos theta, so from this thing 4 cos cube theta becomes cos 3 theta plus 3 cos theta and cos cube theta becomes, so this is divided by 4. So, this becomes cos 3 theta cos 3 theta plus 3 cos theta by 4, so replacing this theta by omega t. So, you can easily get this expression that is cos cube omega t equal to cos 3 omega t plus 3 cos omega t by 4. Similarly the sin cube omega t and this other two terms also can be derived.

So, now substituting this equations there and so, by substituting this equation this cos term and sin term and collecting the terms cos omega t and sin omega t and equating them to 0. So, one can obtain this 2 a dot. So, you can see 2 a dot, we can have this equation a 2 dot here, so a 2 omega a 1 dot here so collecting this terms. So, easily you can find.

So, 2 a 1 dot plus omega square minus omega square by omega into a 2 minus epsilon a 1; 1 minus a 1 square plus a 2 square by 4 equal to 0. Similarly you can get this 2 a 2 dot minus omega square minus omega 0 square by omega a 1 minus epsilon a 2 1 minus a 1 square plus a 2 square by 4 equal to epsilon f. So, for steady state already we know that a 1 as a 1 and a 2 are slowly varying function of time. This a 1 dot and a 2 dot equal to 0 by substituting a 1 dot and a 2 dot equal to 0.

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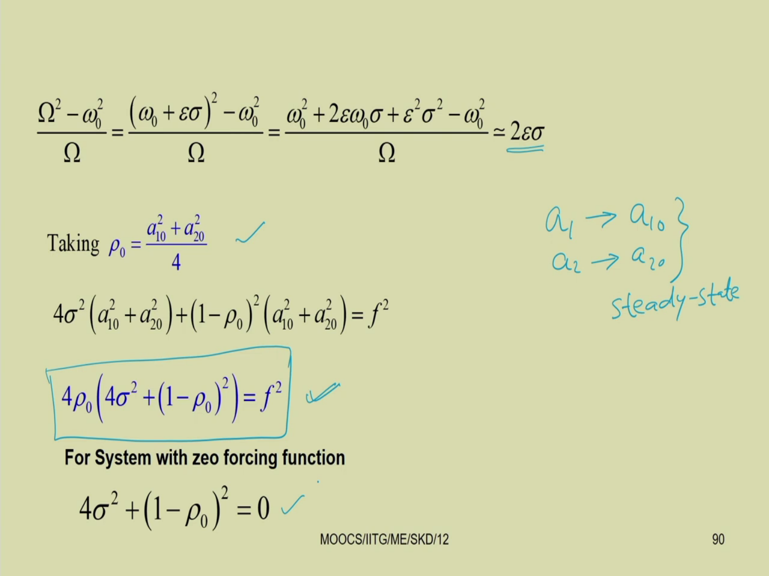

So, one can find this equation that is omega square minus omega 0 square by omega equal to omega 0 plus epsilon sigma whole square minus omega 0 square by omega; equal to this omega 0 square plus 2 epsilon omega 0 sigma plus epsilon square sigma square minus omega 0 square by omega. So, this is equal to simply this is equal to 2 epsilon sigma.

So, now by taking this rho 0 equal to a 1, so for steady state taking a equal to taking a 1 for steady state taking a 1 equal to a 1 0 and a 2 equal to a 2 0. So, this is for steady state, steady state solution. So, for steady state solution taking this thing and taking this rho 0 equal to a 1 0 square plus a 2 0 square by 4.

So, we can conveniently write the equation in this form that is 4 sigma square 4 sigma square into a 1 0 square plus a 2 0 square plus 1 minus rho 0 square into a 1 0 square plus a 2 0 square equal to f square. Or it can be simplified and can be written in this form that is 4 rho 0 into 4 minus 4 sigma square plus 1 minus rho 0 whole square equal to f square.

So, this equation can be conveniently plot plotted for a different values of forcing and taking different system parameters to study the limit cycles obtained in case of the van der Pol oscillator or van der Pol equation. So, for the system with 0 forcing, so the right hand side will become 0. So, this equation can be reduced to 4 sigma square plus 1 minus rho 0 whole square equal to 0. So, one can plot these sigma and rho 0 to obtain the limit cycles conveniently.

So, this way one can solve this van der Pol equation by using method of averaging. So, in this case initially we have taken the term with epsilon equal to 0. So, we have taken the solution and here instead of taking constant term we have taken those term a and so, we have taken the term a 1 and a 2 are slowly varying function of time.

So, then differentiating those things we can we have written down the equation and using the trigonometric functions like the sin cube cos cube and this other terms in their in the form of sin omega t cos omega t or sin 3 omega t cos 3 omega t or in the form of harmonic functions.

So, then by taking the coefficients we conveniently get the equations we reduce to its close form solution. So, this way one can use this technique to study the method of averaging for any other different equations also.

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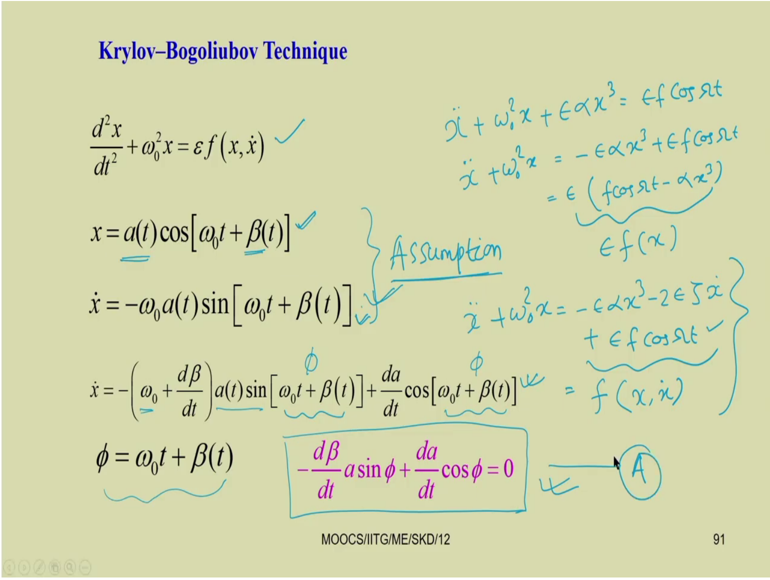

So, let us see another method of averaging proposed by this Krylov and Bogoliubov and it is known as Krylov-Bogoliubov technique. So, we can explain this equation with the help of one example.

So, let us take the example of the or first let us take the general equation then we will take the example of the doping oscillator and see how the solution of the doping of oscillator obtained in case of the by using this Krylov-Bogoliubov technique.

So, a generalized equation can be written in this form d square x by d t square plus omega 0 square x equal to epsilon f x x dot, so in for example, in case of the doping equation if I writing this x double dot plus omega 0 square x plus epsilon alpha x cube equal to epsilon f cos omega t. So, here this equation is written this x double dot plus omega 0 square x equal to minus epsilon alpha x cube plus epsilon f cos omega t by taking this epsilon common.

So, we can write this equal to f cos omega t minus alpha x cube. So, here so this term that is this f cos or this full term can be written as epsilon f x. So, if there is some damping also so this is a function of x. So, if there is some damping then this equation can be written.

So, for example, let us add damping to this equation. So, equation can be written x double dot plus omega 0 square x. So, I will take the damping term to the right hand side. So, in that case it will be minus epsilon alpha x cube minus 2 epsilon zeta. So, this becomes x dot generally it is written 2 epsilon zeta x dot.

So, this taking this damping term to the right hand side one can conveniently write down this equation. So, epsilon f cos omega t, so this becomes. So, this whole term now you can see it is a function of x, x dot and this periodic term. So, it can be written as a function of f, so, it is a function of x and x dot. So, a general equation any nonlinear equation also can be written in this form, that is why it is written d square x by d t square plus omega 0 square x equal to epsilon f x x dot.

So, x equal to here we can take the solution like in the previous case of doping equation here also we can take the solution, in this form that is x equal to a cos omega 0 t plus beta t. So, here you just see so for the secondary differential equation we are using two constant one is a other one is b. So, but here this a and beta a and beta. So, both are slowly varying function of time. So, that is why it is written x equal to a t cos omega 0 t plus beta t.

So, another assumptions actually we have to take that this x dot should be in this form that is a t into minus omega 0 as it is cos then differentiation becomes minus omega 0. So, that is minus omega 0 a t sin omega 0 t plus beta t. So, you just see it is not the direct differentiation of x to get this x dot. Rather it is assumed that this x dot that is x dot equal to minus omega 0 a t sin omega 0 t beta t.

So, these two are the assumptions actually assumptions in this case. So, these two are the assumptions in case of the this Krylov-Bogoliubov technique or. So, then we can write actually differentiating this one differentiating this x, one can write this x dot equal to minus omega 0 plus d beta by d t a t sin omega 0 t plus beta t plus d a by d t cos omega 0 t beta t. So, we can substitute this phi equal to omega 0 t plus beta t. So, that this term we can write equal to phi, so this is sin phi and this becomes also cos phi.

So, now comparing this equation and this equation so, we have assumed this x dot equal to minus omega 0 square a t. So, by comparing these two equation, so we can conveniently write that is, this minus d beta by d t a sin phi plus d a by d t cos phi equal to 0. So, you can take out the term for example, minus omega 0 a sin phi minus omega 0.

So, this term has gone a sin phi then this beta t. So, the other remaining term that is d beta minus d beta by d t into a sin phi and this term that is d a by d t cos phi equal to 0. So, comparing this term and this term easily we can write this equation. So, this is one useful equation which we will use later. So, let me take this equation A.

So, then next we can write this x double dot also. So, now, differentiating this equation x dot equal to minus omega 0 square a t sin omega 0 t into sin omega 0 t plus beta t. So, if you differentiate this thing then you can get this equation that is equal to minus omega 0 d a by d t sin.

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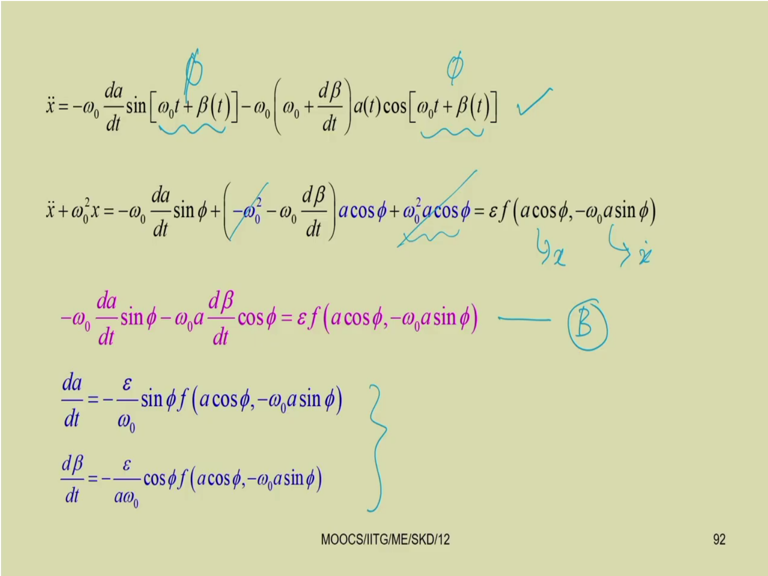

So, this is phi. So, this is sin phi this omega 0 t omega 0 t plus beta t already we have written phi. So, this is sin phi and then minus omega 0 into omega 0 d beta by d t a t cos omega 0 t beta t. So, this thing also we can write equal to phi.

So, now, substituting this x double dot expression and x dot expression and x expression in the original equation, that is x double dot plus omega 0 square x, so this becomes minus omega 0. So, this equation we have substituted for x double dot minus omega 0 d a by d t sin phi plus minus omega 0 square minus omega 0 d beta by d t a cos phi plus omega 0 square. So, this is omega 0 square x.

So, for that thing you can write this is omega 0 square a cos phi. So, that will be equal to in the right hand side. So, for x we have to substitute a cos phi and for x dot we have to substitute minus omega 0 a sin phi. So, this is for, so this is for x and this is for x dot. So, for x we are substituting a cos phi and for x dot equal to minus omega 0 square a sin phi.

So, now this equation then reduces to. So, this equation now reduces you just see this minus omega 0 square a cos phi and this is plus omega 0 square a cos phi they cancel out. So, the remaining term can be written conveniently as minus omega 0 d a by d t sin phi minus omega 0 a d beta by d t cos phi then equal to right hand side epsilon f a cos phi minus omega 0 a sin phi.

So, from this equation, now we can write this d a by d t. So, d a by d t equal to, so taking this equation as b and the previous equation what we have taken that is a. So, we have two equations so, this minus d beta by d t a sin phi plus d a by d t cos phi equal to 0. And another equation minus omega 0 omega 0 d a by d t sin phi minus omega 0 a d beta by d t cos phi equal to epsilon f cos phi minus omega 0 a sin phi.

So, by solving these two equation, so one can write this two equation that is d a by d t so, d a by d t equal to minus epsilon by omega 0 sin phi f a cos phi minus omega 0 a sin phi and this d beta by d t equal to minus epsilon by a omega 0 cos phi f function of a cos phi minus omega 0 a sin phi. So, this way we can write this equation of motion, but now as we have taken this a and phi a and beta are ok.

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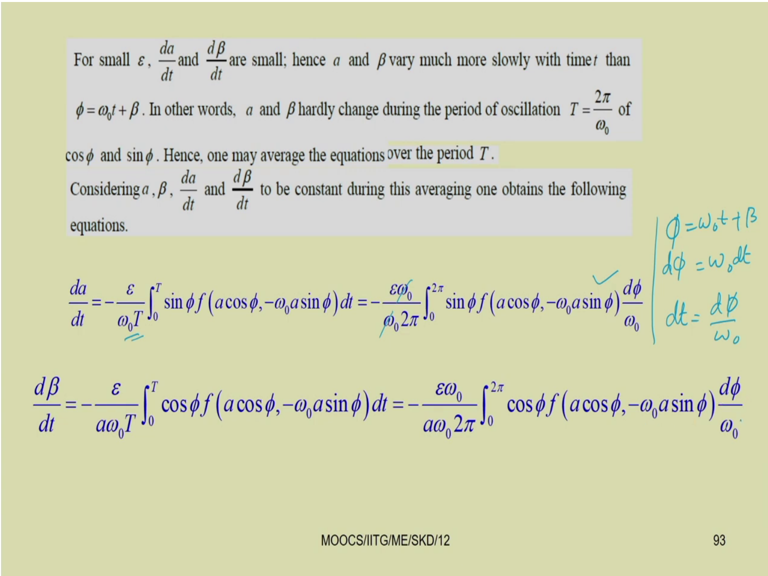

So, for small value of epsilon d a by d t and d beta by d t are small because we are considering a and beta to be slowly varying function of time. So, then a and beta vary much more slowly than time t than this phi omega 0 t plus beta. So, in other word a and beta hardly change during the period of oscillation, if we are taking one period that is T equal to 2 pi by omega 0.

So, of cos phi and sin phi so they vary. So, in a cycle, so this terms vary very slowly hence one can average the equation over one time period T, considering a beta d a by d t and d beta by d t to be constant during this averaging.

So, one obtain the following equation. So, d a by d t so we are assuming this is constant over a period. So, we can conveniently write this d a by d t equal to. So, divide this one by T. So, T is divided here and it is integrated over the time period 0 to T so, minus epsilon by omega 0 T, so integration 0 to T sin phi f a cos phi minus omega 0 a sin phi d t.

So, this T can be written equal to 2 pi by omega 0. So, that thing can be written 2 pi by omega 0. So, then this becomes epsilon omega 0 by omega 0 2 pi integration 0 to 2 pi sin phi f a cos phi minus omega 0 a sin phi d phi by omega 0 as we have taken phi equal to omega 0 t. So, we have taken phi equal to omega 0 t plus omega 0 t plus beta omega 0 t plus beta. So, differentiating this thing, so d phi by d t or d phi becomes omega 0 d t. So, beta we are considering to be constant. So, this d beta equal to 0.

So, now so this d phi by so d t can be replaced by d phi by omega, so d phi by omega 0. So, substituting this d t equal to d phi by omega 0, so we can get this equation so, one more omega 0 here omega 0 omega 0; so this omega 0 omega 0 cancel. Similarly this d beta by d t can be written as. So, over a time period minus epsilon by a omega 0 t. So, this t is divided and then it is integrated 0 to t cos phi f a cos phi minus omega 0 a sin phi d t.

So, this equal to minus epsilon omega 0 a omega 0 2 pi 0 to 2 pi cos pi f a cos phi minus omega 0 a sin phi d phi by omega 0. So, this way one can write averaging over a period. So, this d a by d t can be written.

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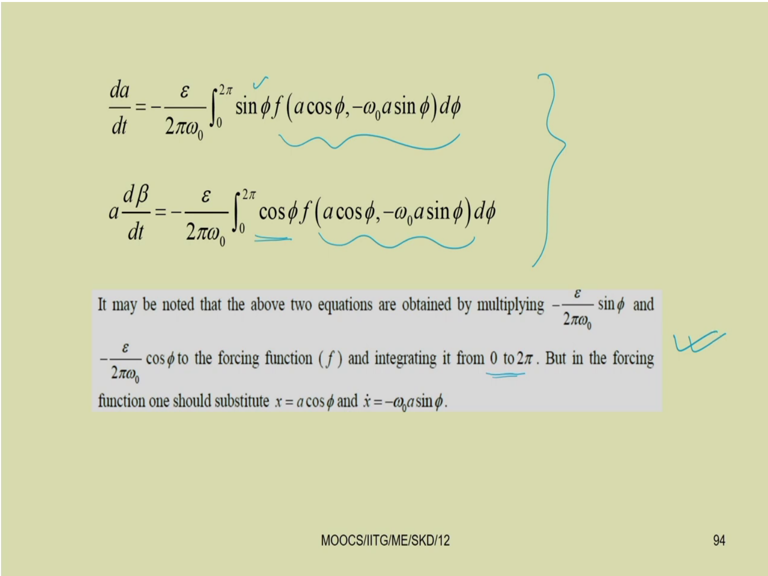

So, d a by d t is written equal to minus epsilon by 2 pi omega 0; 0 to 2 phi sin phi f a cos phi minus omega 0 a sin phi d phi. And this a d beta by d t equal to minus epsilon by 2 pi omega 0; 0 to 2 pi cos phi f cos phi minus omega 0 a sin phi d phi. So, one can conveniently remember this thing in this way that is d a by d t, so this is the function.

So, right hand side we have the function f cos. So, we have replaced this x by a cos phi and x dot by minus omega 0 a sin phi multiplied that by sin phi and integrate over a period 0 to 2 pi and multiplied by epsilon by 2 pi omega 0. So, that will give this d a by d t. Similarly a d beta by d t equal to minus epsilon by 2 pi omega 0 integration 0 to 2 pi. So, this function is multiplied by cos phi.

So, for beta so cos phi is multiplied for a the sin phi is multiplied and overly in both the cases minus epsilon by 2 pi omega 0 is multiplied to get the equation. So, these are the reduced equation obtained in case of the by using this Krylov-Bogoliubov method.

So, it may be noted that the above two equations are obtained by multiplying minus epsilon by 2 pi omega 0 sin phi and minus 2 pi omega 0 cos phi to the forcing function f and integrating it over 0 to 2 pi, but in the forcing function one should substitute x equal to a cos pi and x dot equal to minus omega 0 a sin phi. So, by using this principle this thumb rule so, one can apply this Krylov-Bogoliubov method conveniently to find the solution.

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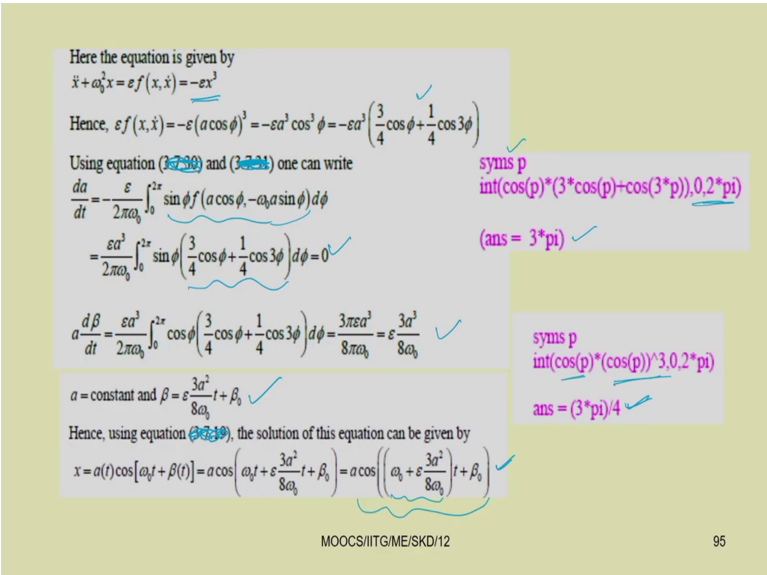

But let us see in one example for example, if we are taking this doping oscillator. So, you can see by using this doping oscillator that is or doping equation that is x double dot plus omega 0 square x equal to epsilon f x x dot.

So, let us take without damp without damping and without forcing. Then in that case this epsilon f x x dot becomes minus epsilon x cube, so minus epsilon x cube. So, here epsilon f x x dot becomes minus epsilon. So, for x we have to substitute a cos phi. So, this becomes minus a cos phi cube. So, this becomes minus epsilon a cube cos cube phi. So, previously we have derived this cos cube phi so, cos cube phi can be written as 3 by 4 cos phi plus 1 by 4 cos 3 phi.

So, this way we can write this epsilon f x dot equal to minus 3 a cube 3 by 4 d a 3 by 4 cos phi plus 1 by 4 cos 3 phi. So, now, by using the previous equations, so what do equations we have written so previous equation. So, now, writing the previous equation. So, we can write this d a by d t equal to d a by d t equal to this forcing function multiplied by sin phi integrate to over this 0 to 2 pi and further multiplying by minus epsilon by 2 pi omega 0. So, this integration you can conveniently use by using the symbolic software.

So, this becomes minus 3 a epsilon a cube by 2 pi omega 0; 0 to 2 pi. So, this sin phi into 3 by 4 cos phi plus 1 by 4 cos 3 phi d phi for example, so if you integrate this thing. So, you can integrate similarly you can write down this a d beta by d t equal to epsilon a cube divided by 2 pi omega 0. So, minus and here we have a minus epsilon x cube that is why this becomes plus.

So, integration 0 to 2 pi cos phi and this becomes 3 by 4 cos phi plus 1 by 4 cos 3 phi d phi. So, if you integrate, so you can check that this integration becomes 0, so this d a by d t equal to 0. Now by integrating this cos this is cos square phi and this becomes cos phi into cos 3 phi cos phi into 3 cos phi plus cos 3 phi. So, if you integrate it, so we can check.

So, you can view this in MATLAB if you are writing simply you just write symp sp. So, p for this phi we have written. So, integrate i and t. So, you can use this i and t function cos phi into

3 cos phi plus cos 3 phi 0 to 2 pi. So, this is 0 to 2 pi that is the in integration that is 0 to 2 pi. So, you can easily get these answer to be 3 phi though you can integrate it manually. So, you can use the symbolic software to conveniently integrate this also.

Similarly, this otherwise so if you are not expanding that thing by using this cos that cos cube phi if you do not want to expand. So, you can keep that way also cos cube phi. So, just simply multiply this cos phi into cos cube phi integration for 2 0 to 2 pi, it will give the same answer that is 3 by 4; 3 pi by 4. So, it will give 3 pi by 4, so you can substitute it in that equation to get the solution.

So, now this a d beta by d t you are getting equal to epsilon 3 a cube by 8 omega 0. So, as d a by d t equal to 0 so a becomes constant similarly as d a d beta by d t becomes this that is epsilon 3 a cube by 8 s omega 0. So, d beta by d t becomes epsilon 3 a square by 8 omega 0. So, integrating this b beta becomes epsilon 3 a square by 8 omega 0 t plus beta 0. So, this is integration constant.

So, by using this equation so you can write. So, you can write also this solution of the equation becomes x a t cos omega 0 t plus 3 beta t equal to a cos omega 0 t plus epsilon 3 a square by 8 omega 0 t plus beta 0. So, this becomes a cos omega 0 plus epsilon 3 a square by 8 omega 0 t plus beta 0.

So, you have solved this equation using this Lindstedt Poincare method, so they that time we have seen. So, there will be some additional term or this is not the correct answer. So, by using this Bogoliubov or this Krylov-Bogoliubov method, so we have seen the result what you are getting is not the correct answer, but it is approximate it approximate the solution of the doping equation.

So, to make it more accurate, so this method has been modified by Mitropolsky. So, this method becomes now we have a method KBM method that is Krylov-Bogoliubov and Mitropolsky method. So, by using this Krylov-Bogoliubov-Mitropolsky method.

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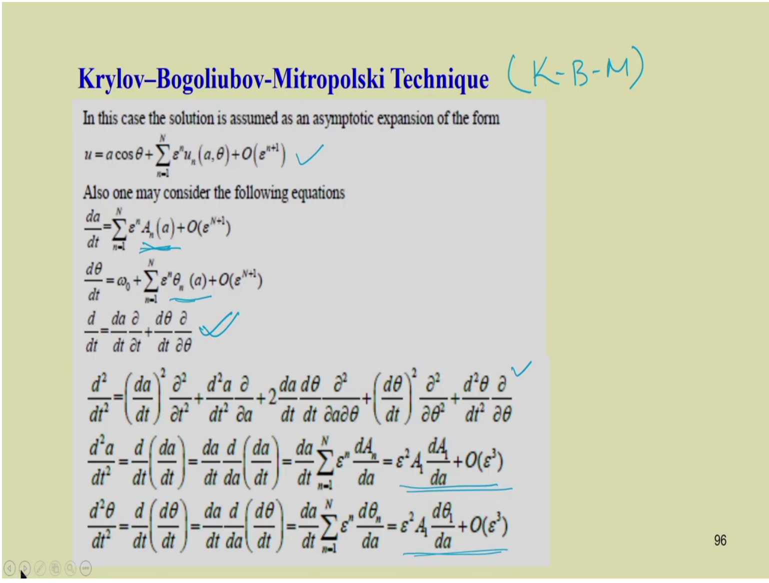

So, we can see, so this is this in this Krylov-Bogoliubov-Mitropolsky method. So, this u is assumed to be in this form that is a cos phi plus or a cos theta plus summation n equal to 1 to n epsilon n u n a theta plus order of epsilon n plus 1.

So, this way we can expand this u so, in this method of Krylov-Bogoliubov Mitropolsky or conveniently known as KBM method, KBM method or KBM technique. So, we can assume u to be equal to a cos theta plus n equal to 1 to n summation.

So, this is epsilon n u n. So, u n is a function of a and theta also one may consider the following equation. So, here so previously we have taken a for example, in case of van der Pol we have taken a and beta are slowly varying function of t and d a by d t equal to 0 for this thing for this ok.

So, but here we are assuming this d a by d t to be. So, d a by d t equal to n equal summation n equal to 1 to n epsilon n a n a. Similarly so you just see this d a by d t is assumed to be a function of a n a. So, a n is not a function of theta, similarly this d theta by d t equal to omega 0 plus n equal to 1 to n epsilon n theta n a plus order of epsilon to the power n plus 1.

So, here theta n is a function of a both a capital A n and theta n are function of a. And d a by d t can be by using this n rule it can be written d a by d t equal to d a by so, d a by del by del t d a by d t equal to d a by d t into del by del t plus d theta by d t into del by del theta.

So, using the chain rule one can conveniently write this way. Similarly d square by d t square can be written in this form that is d a by d t whole square del square by del t square plus del square a by del d t square del by del a plus 2 d a by d t d theta by d t del square by del a del theta plus del theta by d t whole square del square del theta square plus del square theta by d t square del by del theta.

So, now this d square a by d t square can be written d by d t of d a by d t. So, which is nothing, but epsilon square a 1 d a by d a 1 by d a. Similarly d square theta by d t square is nothing, but epsilon square a 1 d theta by d theta 1 by d a. So, this way one can write or use these equations in by using this KBM method one has to use these equations to solve the equations.

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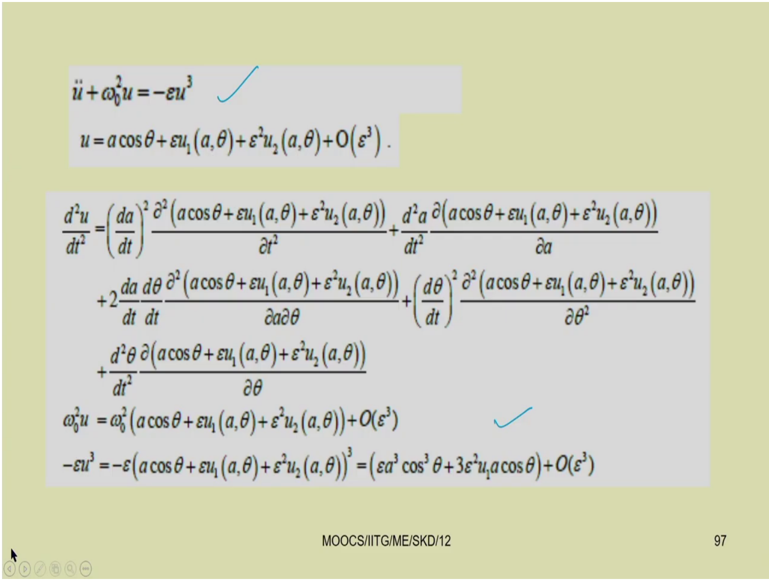

So, for example, take the example of the doping equation same equation we have taken. So, here we have taken u equal to a cos theta plus epsilon u 1 plus epsilon square u 2. So, here u 1 and u 2 are function of a and theta. So, now substituting the previous equation like this d square u by d t square so, we can write down this equation similarly conveniently.

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Or, $$\left(\frac{da}{dt}\right)^2\left(-\left(\frac{da}{dt}\frac{d\theta}{dt}\sin\theta+a\cos\theta\left(\frac{d\theta}{dt}\right)^2+a\sin\theta\frac{d^2\theta}{dt^2}\right)+\frac{d^2a}{dt^2}\cos\theta-\frac{da}{dt}\frac{d\theta}{dt}\sin\theta+\varepsilon\frac{\partial^2u_1}{\partial t^2}+\varepsilon^2\frac{\partial^2u_2}{\partial t^2}\right)$$

$$+\left(\frac{d^2a}{dt^2}\right)\left(\cos\theta+\varepsilon\frac{\partial u_1}{\partial a}+\varepsilon^2\frac{\partial u_2}{\partial a}\right)+2\left(\frac{da}{dt}\right)\left(\frac{d\theta}{dt}\right)\left(-\sin\theta+\varepsilon\frac{\partial^2u_1}{\partial a\partial\theta}+\varepsilon^2\frac{\partial^2u_2}{\partial a\partial\theta}\right)$$

$$+\left(\frac{d\theta}{dt}\right)^2\left(-a\cos\theta+\varepsilon\frac{\partial^2u_1}{\partial\theta^2}+\varepsilon^2\frac{\partial^2u_2}{\partial\theta^2}\right)+\left(\frac{d^2\theta}{dt^2}\right)\left(-a\sin\theta+\varepsilon\frac{\partial u_1}{\partial\theta}+\varepsilon^2\frac{\partial u_2}{\partial\theta}\right)+$$

$$\omega_0^2\left(a\cos\theta+\varepsilon u_1(a,\theta)+\varepsilon^2u_2(a,\theta)\right)=$$

$$-\varepsilon\left(a\cos\theta+\varepsilon u_1(a,\theta)+\varepsilon^2u_2(a,\theta)\right)^3=\left(\varepsilon a^3\cos^3\theta+3\varepsilon^2u_1a\cos\theta\right)+O(\varepsilon^3)$$

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$$
\text{Or, } \left(\varepsilon A_1+\varepsilon^2 A_2\right)^2 \begin{pmatrix} -\left(\left(\varepsilon A_1+\varepsilon^2 A_2\right)\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)\sin\theta+a\cos\theta\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)^2+a\sin\theta\left(\varepsilon^2 A_1\frac{d\theta_1}{da}\right)\right) \\ +\left(\varepsilon^2 A_1\frac{dA_1}{da}\right)\cos\theta-\left(\varepsilon A_1+\varepsilon^2 A_2\right)\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)\sin\theta+\varepsilon\frac{\partial^2 u_1}{\partial t^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial t^2} \end{pmatrix}
$$

$$
+\left(\varepsilon^2 A_1\frac{dA_1}{da}\right)\left(\cos\theta+\varepsilon\frac{\partial u_1}{\partial a}+\varepsilon^2\frac{\partial u_2}{\partial a}\right)+2\left(\varepsilon A_1+\varepsilon^2 A_2\right)\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)\left(-\sin\theta+\varepsilon\frac{\partial^2 u_1}{\partial a\partial\theta}+\varepsilon^2\frac{\partial^2 u_2}{\partial a\partial\theta}\right)
$$

$$
+\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)^2\left(-a\cos\theta+\varepsilon\frac{\partial^2 u_1}{\partial\theta^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial\theta^2}\right)+\left(\varepsilon^2 A_1\frac{d\theta_1}{da}\right)\left(-a\sin\theta+\varepsilon\frac{\partial u_1}{\partial\theta}+\varepsilon^2\frac{\partial u_2}{\partial\theta}\right)+
$$

$$
\omega_0^2\left(a\cos\theta+\varepsilon u_1(a,\theta)+\varepsilon^2 u_2(a,\theta)\right)=
$$

$$
-\varepsilon\left(a\cos\theta+\varepsilon u_1(a,\theta)+\varepsilon^2 u_2(a,\theta)\right)^3=\left(\varepsilon a^3\cos^3\theta+3\varepsilon^2 u_1 a\cos\theta\right)+O(\varepsilon^3)
$$

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We can write the equations by using this d square u and other term and so from that thing.

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Or, $$\left(\varepsilon A_1+\varepsilon^2 A_2\right)^2\left(\begin{array}{l}-\left(\left(\varepsilon A_1+\varepsilon^2 A_2\right)\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)\sin\theta+a\cos\theta\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)^2+a\sin\theta\left(\varepsilon^2 A_1\frac{d\theta_1}{da}\right)\right)\\+\left(\varepsilon^2 A_1\frac{dA_1}{da}\right)\cos\theta-\left(\varepsilon A_1+\varepsilon^2 A_2\right)\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)\sin\theta+\varepsilon\frac{\partial^2 u_1}{\partial t^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial t^2}\end{array}\right)$$

$$+\left(\varepsilon^2 A_1\frac{dA_1}{da}\right)\left(\cos\theta+\varepsilon\frac{\partial u_1}{\partial a}+\varepsilon^2\frac{\partial u_2}{\partial a}\right)+2\left(\varepsilon A_1+\varepsilon^2 A_2\right)\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)\left(-\sin\theta+\varepsilon\frac{\partial^2 u_1}{\partial a\partial\theta}+\varepsilon^2\frac{\partial^2 u_2}{\partial a\partial\theta}\right)$$

$$+\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)^2\left(-a\cos\theta+\varepsilon\frac{\partial^2 u_1}{\partial\theta^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial\theta^2}\right)+\left(\varepsilon^2 A_1\frac{d\theta_1}{da}\right)\left(-a\sin\theta+\varepsilon\frac{\partial u_1}{\partial\theta}+\varepsilon^2\frac{\partial u_2}{\partial\theta}\right)+$$

$$\omega_0^2\left(a\cos\theta+\varepsilon u_1(a,\theta)+\varepsilon^2 u_2(a,\theta)\right)=$$

$$-\varepsilon\left(a\cos\theta+\varepsilon u_1(a,\theta)+\varepsilon^2 u_2(a,\theta)\right)^3=\left(\varepsilon a^3\cos^3\theta+3\varepsilon^2 u_1 a\cos\theta\right)+O(\varepsilon^3)$$

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$$\text{Or, } \underbrace{\left(\varepsilon A_1+\varepsilon^2 A_2\right)^2 \begin{pmatrix} -\left(\left(\varepsilon A_1+\varepsilon^2 A_2\right)\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)\sin\theta + a\cos\theta\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)^2 + a\sin\theta\left(\varepsilon^2 A_1 \frac{d\theta_1}{da}\right)\right) \\ +\left(\varepsilon^2 A_1 \frac{dA_1}{da}\right)\cos\theta - \left(\varepsilon A_1+\varepsilon^2 A_2\right)\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)\sin\theta + \varepsilon\frac{\partial^2 u_1}{\partial t^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial t^2} \end{pmatrix}}_{\varepsilon^2 A_1^2 a\omega_0^2\cos\theta}$$

$$+\underbrace{\left(\varepsilon^2 A_1\frac{dA_1}{da}\right)\left(\cos\theta+\varepsilon\frac{\partial u_1}{\partial a}+\varepsilon^2\frac{\partial u_2}{\partial a}\right)}_{\varepsilon^2 A_1 \frac{dA_1}{da}\cos\theta} + \underbrace{2\left(\varepsilon A_1+\varepsilon^2 A_2\right)\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)\left(-\sin\theta+\varepsilon\frac{\partial^2 u_1}{\partial a\partial\theta}+\varepsilon^2\frac{\partial^2 u_2}{\partial a\partial\theta}\right)}_{-2\varepsilon A_1\omega_0\sin\theta-2\varepsilon^2 A_1\theta_1\sin\theta+2\varepsilon^2\omega_0 A_2\sin\theta+2\varepsilon^2 A_1\omega_0\frac{\partial^2 u_1}{\partial a\partial\theta}}$$

$$+\underbrace{\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)^2\left(-a\cos\theta+\varepsilon\frac{\partial^2 u_1}{\partial\theta^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial\theta^2}\right)}_{-a\omega_0^2\cos\theta+\omega_0^2\varepsilon\frac{\partial^2 u_1}{\partial\theta^2}+\omega_0^2\varepsilon^2\frac{\partial^2 u_2}{\partial\theta^2}-\varepsilon^2\theta_1^2 a\cos\theta-2\omega_0\varepsilon\theta_1 a\cos\theta+2\omega_0\varepsilon^2\theta_1\frac{\partial^2 u_1}{\partial\theta^2}-2\omega_0\varepsilon^2\theta_2 a\cos\theta}$$

$$+\underbrace{\left(\varepsilon^2 A_1\frac{d\theta_1}{da}\right)\left(-a\sin\theta+\varepsilon\frac{\partial u_1}{\partial\theta}+\varepsilon^2\frac{\partial u_2}{\partial\theta}\right)}_{-a\varepsilon^2 A_1\frac{d\theta_1}{da}\sin\theta}+\omega_0^2\left(a\cos\theta+\varepsilon u_1(a,\theta)+\varepsilon^2 u_2(a,\theta)\right)$$

$$=\underbrace{-\varepsilon\left(a\cos\theta+\varepsilon u_1(a,\theta)+\varepsilon^2 u_2(a,\theta)\right)^3}_{\varepsilon a^3\cos^3\theta+3\varepsilon^2 u_1 a\cos\theta}$$

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So, separating order of epsilon so, we can or now we can eliminating or taking the without taking the higher order term. So, all this terms can be conveniently written equal to epsilon square for example, this whole term can be written keeping up to epsilon square order. So, keeping up to epsilon square order one can follow this derivation and one can find the equation can be written in this formulae.

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Or,

$$\varepsilon^2 A_1^2 a\omega_0^2 \cos\theta + \varepsilon^2 A_1 \frac{dA_1}{da}\cos\theta - 2\varepsilon A_1\omega_0 \sin\theta - 2\varepsilon^2 A_1\theta_1 \sin\theta + 2\varepsilon^2\omega_0 A_2 \sin\theta$$

$$+2\varepsilon^2 A_1\omega_0 \frac{\partial^2 u_1}{\partial a\partial\theta} - a\omega_0^2 \cos\theta + \omega_0^2\varepsilon \frac{\partial^2 u_1}{\partial\theta^2} + \omega_0^2\varepsilon^2 \frac{\partial^2 u_2}{\partial\theta^2} - \varepsilon^2\theta_1^2 a\cos\theta - 2\omega_0\varepsilon\theta_1 a\cos\theta$$

$$+2\omega_0\theta_1\varepsilon^2 \frac{\partial^2 u_1}{\partial\theta^2} - 2\omega_0\varepsilon^2\theta_2 a\cos\theta - a\varepsilon^2 A_1 \frac{d\theta_1}{da}\sin\theta + \omega_0^2\left(a\cos\theta + \varepsilon u_1 + \varepsilon^2 u_2\right)$$

$$= -\left(\varepsilon a^3 \cos^3\theta + 3\varepsilon^2 u_1 a\cos\theta\right)$$

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Or,

$$a\omega_0^2\cos\theta - a\omega_0^2\cos\theta + \varepsilon\left(\omega_0^2\frac{\partial^2 u_1}{\partial\theta^2} + \omega_0^2 u_1 - 2\omega_0\varepsilon\theta_1 a\cos\theta - 2\varepsilon A_1\omega_0\sin\theta + a^3\cos^3\theta\right)$$

$$+\varepsilon^2\left(\begin{array}{l}\omega_0^2\dfrac{\partial^2 u_2}{\partial\theta^2} + \omega_0^2 u_2 + 2\omega_0\theta_1\dfrac{\partial^2 u_1}{\partial\theta^2} + A_1^2 a\omega_0^2\cos\theta + A_1\dfrac{dA_1}{da}\cos\theta - 2\omega_0\theta_2 a\cos\theta - \theta_1^2 a\cos\theta \\ -2A_1\theta_1\sin\theta + 2\omega_0 A_2\sin\theta - aA_1\dfrac{d\theta_1}{da}\sin\theta + 2\omega_0 A_1\dfrac{\partial^2 u_1}{\partial a\partial\theta} + 3u_1 a^2\cos^2\theta\end{array}\right) = 0$$

$$\omega_0^2\frac{\partial^2 u_1}{\partial\theta^2} + \omega_0^2 u_1 = 2\omega_0\theta_1 a\cos\theta + 2\omega_0 A_1\sin\theta - a^3\cos^3\theta$$

$$\omega_0^2\frac{\partial^2 u_2}{\partial\theta^2} + \omega_0^2 u_2 = \left[(2\omega_0\theta_2 + \theta_1^2)a - A_1\frac{dA_1}{da}\right]\cos\theta + \left[2(\omega_0 A_2 + A_1\theta_1) + aA_1\frac{\partial\theta_1}{\partial a}\right]\sin\theta$$

$$-3u_1 a^2\cos^2\theta - 2\omega_0\theta_1\frac{d^2 u_1}{d\theta^2} - 2\omega_0 A_1\frac{d^2 u_1}{\partial a\partial\theta}$$

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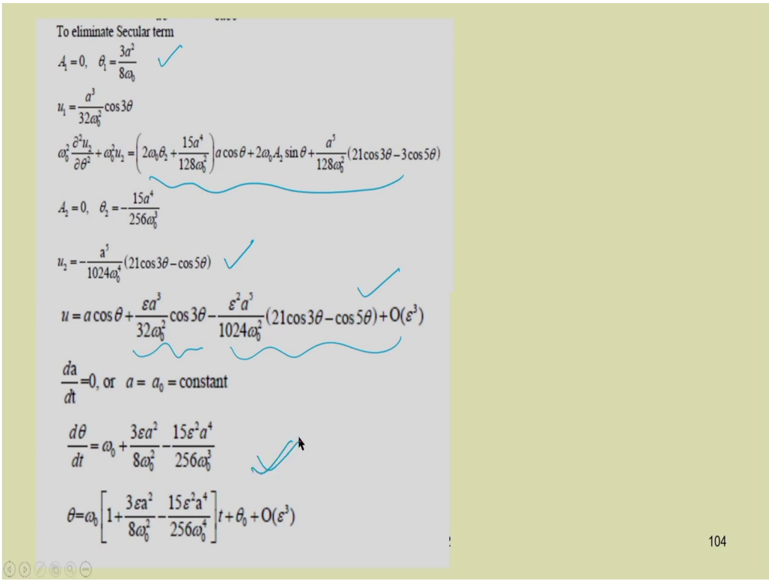

So, this is the equation one can get now separating the order of epsilon one can get the secular term. And from the secular term one can see this A 1 equal to 0 theta 1 equal to 3 a square by 8 0 8 omega 0 u 1 equal to a cube by 32 omega 0 square cos 3 theta.

Similarly, this omega 0 square d square u 2 by del theta square plus omega 0 square u 2 equal to. So, this is the term one can get. So, from these equations, so one can get these expression for. So, A 2 equal to 0 theta 2 equal to minus 15 a 4th by 256 omega 0 cube.

Similarly one can get u 2 also. So, from this equation u 2 can be one can obtain. So, by substituting all this one can get the solution in this form that is u equal to a cos theta. So, this is this term for this u 1 and this term for u 2 that is we have taken a u equal to a cos theta plus epsilon u 1 plus epsilon square u 2.

So, by substituting this u 1 u 2 so, one can get this thing and here we got the condition that d a by d t equal to 0, so a equal to a 0 is constant. So, d theta by d t equal to omega 0 3 epsilon a square by 8 omega 0 square minus 15 epsilon square a 4th by 250 omega 0 cube.

So, theta equal to omega 0 into 1 plus 3 epsilon a square by 8 omega 0 square minus 15 epsilon square a 4th by 256 omega 0 4th. So, this equation closely matches with the solutions what we obtained by using these Lindstedt Poincare method.

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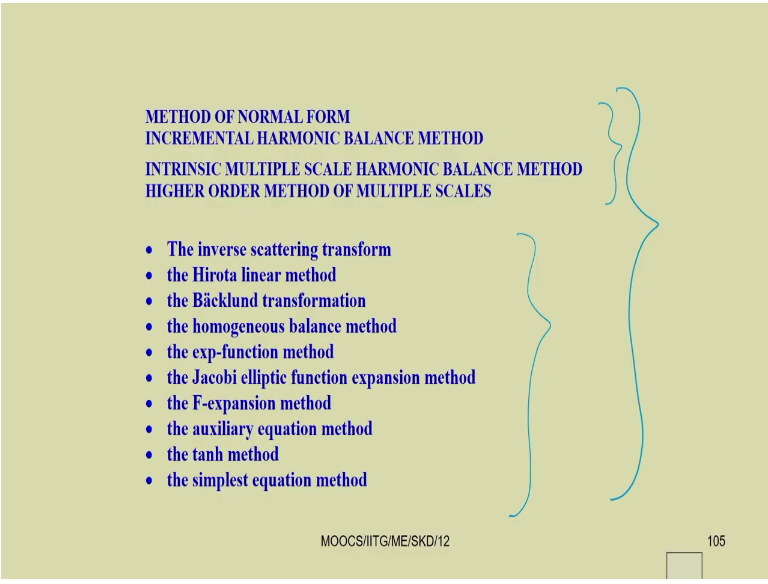

So, this way one can use this KBM method to find the solution of this nonlinear differential equation. So, there are several other methods of solving this nonlinear differential equation. So, for example, you can have the method of normal form incremental harmonic balance method, intrinsic multiple scale harmonic balance method, higher order method of multiple scales the.

So, there are so these are perturbation methods actually and but some exact solution methods are also available. So, these are this inverse scattering transform, the Hirota linear method, the Backlund transformation, the homogenous balance method the exp-function method, the Jacobi elliptic function expansion method, the F expansion method, the auxiliary equation method, the tanh method, the simplest equation method.

So, by using these different methods one can solve this governing given a governing equation of non-linear equation. So, all these methods which have not been covered in this module, so will be taken as an example in the assignments and they will be solved. And you will know all this methods which are applicable to solve this non-linear governing equation.

So, with this we complete this module and next class we will study the other module. Where we may study different equations or we can study the application of these nonlinear equations for single degree of freedom system and multi degree of freedom system.

So, Thank you.