Nonlinear Vibration
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Lecture – 11
Method of Multiple Scales

Welcome to today class of Non-linear Vibration. So, in this module we are studying regarding the solution of non-linear differential equations. So, we known different methods and last few classes we are studying the perturbation methods and in perturbation methods particularly we have seen the straight forward expansion. Then we have studied this Lindstedt Poincare method and last class we have studied regarding the method of multiple scales.

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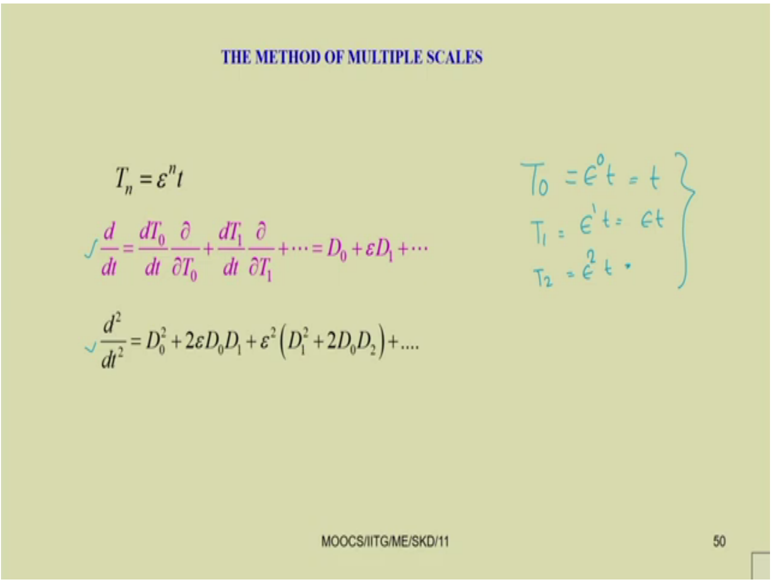

So, in method of multiple scales so we have taken different time scales. So, this T n capital T n is the n can be 0 1 2 so then we can have T 0. So, T 0 will be equal to epsilon to the power 0 t so epsilon to the power 0 equal to 1 so this is equal to t. Then T 1 so this will be equal to epsilon to the power 1 t so this is equal to epsilon t. So, T 2 will be equal to epsilon square t so this is ok. So, this way we can write different time scale.

So, by using different time scale like these T 0 T 1 T 2 similar to our hour hand minute hand and second hand. So, we can write down the derivative and first derivative and second derivative and substitute all those things in the governing equation of motion by also expanding the primary variable.

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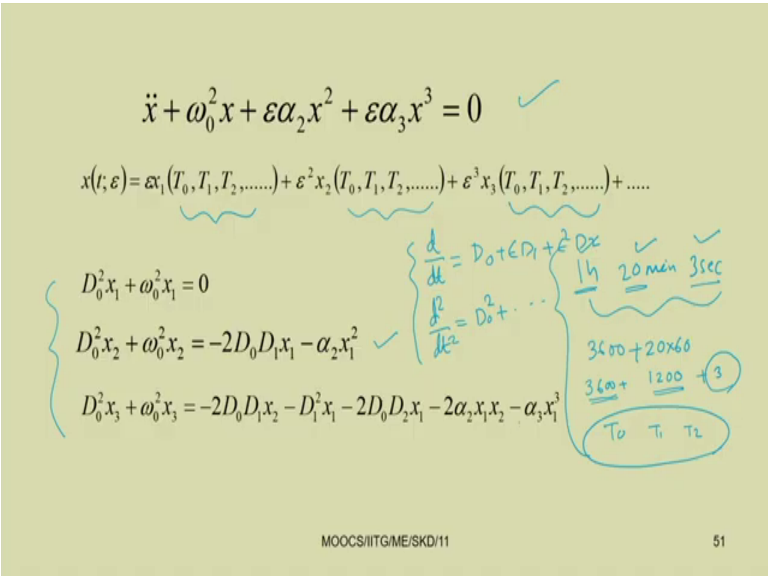

For example, in this case of this equation doping equation with quadratic and cubic non-linearity. So, initially we have expanded this x equal to epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3 where, this x 1 x 2 x 3 we have taken as different time scales. So, instead of defining them in one time scale for example, in second or in minute or in hour so, we are taking different time scales.

For example 1 hour 20 minutes 3 second so, 1 hour so 20 minutes and 3 second. So, if this time we are taking so here we are defining. So, this is 1 hour 20 minutes this is one time scale so this is the other time scale and this is the third time scale. So, T 0 T 1 T 2 that way we are defining. So, somebody can define this thing by using only the second. So, in that case these 3 second can be neglected with respect to the overall time. For example, this 1 hour so 1 hour equal to 3600 30 minutes and 30 minute 1 minute equal to 30 second.

So, this becomes 60 into 60 that is 36 00 second plus 20 minutes is 20 into 60. So, 20 into 60 this becomes 12 00 second so plus 3 second. So, this thing if you see so this becomes 36 00 second plus 1200 second plus 3 second. So, when you are comparing then always there is a tendency to neglect this term so these 3 second term. So, to take care of these 3 second so here what you are doing. So, we are defining different time scales and we are telling so for example, it is 1 hour 20 minutes 3 second.

So, here all the time terms are getting the prominence. So, hour is though 1 hour equal to 36 00 second so here we are telling that is in terms of hour instead of telling in terms of second. Similarly, this minute so 20 minutes is 1200 second, but we are telling that thing as 20 minutes. Similarly these 3 second we are telling in second so that it is getting its prominence that this is another time scale.

So, similar to this so we are taking different time scale that is T 0 T 1 and T 2. And by taking different time scales so then we are giving importance to each scale scaling term and we are not neglecting some time scales with respect to the other. So, here this x 1 x 2 x 3 can be written using different time scale like this T 0 T 1 T 2, x 2 is also written in the same T 0 T 1 T 2 and x 3 is also written. And by substituting these in this governing equation this x and also this we have already written this d by d t; d by d t and d square by d t square so these two terms already we have written.

So, this is equal to so it can be written as D 0 plus epsilon, D 1 plus epsilon square D 2. And, similarly this thing can be written as D 0 square plus so already we have written this term. So, D 0 square plus 2 epsilon D 0 D 1 plus epsilon square to epsilon D 0 D 1 plus epsilon square D 1 square plus 2 D 0 D 2.

So, this way we can write down this terms other terms also can be written and substituting these things in this original equation and separating the order of epsilon. So, we can get so this is epsilon to the power 1 D 0 square x 1 plus omega 0 square x 1 equal to 0. D 0 square x 2 plus omega 0 square x 2 equal to this equation.

So, here you just see left side we are keeping the unknown variable and right side we are keeping the known variable and we have expanded this thing up to our epsilon cube order. So, higher order terms with epsilon 4 and higher have been neglected in this analysis. So, this way or we have kept it up to epsilon square term epsilon cube and higher order terms have been neglected.

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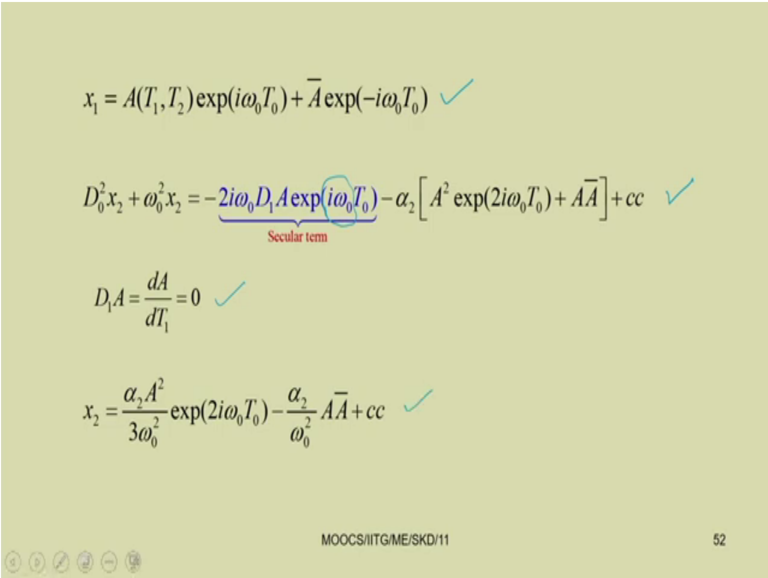

So, by taking this way so we can now find the solution of the first equation in this form that is x 1 equal to a T 1 T 2. So, A is a constant with respect to T 0 so that is why a is written T 1 T 2 A to the power i omega 0 T 0 plus A bar e to the power minus i omega 0 T 0. So, here A bar is the complex conjugate of A.

Now, substituting this x 1 equation in this equation of D 0 square x 2 plus omega 0 square x 2 that is the second equation so we are getting this term. And if you look it carefully you can see this term at we have a term that is minus 2 i omega 0 D 1 A to the power i omega 0 T 0.

So, its particular integral as this term contains the term A to the power i omega 0 T 0. So, its particular integral so which you can find wise substituting so these divided by D 0 square omega 0 square and replacing this D 0 square by minus omega 0 square of these term square of this i omega 0.

So, square of this term you have to find. So, this is i omega 0 square equal to minus omega 0 square. So, minus omega 0 square plus omega 0 square equal to 0. So, in the denominator it becomes 0 so this term tends to infinite. So, as this terms tends to infinite so this is known as the secular term.

So, now eliminating so as the solution actually is bounded. So, there will be no presence of this type of term. So, this type of term should be eliminated. So, to eliminate this term so we must have as this omega 0 not equal to 0 and e to the power i omega 0 T 0 is not equal to 0. So, the only term which is 0 is D 1 A, that is equal to D A by d T 1 equal to 0. So, this way we got this thing so now, substituting this D 1 A equal to 0. So, we can get the or putting the secular term equal to 0 so we can find the expression for x 2 which is nothing, but the particular solution of that expression.

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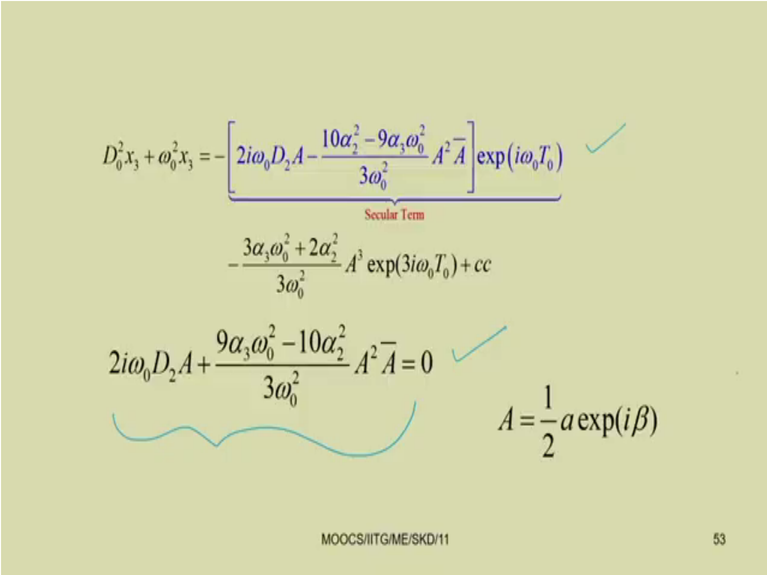

So, now substituting the value of x 1 x 2 in the 3rd equation we can get this equation and here also we have to check the term with e to the power i omega 0 T 0. So, that term will leads to a secular term and that term has to be eliminated.

So, by substituting that thing so we can see. So, this putting this thing 0 so this is the secular term. So, now, by substituting this secular term to be 0 so we can find the required expressions so which will give rise to or which will give our condition to find this coefficients A.

So, here now this 2 i omega 0 D 2 A plus 9 alpha 3 omega 0 square minus 10 alpha 2 square by 3 omega 0 square, A square A bar equal to 0. So, already you know A is a complex number so as a is a complex number we can write this thing by using half a e to the power i beta that is its polar form where a and beta are real numbers.

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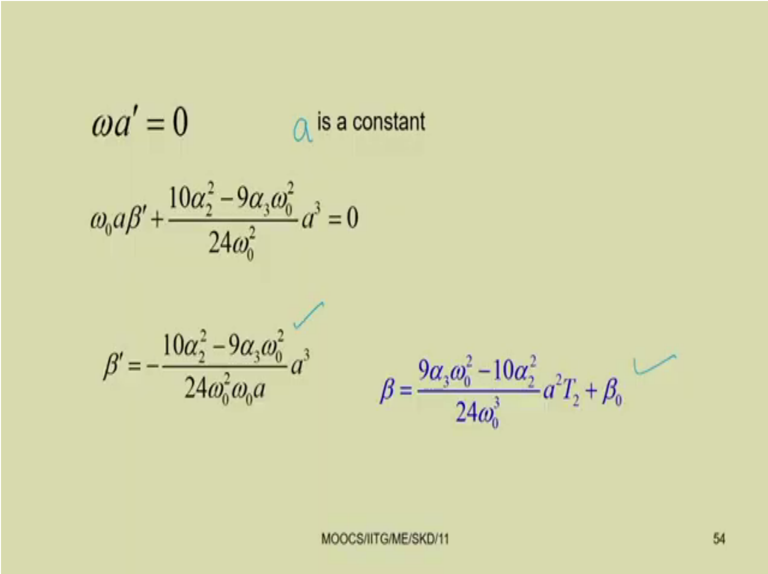

So, now substituting A equal to half a e to the power i beta so in this equation so we can get omega a dash equal to 0. So, a must be a constant as a must be a constant and the second equation leads to omega 0 a beta dash plus 10 alpha 2 square minus 9 alpha 3 omega 0 square by 24 omega 0 square a cube equal to 0.

So, from this thing so we can get beta dash expression so now, you just see in the right hand side so this is a constant so beta dash is a constant so beta must be; so expression for beta will be equal to this. So, it must so as it is a constant so integration of this thing equal to this into T 2 so you just see this dash with respect to T 2 so this beta equal to this one.

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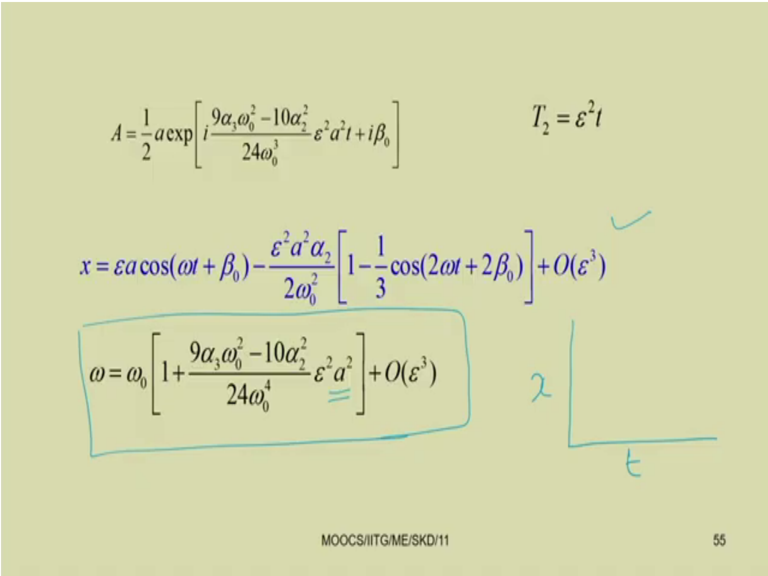

So, now I substituting this beta value in this original equation so we can get the expression for A and also get the expression for x. So, from this thing so we can get a relation between omega and a. So, like in linear system where omega and a are independent here omega is a function omega that is frequency of the non-linear response is a function of the amplitude of oscillation.

So, we can plot this equation to get the relation between omega and a and then knowing this relation between omega and a we can find the value of x and we can plot x verses t to get the free vibration response of the system. So, this is an assignment to plot or find the response of free vibration of a system with quadratic and cubic non-linearity you can change different value of the system parameters like this alpha 1; alpha 1 equal to omega 0 square alpha 2 and other parameters and you can plot different response of the system.

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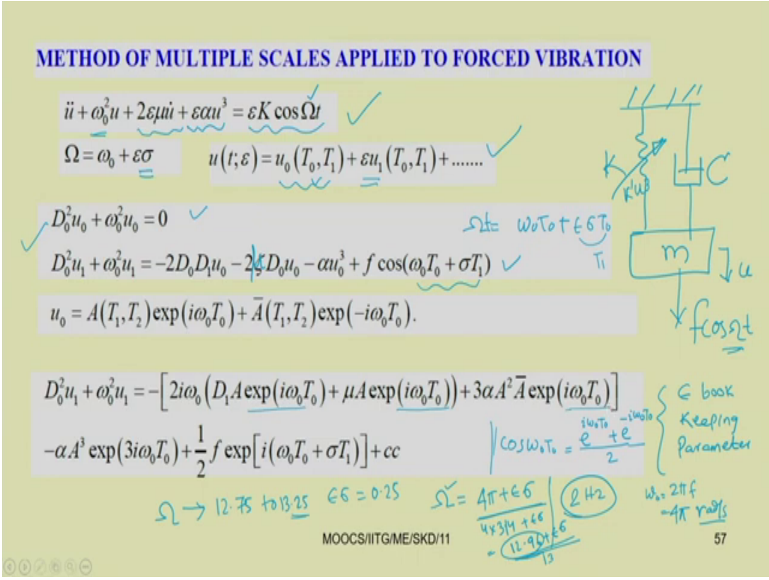

So, today class we are going to start. So, this is for the free vibration case. So, let us take one example one more example so two more two or three more examples we will see today class and then we will start the method of averaging also. So, let us apply this method of multiple scale to forced vibration of the system.

So, in case of forced vibration so let us take a spring mass damper system. So, we have a spring mass damper system subjected to some forcing. So, this is the spring mass damper system. So, this is the mass this is the spring and we have taken the spring to be non-linear. So, the spring is taken to be non-linear and this is the damping C this is mass m and it is subjected to a force. So, this type of force f cos omega t; so, this type of system equation can be written already you know different methods to write down find the equation of motion. So, it can be conveniently written.

So, assuming we have a cubic non-linear term or cubic non-linear spring then this equation can be written by using this equation that is u double dot plus omega 0 square u plus 2 epsilon mu u dot plus epsilon alpha u cube equal to epsilon k cos omega t. Here epsilon is a book keeping parameter whose value book keeping parameter. And it is used to show the terms are one order less than the linear term that of the linear term.

So, this is the linear term omega 0 square u. So, this is the damping term 2 epsilon mu u dot. So, here damping is one order less than the linear that is why we are using this epsilon here. Similarly, this non linearity we are assuming one order less than the linear one that is why we are using 1 epsilon here. Similarly, the forcing we are assuming to be 1 order less than that of the linear one. So, that is why you know our equation we are writing u double dot so here u is the displacement. So, u is the displacement of the mass so u double dot. So, that is the acceleration term until or think in that way.

So, actually m u double dot is the inertia force and then this m is divided everywhere so that k by m gives rise to this omega 0 square and then c by m give rise to 2 epsilon mu u dot generally it is written also zeta 2 epsilon zeta u dot either you can write mu or zeta. So, plus these so if we are using some non-linear stiffness term for example, k dash. So, this is k dash u; u k dash u cube. So, here k dash u cube divided by m so that will give you epsilon alpha. So, this k dash by m will be epsilon alpha.

So, that is why this term becomes epsilon alpha u cube and in the forcing side forcing f cos omega t is there. So, divided by this m; m is divided everywhere so this f by m can be written. So, this f by k f by m can be written as epsilon k and cos omega t. So, this way you can derive this equation of motion. So, after knowing this equation of motion so let us apply the method of multiple scale to find the response of the system.

So, here we are interested to study near the resonance condition simple resonance condition that is when this natural frequency, when this external frequency omega, omega is the external frequency. When the external frequency omega is near to the natural frequency omega 0. So, omega 0 that is equal to k by root over k by m so we are taking so equal to omega 0.

So, by using a detuning parameter; detuning parameter sigma is known as the detuning parameter. So, using a detuning parameter so we can write the nearness of the external frequency to the natural frequency. For example, so let the natural frequency is 2 Hertz if the natural frequency is 2 Hertz or we can write either you can write so natural frequency is 2 Hertz or so you can write in omega also. So, 2 pi f is the natural frequency 2 pi f; f equal to 2 here we have taken so omega 0 equal to so this is equal to 4 pi.

So, you have taken 4 pi is the, 4 pi radian per second is the natural frequency of the system. So, if you are taking omega 0 equal to 4 pi radian per second so we are interested to investigate near omega equal to 4 pi plus minus epsilon; plus minus epsilon or we can write only the plus also. So, plus minus epsilon can be written. So, epsilon sigma plus minus epsilon sigma so this sigma either you write plus minus epsilon sigma where sigma will be positive or you take a 4 pi plus epsilon sigma where sigma can take a negative value also so; that means, near this 4 pi you are interested to study the value of or study the external frequency.

So, pi equal to 3.14 so 4 into 3.14 plus epsilon sigma. So, this becomes 12.96 plus minus epsilon sigma. So, we can take near to this 12.96 that is we can for example, we may take we can vary the value of omega. So, we can vary the value of omega for example, from 12.75.

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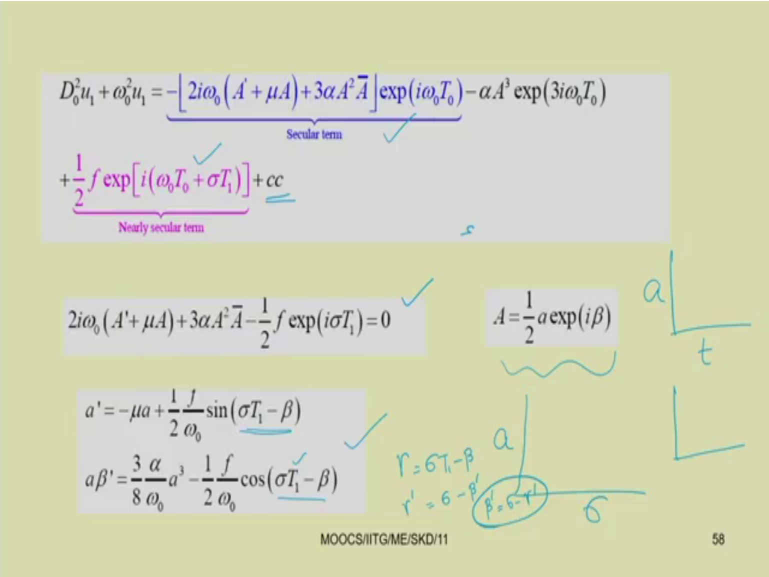

So, let us vary this omega from 12.75 to 13.25. So, this way we can vary so here so by varying this way so here you can see the sigma is nothing, but. So, this 12.96 approximately you can write equal to 13. So, 13 plus epsilon sigma. So, 13 plus epsilon sigma if you are taking. So, these sigma we are taking to be so 0.25. So, epsilon sigma epsilon sigma equal to 0.25.

So, taking epsilon so what we have taken so epsilon sigma equal to 0.25. So, taking this epsilon equal to 0.1 so we can have the sigma value. So, epsilon sigma equal to 0.25 so epsilon equal to 0.1. So, if you write epsilon equal to 0.1, so you can find sigma equal to 0.25 by 0.1. So, this way you can find the value of sigma and you can use that value that variation of sigma to show the nearness of the; nearness of the external frequency to the natural frequency.

So, we by using this detuning parameter sigma we can vary the system parameter or the frequency to show the nearness of that frequency to that of the natural frequency. Now again expanding this u t epsilon in this form u 0 t T 0 T 1 plus epsilon u 1 T 0 T 1.

So, up to we can take two terms or more terms also we can take depending on our applications we can take more and more number of terms. So, when you are taking more number of terms so it will be difficult to solve manually. So, if one to solve manually you can take few number of terms, but if you can use the symbolic software and write your own code so you can take more number of terms also.

So, by substituting this u equal to u 0 plus epsilon u 1 so you can get the expression for u double dot equal to u 0 double dot plus epsilon u 1 double dot, here double dot also this D square by D T square. So, you can replace it by the expression you have already seen that is D 0 square plus other terms are there so already you have seen that expression.

So, by substituting this thing and separating of the order of epsilon so you can get this equation. So, this equation becomes D 0 square u 0 plus omega 0 square u 0 equal to 0. And so this is you can see this is the order of epsilon to the power 0. So, here unlike in the previous example you just see we have taken a term u 0 plus epsilon u 1, previously we have taken epsilon u 0 plus epsilon square u l u 1 plus epsilon cube use 2 so that way we have taken.

But now we have taken a constant term plus this epsilon u 1. So, [FL] we are studying this force vibration so in this case our response study state response cannot be 0. So, for that purpose there will be so we are taking non zero value of u 0. So, that is why you are putting u 0 plus epsilon u 1 instead of putting this term epsilon u 0 plus epsilon square u 1. So, now the first equation becomes D 0 square u 0 plus omega 0 square u 0 equal to 0 and second equation becomes D 0 square u 1 plus omega 0 square u 1 equal to minus 2 D 0 D 1 u 0 minus 2.

So, here we can take so it is written in terms of zeta or you can write this thing using this 2mu so instead of zeta so it will be mu so this is 2 mu D 0 u 0 minus alpha u 0 q plus f cos omega 0 T 0 plus sigma. So, here you just see this sigma is replaced by omega 0 T 0 plus omega 0 T 0 plus ok. So, we can see this so we can write this capital omega equal omega t, omega t equal to omega 0 T 0 plus epsilon sigma t 0. So, epsilon T 0 is T 1 epsilon T 0 becomes t 1so that is why it is written omega 0 T 0 plus sigma T 1.

So, epsilon T 0 becomes t 1 so that is why this is epsilon T 1 ok. The solution of D 0 square u 0 plus omega 0 square u 0 equal to so we can write equal to a e to the power e to the i omega 0 T 0 plus A bar e to the power minus i omega 0 T 0 and here A is a constant with respect to T 0. So, it can be a function of T 1 T 2 in later we can see whether it is a function of T 1 T 2 or not.

In the previous case we have seen a becomes a constant finally, but in this case let us see so what will happen to A. So, here initially this A is not a function of T 0 because this D 0 square u 0 plus omega 0 square u 0 equal to 0 it solution A so it will not be a function of T 0.

So, knowing this a u 0 so now we can substitute this u 0 in this expression; so by substituting u 0 in this expression. So, we can write D 0 square u 1 plus omega 0 square u 1 equal to minus 2 i omega 0 D 1 A e to the power i omega 0 T 0 plus mu A e to the power i omega 0 T 0 plus 3 alpha A square A bar e to the power, A bar e to the power i omega 0 T 0 minus alpha A cube e to the power 3 i omega 0 T 0 plus half f.

So, this f cos omega T cos cos term can be written as cos expression can be written cos omega T can be written cos omega 0 T 0. So, it can be written as e to the power, e to the power i omega 0 T 0 plus e to the power minus i omega 0 T 0 divided by 2. So, this way you can write this expression for cos omega T 0 because e to the power i theta equal to cos theta plus i sin theta. So, e to the power i theta equal to cos theta plus i sin theta. So, using that expression you can write e to the power minus i theta i equal to cos theta plus minus i sin theta.

So, adding these 2 e to the power i theta plus e to the power minus i theta. So, this becomes 2 cos theta. So, cos theta becomes e to the power i theta plus e to the power minus i theta by 2. So, by substituting this thing so in this forcing term so you can write or you can find the expression for.

So, we can find the expression and here you can note so there are many terms we e to the power i omega T 0. So, we can combine all these things and we can write this expression next we can write this expression again where we can write this D 0 square u 1 plus omega 0 square u 1 equal to minus 2 i omega 0 A dash plus.

So, this D 1 a is written as A dash so plus mu A plus 3 alpha A square A bar e to the power i omega 0 T 0 minus alpha A cube e to the power 3 i omega 0 T 0, plus half f e to the power i omega 0 T 0 plus sigma T 1 plus its complex conjugate of the preceding term. Here the c c represent the complex conjugate of the; of all the preceding terms. So, here you have 3 terms so c c represent the complex conjugate of all these 3 terms.

So, here now you already you know that the coefficient has this is the coefficient of e to the power i omega 0 T 0 or the first term contain the term e to the power i omega 0 T 0. So, this term contain e to the power i omega 0 T 0 its particular integral will tends to infinite. So, that is why this is known as secular term, but now you look this term. So, here also when sigma equal to 0; sigma equal to 0 so this exactly equal to becomes i omega 0 T 0.

So, this will also be a secular term provided the sigma is 0, but if sigma is nearer to 0 also. So, it will tends to infinite or it will leads to a secular term that is why it is known as nearly secular term or mixed secular term. So, it is nearly secular term. So, due to the presence of sigma T 1 so this is a mixed or nearly secular term. So, this term will be secular when sigma tends to 0; that means, when you are studying or taking this term sigma nearer to the omega 0 term then these external frequency will leads to a secular term. So, this term will lead to a secular term.

So, to get a bounded solution we must kill this secular and nearly secular term or we must eliminate the secular and nearly secular term. So, eliminating this secular and nearly secular term so we have this expression.

So, 2 i omega 0 A dash mu A, plus mu A plus 3 alpha A square A bar minus half f e to the power i sigma T 1 equal to 0. So, you just see so this term actually this term into e to the power i omega 0 T 0 equal to 0. So, e to the power i omega T; T 0 e to the power i omega 0 T 0 cannot be 0 exponential function cannot be 0 as t tends to infinite then only it will become 0.

So, these term the product are so 2 i omega 0 A dash plus mu A plus 3 alpha A square A bar minus half f e to the power i sigma T 1 become 0. So, as A is a complex number as explained before so A can be substituted by half A e to the power i beta and substituting this and separating the real and imaginary terms so we can get this 2 expression. So, that is A dash equal to minus mu A plus half f by omega 0 sin sigma T 1 minus beta and a beta dash equal to 3 by 8 alpha by omega 0 a cube minus half f by omega 0 cos sigma T 1 minus beta.

So, for steady state so you can solve this equation by using Runge Kutta method to find the relation between this a and omega or a and sigma. So, this is known as your frequency response plot. So, you can plot the frequency response plot and you can plot this so, this will be given as an assignment. And for steady state or you can plot also a verses t how a is varying with respect to time and how this gamma ok, beta is also varying with time you can plot ok.

So, now, to write down this equation in its autonomous form. So, you just see explicitly so it is a time term is coming in the right hand side. A time term T 1 which is equal to epsilon T. So, is coming in the right hand side. So, by writing another term gamma equal to sigma T 1 minus beta so we can write down this first expression equal to gamma a dash equal to minus mu a plus half f by omega 0 sin gamma. Similarly, the second equation can be written so you just see gamma equal to this.

So, gamma dash that is D gamma by D T 1 will be equal to sigma minus beta dash D beta by D T equal to D beta by D T 1 equal to beta dash. So, gamma dash equal to sigma minus beta dash or beta dash can be written beta dash can be written sigma minus gamma dash. So, by substituting this sigma minus beta dash here so we can write the second expression.

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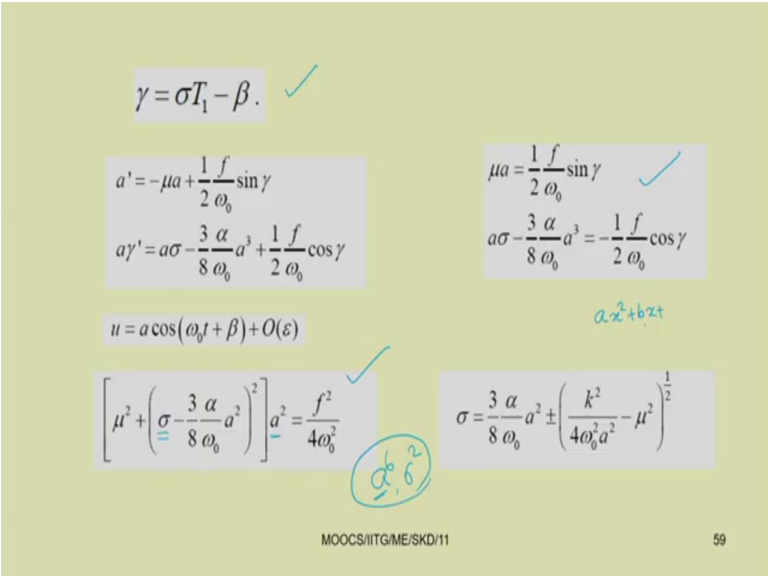

So, by substituting this way you can write the equation. So, substituting gamma equal to sigma T 1 minus beta so we can write a dash equal to minus mu a plus half f by omega 0 sin gamma and a gamma dash equal to a sigma minus 3 by 8 alpha by omega 0 a cube plus half f by omega 0 cos gamma.

So, for steady state this a dash and gamma dash will be equal to 0 so the resulting solution we can write in terms of a ok. So, now by putting that is equal to. So, this is the expression we will get for steady state solution a dash we have substituted 0 and gamma dash also substituted 0.

So, our equation becomes mu a equal to half f by omega 0 sin gamma a sigma minus 3 by 8 alpha by omega 0 a cube equal to minus half f by omega 0 cos gamma. Now, by squaring and adding so sin square gamma and cos square gamma so we have. So, this you can square and add so, by squaring and adding so we can get an expression like this where it is mu square plus sigma minus 3 by 8 alpha by omega 0 a square whole square into a square.

So, you just see so this equation contain a square whole square a to the power 4 and outside also you have a square. So, a term where we have a to the power 6 and the same expression so is in the form of a quadratic that is sigma square quadratic in sigma.

So, this expression is quadratic in sigma, but 6th power in terms of displacement a. So, we can write down this quadratic equation and we know the solution of a quadratic equation. And so that is if we have a equation for example, a x square plus b x plus c equal to 0 or, here we have the sigma.

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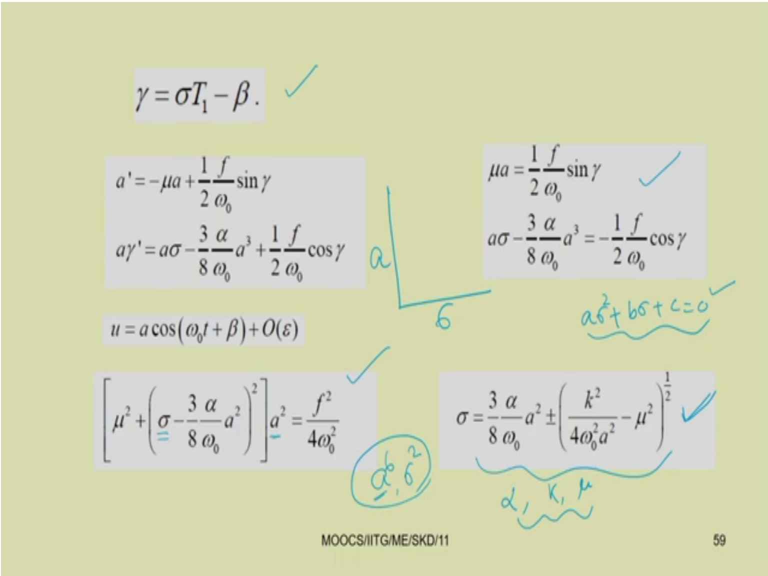

So, we can write in terms of sigma also or let a sigma plus b a sigma square plus b sigma plus c equal to 0. So, you know the solution which is can be written where sigma equal to minus b plus minus root over b square minus 4 a c by 2 a using that expression so you can find the expression for sigma.

So, you can find the expression for sigma which is equal to sigma equal to 3 by 8 alpha by omega 0 a square plus minus k square by 4 omega 0 square a square minus mu square to the power half. So, you can study the response that is the frequency response of the system by plotting a verses sigma for different value of for different value.

So, you can have different value of alpha, different value of k, different value of mu. So, by changing all these parameters so you can study the frequency response plot. So, this will be given as an assignment for you to plot these a verses sigma and study the frequency response of the system and its effect due to damping due to this non-linearity and due to so you just see so you have a term k so which is for the forcing amplitude of forcing.

So, those things we will study when we are going to study exactly on the response of the system. So, this way you can solve this differential equation by using method of multiple scale. So, in this example we have seen the study often 4th vibrated system if the system is subjected to forcing a weak forcing. So, here you have seen the system is subjected to one weak forcing weak forcing means the forcing term is one order less than that of the linear term.

So, if it is not one order less than that of the linear term so we can tell that the forcing is strong. So, in case of the strong forcing so later we will see how we can use this method of multiple scale or other method to find the response of the system.

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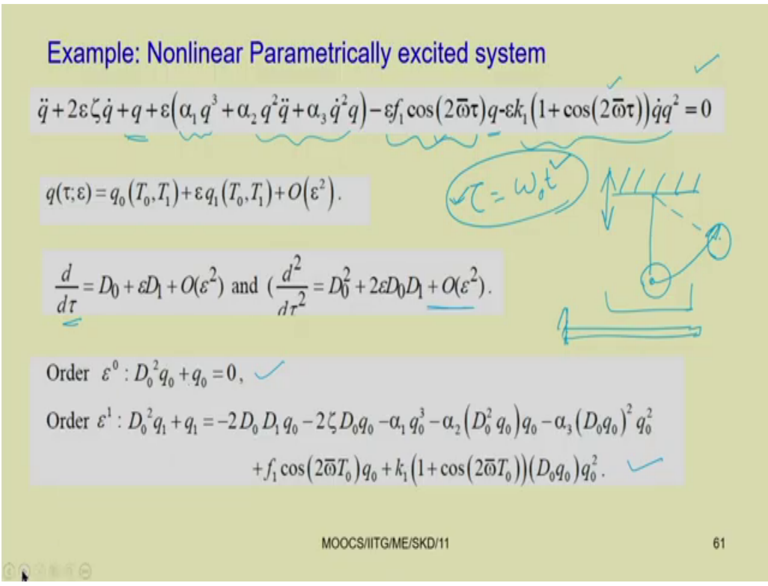

So, next so let us we can take another example also. So, let us take another example where we have studied a system a parametrically excited system. So, already I have given you the example of a. So, for example, we have seen the example. So, example of a simply; of a simple pendulum where the platform is moving up and down. So, if the platform is moving up and down. So, the motion of the platform is taking place in this direction, but the motion of this pendulum is taking place in this direction.

So, the pendulum is moving, so the pendulum is moving in a direction that is perpendicular to that of the motion of the platform. So, in such a cases the equation can be written where the forcing term is the coefficient of the response and this type of excitations are generally known as parametric excitation. So, the let us take the example of a parametrically excited system and let us see how we can use this method of multiple scale to solve this type of equation.

So, actually this equation is written for a system base excited system or ok. So, there are several types of systems can be taken which will represent this thing. So, this is generalized case is taken so in this case. So, we have a cantilever beam subjected to; so this cantilever beam subjected to magnetic field so this is a conducting beam and subjected to magnetic field.

So, due to this presence of this magnetic field so it will be subjected to a compressive load and so due to that compressive load and considering this large amplitude of vibration. So, the system can be considered as that of a forced and parametrically excited systems also or sometime the that forcing is weak then it can be that of a parametrically excited system. As the system is considered to be having large deflection so it can be considered as a non-linear systems also.

So, you can see physically this equation contains, like I previously told you have divided this m here in this equation and you can write this thing by using this non dimensional form. So, by writing this non dimensional form so you can have this equation in this way; so that is q double dot plus 2 epsilon zeta q dot plus q plus epsilon in to alpha 1 q cube, plus alpha 2 q square q double dot plus alpha 3 q dot square q minus epsilon f 1 cos 2 omega bar tau into q ok.

So, minus epsilon k 1 into 1 plus cos 2 omega bar tau into q dot q square. So, here clearly you can observe this equation. So, in this equation you just observe the non-linear term. So, this is alpha 1 q cube so this is geometric non-linear term and these two terms clearly you can see. So, here q square into q double dot q double dot is acceleration so it is multiplied by this displacement square. So, due to the presence of acceleration so this term is known as inertia non-linear term and this is product of 2 velocity, product of two velocity is acceleration.

So, due to the presence of acceleration so this is also known as inertia non-linear term. So, these two terms are inertia non-linear term and this is geometric non-linear term. So, this term which is a time barring term is coefficient of q so that is why it is known as a parametrically excited term.

And this part of this equation you can see this forcing term is the coefficient of this non-linear variable that is q dot into q square. So, this q due to presence of this q dot and q square so this is a non-linear term and as this time barring term is coefficient of this variable so that is why also this is non-linear parametrically excited system.

So, now proceeding in the similar way as we did before taking this q equal to q 0 plus epsilon q 1 and taking this d by d tau; so. So, here actually when we have derived this equation so we have taken this tau equal to omega 0 t. So, that is why this tau is the non dimensional time so you can have a term k by m which is equal to omega 0 square. So, now this d square by d t square can be written as omega 0 square d square by d tau square.

So, dividing that term everywhere that is why this coefficient of this q becomes 1, that is why we are using this non dimensional time tau here instead of the real time t. So, d by d tau can be written here as D 0 plus epsilon D 1, similarly this d square by d tau square can be written as D 0 square plus 2 epsilon D 0 D 1. So, here you just note that we have neglected the term with epsilon square. So, when you are manually doing this thing you may neglect that higher order terms, but if we are doing using the symbolic software. So, conveniently you can use this higher order term to write down this equation also.

So, this epsilon now separating the terms with coefficient of different order of epsilon so we can write down these equation. So, order of epsilon 0 so this is the equation with order of epsilon 0 so this is the equation with order of epsilon 1. So, we know the solution of this order of epsilon 0 that is q 0 equal to like previous case we can here also write q 0 equal to a T 1 T 2 e to the power i T 0 plus A T 1 T 2 e to the power minus i T 0. So, here you just see omega 0 equal to 1 that is why it is i T 0.

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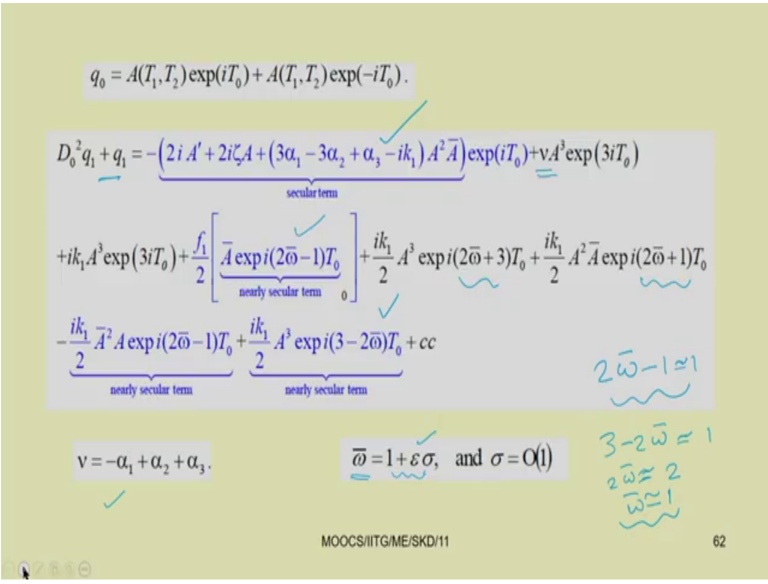

So, now we can write this D 0 square q 1 plus q 1 equal to minus 2 i A dash plus 2 i zeta A plus 3 alpha 1 minus alpha 3 alpha 2 plus alpha 3 minus i k 1 into a square A bar e to the power i T 0 plus nu into A cube e to the power 3 i T 0 plus i k 1 A cube e to the power 3 i T 0 plus f 1 by 2, A bar e to the power i 2 omega bar minus 1 T 0 with this higher order.

So, higher order frequency or higher frequency 2 omega bar here and 2 omega bar here also and here you are taking another term that is minus i k 1 by 2 A bar square A e to the power i 2 omega bar minus 1 T 0 ok. So, here omega bar that is the external frequency we are taking equal to 1 plus epsilon sigma. So, like previously explanation so this sigma is the detuning parameter. So, here nu we have taken nu equal to minus alpha 1 plus alpha 2 plus alpha 3 which is used here.

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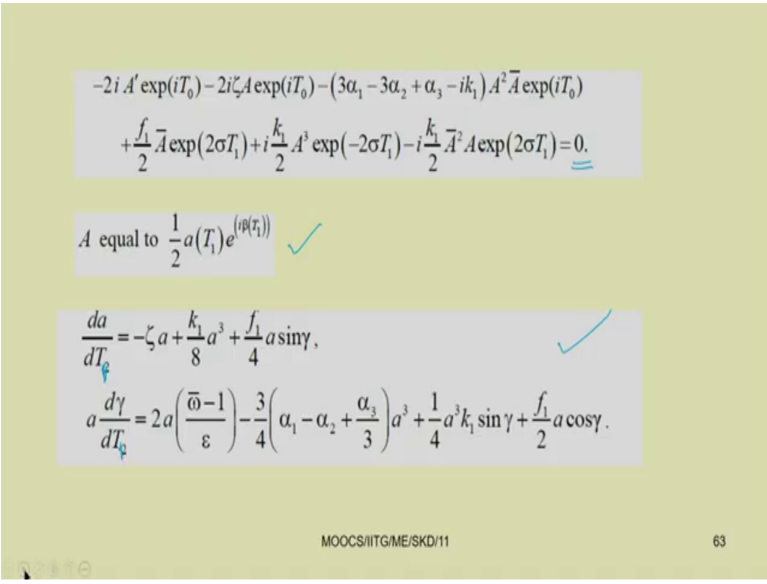

So, now you can see the terms clearly you can observe from the terms that coefficient of e to the power i T 0. So, here coefficient of T 1 is 1 that is why it is i T 0. So, that due to the presence of the term e to the power i T 0. So, this is a secular term and this term become secular so this 2 i omega 1 minus omega 1 so when this 2 omega bar minus omega 1 becomes 1. So, then this term will become secular.

So, from here, so we know this omega bar equal to 2 or 2 omega bar equal to 2 or omega bar nearly equal to 1. So, we can take the condition that is omega bar nearly equal to 1. So, to write down the nearness of 1 so we can use this detuning parameter. So, this is a nearly secular term.

Similarly, here we can see 2 omega bar minus 1 so already we have seen this term. And another term we are seeing this is 3 minus 2 omega bar also. So, 3 minus 2 omega bar can be nearly equal to 1 so in that case; in that case also we can have this 2 omega bar equal to, 2 omega bar equal to 2 or omega bar nearly equal to 2 nearly equal to 1.

So, this term and this term conditionally gives the a frequency equal to that is omega bar equal to 1. So, as conditionally they are becoming 1 so they are known as nearly secular term. So, taking so, as already we have explained, so this secular terms and mixed secular term or nearly secular term must be equal to 0 to have a bounded solution because in our actual system the solution is bounded.

So, by eliminating those terms to eliminate means by putting right hand side this is equal to this term equal to 0 and by taking this a equal to half a e to the power i beta. So, here a and beta are function of t 1. So, we can write by separating the real and imaginary term we can write down this equation. So, that is equal to d a by d T 2 equal to minus zeta a k 1 by 8 a cube plus f 1 by 4 a sin gamma we can use this gamma term by in a similar way. So, we can write down these 2 equation.

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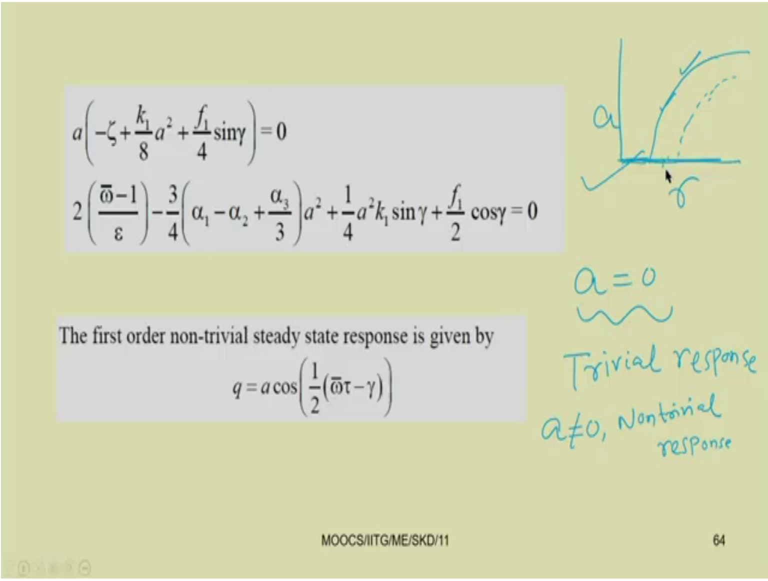

So, after writing these 2 equation so for steady state. So, we can write this d a by d T equal to 0 and d gamma by d T equal to 0 also. So, this T 1 so this will be T 1 I think so this is T 1 T 1 not T 2. So, by putting this equal to 0. So, we can get the expression for the steady state response. So, the first order non trivial response. So, here you can see from the first equation a into this thing equal to 0. So, either a will be 0 or this part will be 0 so when a equal to 0. So, when a equal to 0 so the response is known as trivial response so trivial response.

So, if a not equal to 0 so a not equal to 0 so this is non trivial response. So, we may have so from the first equation so it is clear that we will have both trivial state and non trivial state; that means, if you plot this a verses sigma. So, up to certain distance so it will be stable or unstable; that means, this 0 line is also a solution. So, when we will steady stability we will know more about this thing. So, for example, you may have a response like this. So, here so this is a stable plan so up to certain distance the response equal to 0. So, and then this becomes unstable you have the non trivial response then this is the trivial state is unstable and you can have a stable and unstable response.

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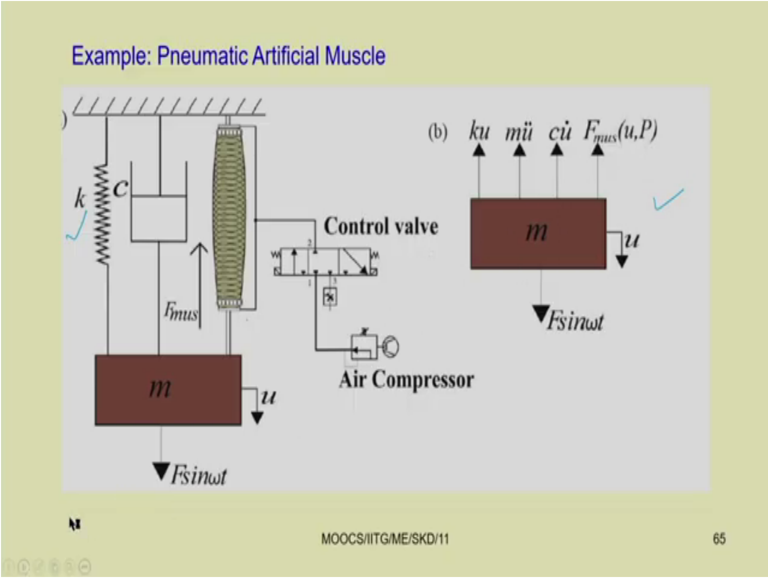

So, later we will steady that you can have a pitch for bifurcation here you have a pitch for bifurcation here, and those things we will see and we will also study regarding the jump up phenomena jump down phenomena. When we will study detail regarding the response of a particular system. So, the first order non trivial steady state response is given by this equation.

So, now you can take another example for example, let us take that of a pneumatic artificial muscle. So, this is a pneumatic artificial muscle so we can study more regarding this pneumatic artificial muscle. So, it has a breaded ring you can see out of this thing and a muscle rubber like or balloon like rubber is there inside and a breaded things is there.

So, we can see different type of artificial muscles one example is the (Refer Time: 49:42) muscle. So, and you have a spring here and damper here. So, you can write down by using this Newton second law. So, you can write down this equation of motion so here by using a pump and a controller. So, you are feeding you are giving this pressure to the muscle. So, due to that pressure of the muscle the muscle get contracted and due to that it will pull this mass and the mass gets its motion.

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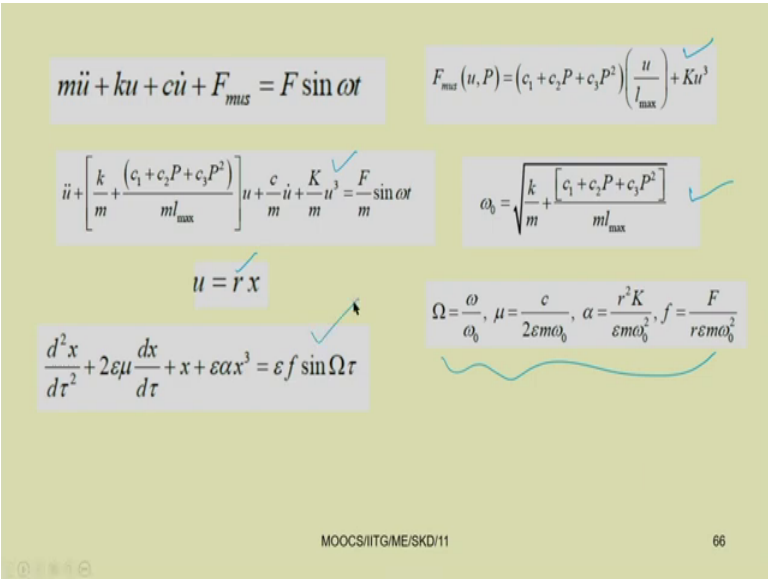

So, the equation of motion you can write conveniently by drawing this or drawing the free body diagram and this F muscle also you can characterize you can write down this F muscle expression by doing some experiment how this muscle force is varying with different pressure.

So, here one expression is written for F muscle. So, you can write down this equation of the system in this form mu double dot plus K u plus cu dot plus F muscle equal to f sin omega t. Now, putting this F muscle expression in this way so we have a non-linear equation of motion. So, you just see this equation by substituting this omega 0 equal to this term.

So, this equation and u equal to r x where r is a scaling parameter already we have defined what is a scaling parameter. And defining these non dimensional parameter the like this non dimensional frequency omega by omega 0 damping equal to c by 2 epsilon m omega 0 alpha equal to r square K by epsilon m omega 0 square and f equal to capital F by r epsilon m omega 0 square.

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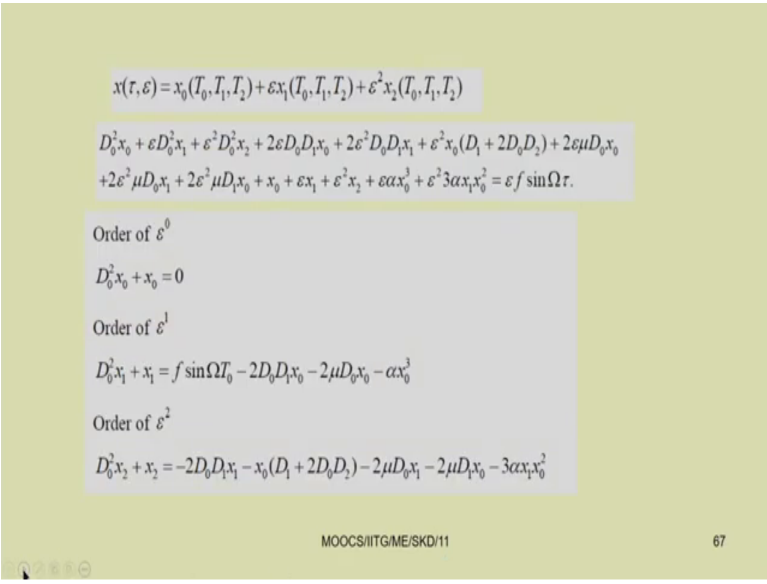

So, you can write this equation in a very convenient way like this d square x by d tau square plus 2 epsilon mu d x by d tau plus x plus epsilon alpha x cube equal to epsilon f sin omega t.

Like the previous cases also here you can use this same method of multiple scale to write down x equal to x 0 plus epsilon x 1 plus epsilon square x 2, previous example we have taken only 2 terms here 3 terms have been taken.

So, then substituting in this equation and separating of the order of epsilon 0 epsilon 1 and epsilon 2 so we can get this expression. So, then solution of this D 0 square x 0 plus x 0 you can write in the similar way.

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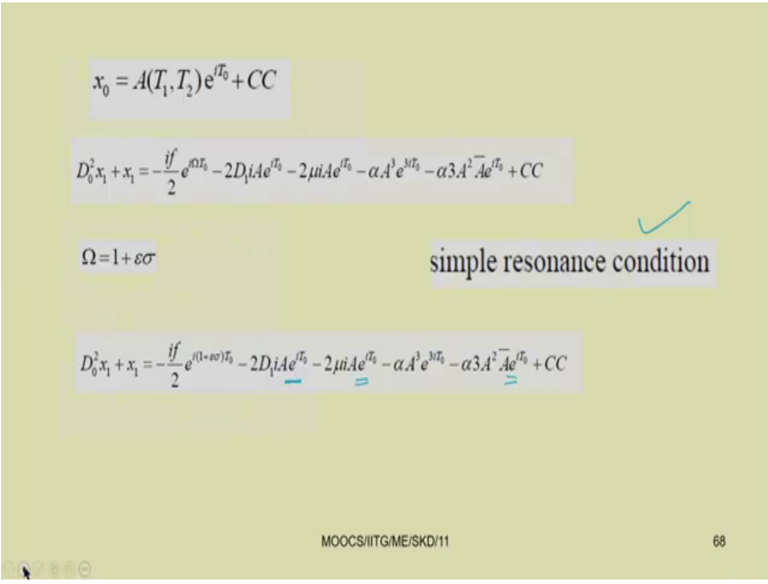

So, this is equal to a e to the power i T 0 plus its complex conjugate then D 0 square x x 1 plus x 1 can be written in this form and by taking this omega equal to 1 plus epsilon sigma like the previous case where we have studied the simple resonance condition.

So, we can write this D 0 square x 1 plus x 1 equal to i f by 2; minus i f by 2 e to the power i into 1 plus epsilon sigma T 0 minus 2 D 1 i a e to the power i T 0 minus 2 mu i a e to the power it 0 minus alpha A cube e to the power minus e to the power 3 i T 0 minus alpha into 3 A square A bar e to the power i T 0 plus its complex conjugate, complex conjugate of the preceding terms.

So, you just see the terms containing e to the power i T 0 e to the power i T 0 e to the power i T 0 will give rise to secular terms.

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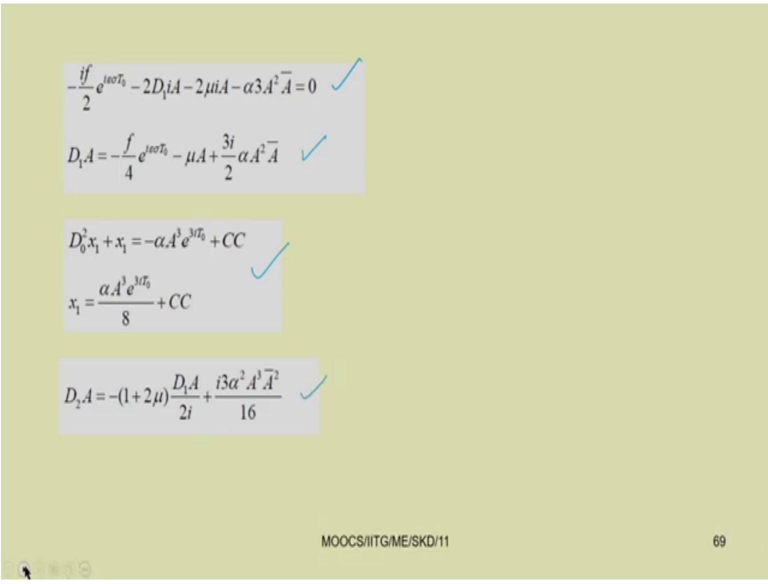

So, we must eliminate the secular terms to get the bounded solution. So, if you eliminate the secular term so this is the expression you got similarly here you got that means. So, from this thing you can write this D 1 A equal to this way. Now, substituting this in the second equation we get the expression for x 1 similarly we can get D 2 A by substituting this x 1 x 2 in 3rd equation and eliminating the secular term we can get D 2 A, then we can write you can note this point actually.

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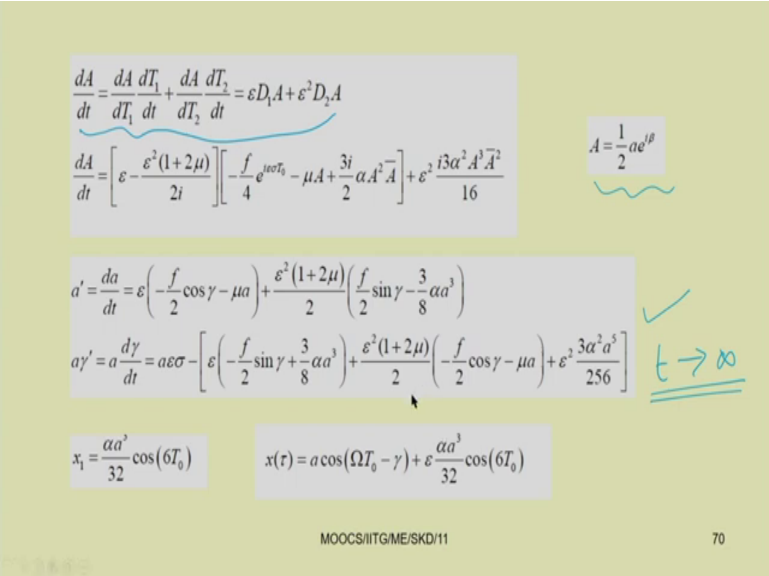

So, when you are doing this higher order expansion so you can note this d A by d t equal to so d A by d T equal to d A by d T 1 into d T 1 by d T plus d A by d T 2 into d T 2 by d T so that is equal to. So, as you know this d T 1 by d T equal to epsilon.

So, this becomes epsilon D 1 A plus epsilon square D 2 A, already we know the expression for D 1 A and D 2 A by substituting these things we got this expression d A by d T is this. Then by substituting this A equal to half a e to the power i beta like the previous case and separating the real and imaginary term. So, we can have the expression for a dash and gamma dash.

So, for steady state you know this a dash and gamma dash they will not be a function of time steady state where it is not a function of time t tends to infinite; t tends to infinite is the steady state response. So, then so the system this a and gamma must not be a function of time.

So, that is why we can get these two equation and by solving these two equation. So, you just see previous case the expressions are very simple. So, you got some closed form solution, but in the present case the expression are not simple. So, you cannot get a simple equation. So, you have to solve it numerically to get the expression.

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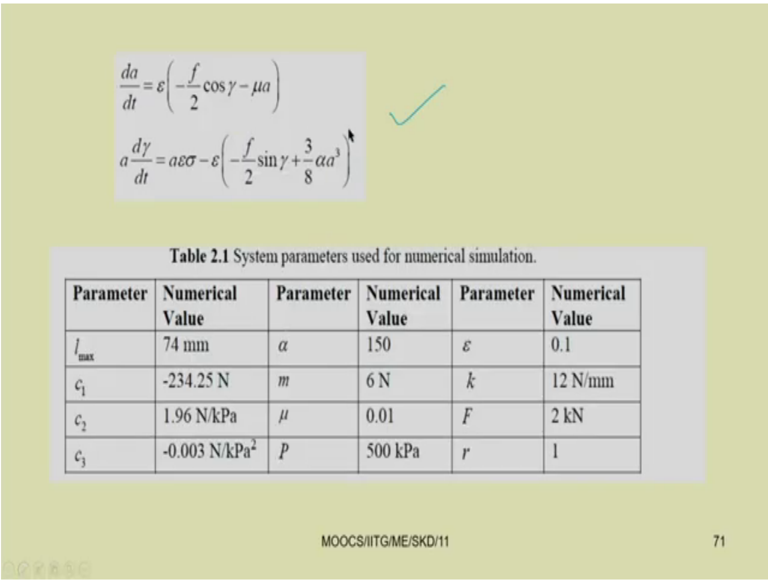

So, if you take the first order solution then it will be simplified to this d a by d t equal to epsilon into minus f by 2 cos gamma minus mu a; a d gamma by d t equal to a epsilon sigma minus sigma minus epsilon into minus f by 2 sin gamma. Plus 3 by 8 alpha a cube so when you are taking higher order. So, you just see you are getting additional terms.

But if you are taking only first order then you are getting a simplified equation. So, if you take the these are the numerical parameters so actually this work has been done by my PhD student mister Bhavan Kolida.

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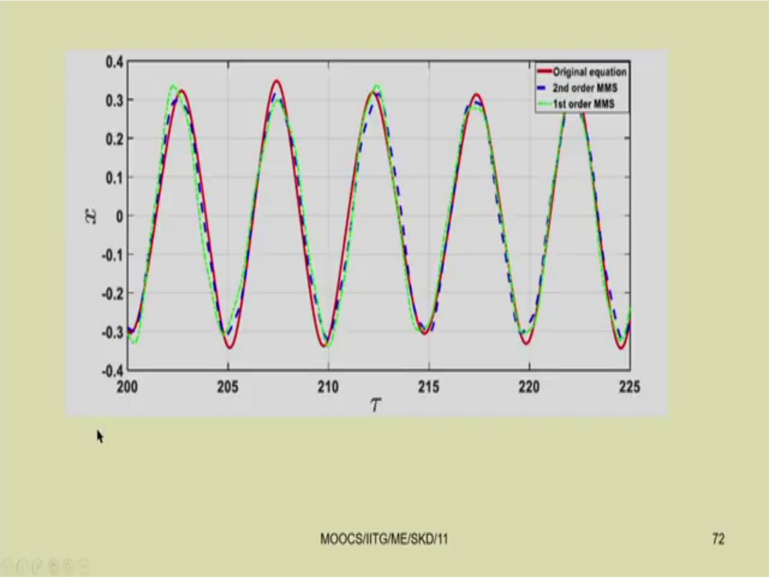

So, if you take these numerical value so you can see you can plot the original equation by solving this original differential equation. Then 2nd order MMS and 1st order MMS, and you can see the 2nd order MMS is giving slightly closer result to that of the numerical solution what you have obtained.

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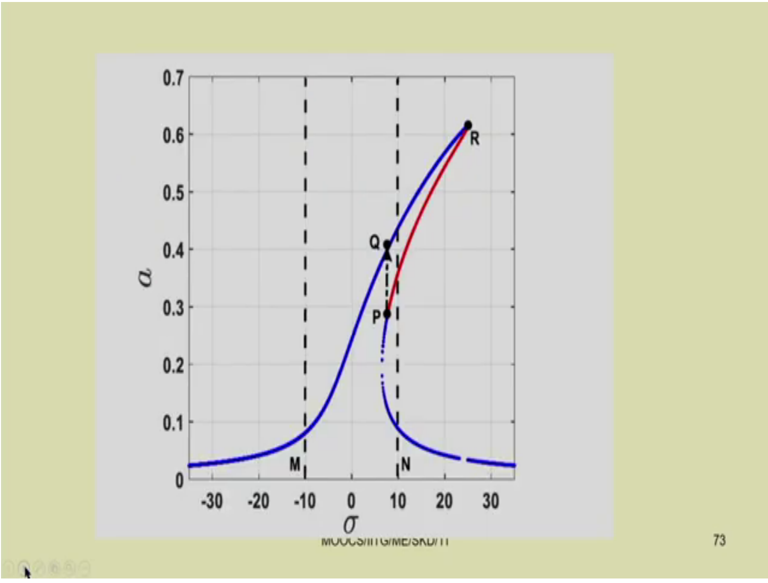

So, now you can plot this already I have told you how to plot by solving numerically you can plot this a verses sigma. So, now you can see when you are plotting this a verses sigma here up to these up to this parameter so up to this thing. So, you have only one solution, but after that thing you have multiple that is 3 solutions are there.

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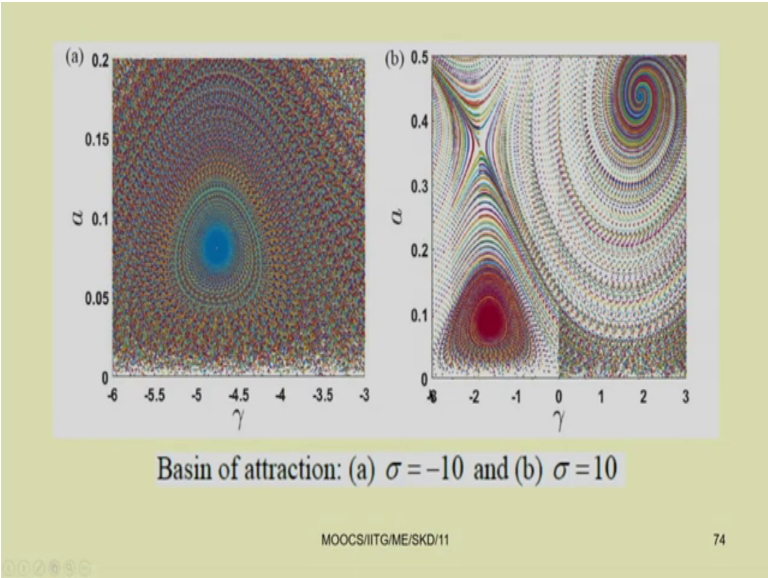

So, as you have 3 solutions so depending on initial conditions actually you can get this solution. For example, if you take this minus 10 so if you take different initial condition and plot these a verses gamma. So, you can see clearly you can see this is the point you have a stable point stable value of response amplitude, stable value of response amplitude is this value we are getting. So, less than 0.1 you are getting which you are clearly seeing from the basin of attraction this is the value less than 0.1.

But, if you are taking another point let us take another point here if you take another point plus 10 so you just see you have 1 2 and 3 solution. So, out of which two solutions are stable and one is unstable, that stability part we will study later. So, you can clearly see this is one stable solution this is one stable solution and this is another stable solution and this is the unstable solution which is a saddle point.

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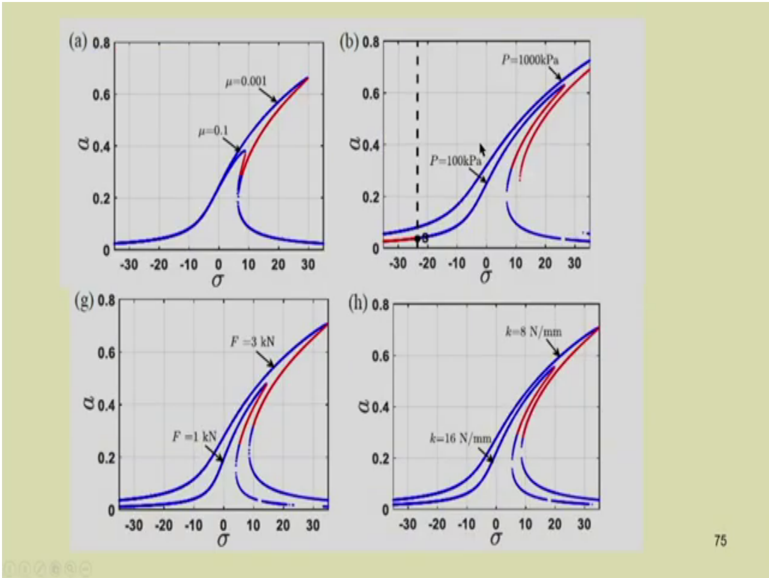

So, this way you can study or you can also use study the system by using different system parameter for example, damping changing the pressure changing the force and changing the stiffness parameter and you can study the system.

So, this way you can study different system different non-linear systems by using this method of multiple scale. So, next class we are going to study regarding this method of averaging and also briefly I will tell what are the other different methods recently available to solve this non-linear differential equations. So, thank you very much for listening to this class.

Thank you.