Theory of Composite Shells

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Week - 02
Lecture - 03
Derivation of strain-displacement relation

Dear learners welcome to lecture-03 of the second week. Classification of Shell Surfaces we have already done.

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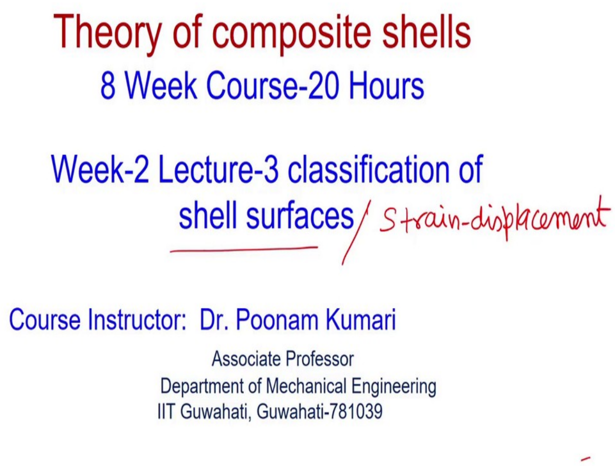

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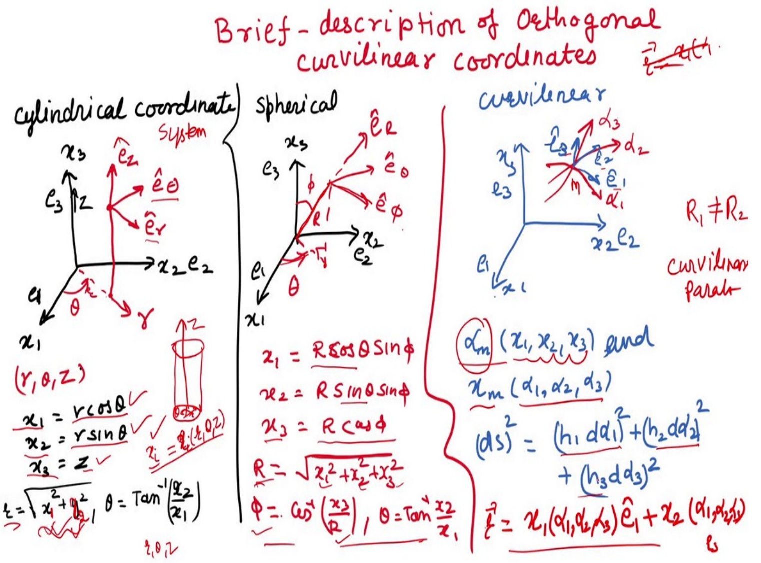

Today we will discuss the Strain Displacement Relations and representing the strain displacement in a curvilinear coordinate system. I will explain briefly about the orthogonal curvilinear coordinate system. The very first system all of you aware of is the cylindrical coordinate system.

In the cylindrical coordinate system; the Cartesian coordinates are x_1, x_2, and x_3, and unit vectors associated with that are $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$. In this system, this is the radial coordinate and radius r making an angle θ with respect to x_1 and this is the length or longitudinal axis of the cylindrical coordinate, means the cylinder is like this. Now we are going to analyze the z-axis, R and θ.

How do we represent the cylindrical coordinate system? Along r direction unit vector will be $\hat{e}_r$, along θ direction unit vector will be $\hat{e}_\theta$ and along z-direction unit vector will be $\hat{e}_z$. $\hat{e}_r$ and $\hat{e}_\theta$ are not constant, they are a function of θ.

In the very first week, I already explained during the transformation that the component along the $x_1 = r\cos\theta$; $x_2 = r\sin\theta$, and $x_3 = z$; then, r can be represented in terms of cartesian coordinates.

From here, we can say that these x_1, x_2, and x_3 can be represented in terms of r, θ, and z. The x_i can be represented as $x_i(r, \theta, z)$. We can say that r, θ and z can be expressed in terms of cartesian coordinates.

This is the relation in the cylindrical coordinate system and this is also a curvilinear orthogonal curvature in one direction singly curved surfaces. If you are interested to solve the problem of singly curved surfaces, then cylindrical coordinate is the best choice. Whether you talk about a circular cylinder, elliptical cylinder, or cone. We can analyze these problems in the cylindrical coordinate system.

For a doubly curved or spherical coordinate system, where the radius in both the directions is the same $R_1 = R_2 = R$. How do we represent this spherical coordinate system in Cartesian coordinates? Like the same way we represent in cartesian coordinates x_1, x_2, and x_3; let us say, this is R single radius in the spherical system and the point is here.

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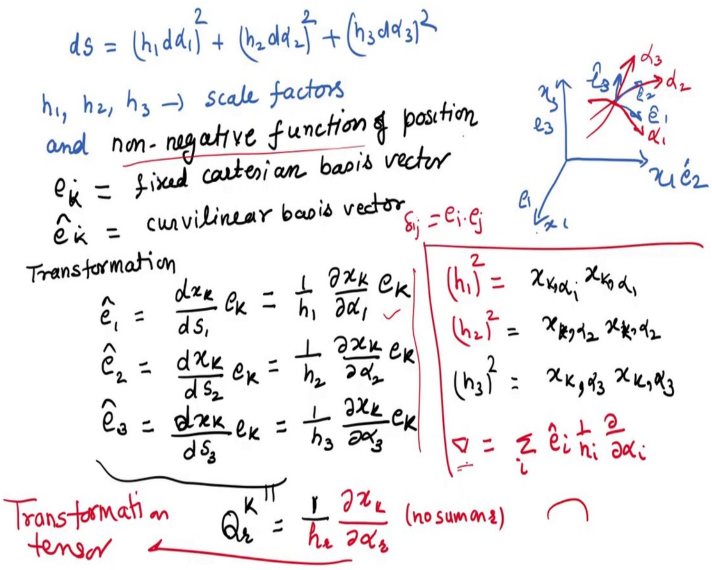

Sometimes, this R is denoted as r or r_0. It will be θ and R is making ϕ angle with respect to x_3. The very first equation:

$$x_3 = R\cos\phi; \quad x_2 = R\sin\theta\sin\phi; \quad \text{and } x_1 = R\cos\theta\sin\phi.$$

In this way x_1, x_2, and x_3 can be represented in terms of R, θ, and ϕ.

$$R = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Similarly, instead of y, you can say this is x_2 and ϕ will be obtained. Here the unit vectors are in R, θ, and ϕ directions.

Similarly, the curvilinear coordinate system; these are those special cases where the curvature is either in one direction or both directions, but the radius remains the same. Now, we are coming to the general case of the doubly curved surface, where the radius $R_1 \neq R_2$. How do we express that system? Already I have given you enough information about the curvilinear parameters.

Let us say, a point in 3-dimensional space is represented by m and this point can be defined in terms of α_1, α_2, and α_3. Along α_1 the unit vector is $\hat{e}_1$, along α_2 unit vector is $\hat{e}_2$, along α_3 unit vector is $\hat{e}_3$. We can represent a point and ultimately, we aim to represent in the Cartesian system x_1, x_2, and x_3. α_m the curvilinear system can be expressed in terms of x_1, x_2, and x_3, r and θ can be expressed in terms of x_1 and x_2, R can be expressed in terms of x_1, x_2, and x_3 and ϕ can be expressed in terms of that.

This Cartesian system can be expressed in terms of α_1, α_2, and α_3. If you remember the position vector r, we can write:

$$r = x_1(\alpha_1, \alpha_2, \alpha_3)\hat{e}_1 + x_2(\alpha_1, \alpha_2, \alpha_3)\hat{e}_3$$

Previously I expressed in terms of a 2-dimensional curvilinear parameter. But now I have expressed in terms of this. Distance ds square can be expressed:

$$\left(ds\right)^2 = \left(h_1 d\alpha_1\right)^2 + \left(h_2 d\alpha_2\right)^2 + \left(h_3 d\alpha_3\right)^2$$

If you remember, in the first fundamental form I explained those E, F, G. Similarly, in a 3-dimensional case; it is extended to h_1, h_2, and h_3. Most of the books are based on the theory of shells or curvilinear coordinate system; h_1, h_2, and h_3 are known as scale factors and non-negative functions of positions. e_k is normal fixed Cartesian system basis vector, whereas, $\hat{e}_k$ is a curvilinear basis system.

Then, how do we transform that $\hat{e}_1$ in terms of the Cartesian system?

$$\hat{e}_1 = \frac{dx_k}{ds_1} e_k = \frac{1}{h_1} \frac{\partial x_k}{\partial \alpha_1}$$

Similarly, we can write in terms of $\hat{e}_2$ and $\hat{e}_3$.

$$\hat{e}_2 = \frac{dx_k}{ds_2} e_k = \frac{1}{h_2} \frac{\partial x_k}{\partial \alpha_2} e_k$$

$$\hat{e}_3 = \frac{dx_k}{ds_3} e_k = \frac{1}{h_3} \frac{\partial x_k}{\partial \alpha_3} e_k$$

It will be a transformation matrix for transforming from second-order tensor or third-order tensor or fourth-order tensor. That is why we use this transformation system. What are h_1, h_2, and h_3? These are the same as in 2-dimensional

$$(h_1)^2 = x_{k,\alpha_1}.x_{k,\alpha_1}; \quad (h_2)^2 = x_{k,\alpha_2}.x_{k,\alpha_2}; \quad (h_3)^2 = x_{k,\alpha_3}.x_{k,\alpha_3}$$

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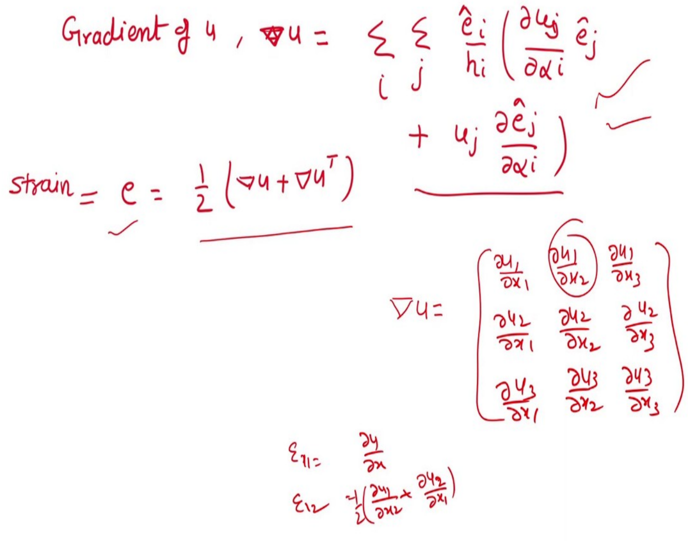

The del operator can be represented like this $\sum_i \hat{e}_i \frac{1}{\eta_i} \frac{\partial}{\partial x_i}$.

The purpose of explaining this thing is to make you aware that it is not very difficult, it is just a system to represent a curved surface. These are some terminologies used in the theory of elasticity to develop the strain displacement relations.

Derivation of strain displacement relation is not the part of this course, but for the sake of completeness, I will briefly explain the basic steps. The gradient of u can be obtained by this relation $u = \sum_i \sum_j \frac{\hat{e}_i}{\eta_i} \left(\frac{\partial u_j}{\partial \alpha_i} \hat{e}_j + u_j \frac{\partial \hat{e}_j}{\partial \alpha_i} \right)$ and strain can be obtained by this relation

$e = \frac{1}{2}\left(\nabla u + \nabla u^T \right)$. The gradient of u plus transpose of the gradient of u will give you the linear set of strain displacement relations in a curvilinear system.

Strain = $e = \frac{1}{2}\left(\nabla u + \nabla u^T \right)$

Already, I explained that in the Cartesian system or a rectangular coordinate system that what is ∇u . ∇u is deformation sometimes the deformation matrix, or deformation gradient: $\nabla u = \frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_1}{\partial x_3}, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_2}{\partial x_2}, \frac{\partial u_2}{\partial x_3}, \frac{\partial u_3}{\partial x_1}, \frac{\partial u_3}{\partial x_2}, \frac{\partial u_3}{\partial x_3}$

By using this, we can get $\varepsilon_{11} = \frac{\partial u}{\partial x}$ and $\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right)$.

Similarly, if we open it explicitly and add it together, we will get the strain in the curvilinear system. But definitely, this will be an entirely different and huge expression.

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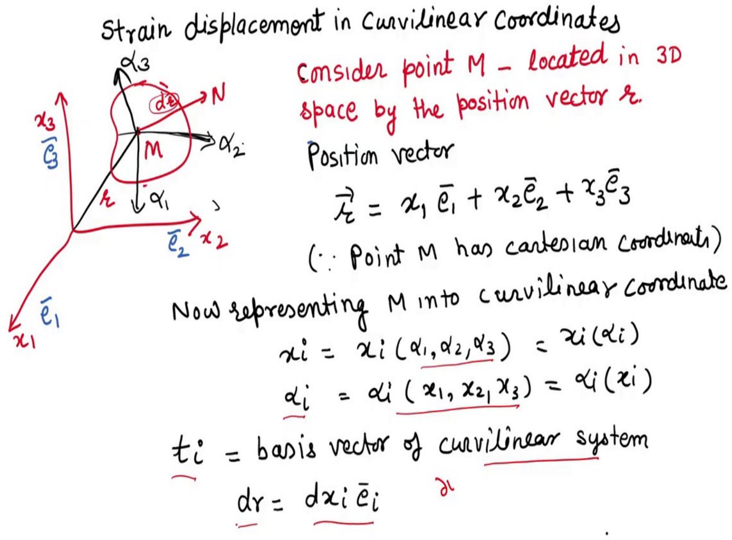

The basic steps to derive a strain displacement relation in the curvilinear coordinate system is let us say point M, located in the 3- D space having position vector r and the curvilinear parameters α_1, α_2, α_3 and this position vector $r = x_1\overline{e_1} + x_2\overline{e_2} + x_3\overline{e_3}$ then, point M has the Cartesian coordinate system.

Now, representing M into a curvilinear system, I already explained that $x_i = x_i(\alpha_1, \alpha_2, \alpha_3) = x_i(\alpha_i)$ and $\alpha_i = \alpha_i(x_1, x_2, x_3) = \alpha_i(x_i)$. t_i is the basis vector of curvilinear system, the change in length dr can find out by $dr = dx_i \overline{e_i}$.

A point M or its nearby point is N. Up to that length is changed, a small change in $r + dr$. A change in length in the material coordinate system can be represented like this.

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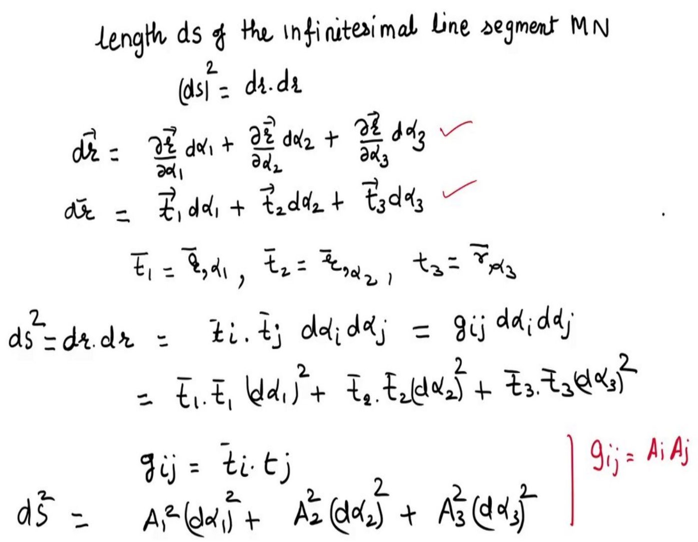

The length of a very small segment $(ds)^2 = dr.dr$.

By following the same procedure, we will get

$$\vec{dr} = \frac{\partial r}{\partial \alpha_1} d\alpha_1 + \frac{\partial r}{\partial \alpha_2} d\alpha_2 + \frac{\partial r}{\partial \alpha_3} d\alpha_3 = \vec{t}_1 d\alpha_1 + \vec{t}_2 d\alpha_2 + \vec{t}_3 d\alpha_3$$

Ultimately, $(ds)^2$ can be represented as g_{ij}.

Here, $\left(ds\right)^2 = t_1.t_1\left(d\alpha_1\right)^2 + t_2.t_2\left(d\alpha_2\right)^2 + t_3.t_3\left(d\alpha_3\right)^2 \; dr.dr$

It will give you g_{ij}. In most of the books of theory of shells, it is represented by g_{ij}. And ultimately it further can be expressed in terms of lame's parameter $A_i.A_j$.

$$g_{ij} = t_i.t_j = A_i.A_j$$

$$\left(ds\right)^2 = A_1^2\left(d\alpha_1\right)^2 + A_2^2\left(d\alpha_2\right)^2 + A_3^2\left(d\alpha_3\right)^2$$

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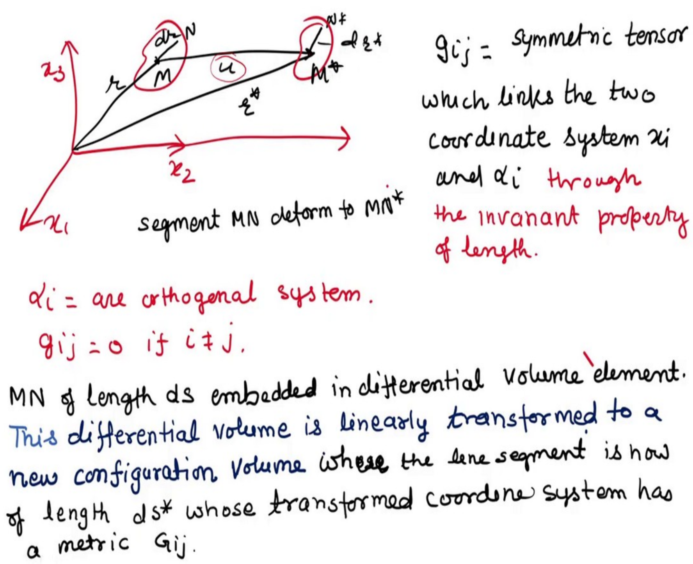

This change in dr; let us say point M changes to M^* and N changes to N^*. We aim to find the strain. From r to r^* this will be u (displacement vector). So, g_{ij} is a symmetric tensor which links two coordinate systems x_i and α_i through the invariant property of the length.

For an orthogonal system; when i $\neq$ j, then g_{ij} = 0. M & N of length ds embedded in a differential volume element, this differential volume element is linearly transformed to a new configuration whose length segment is ds^* and now it is dr^*, M^* and N^*

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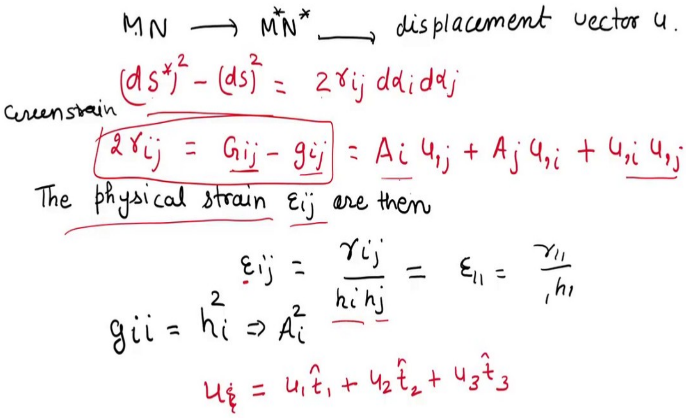

As per the definition, the final length minus the original length will give you the strain.

$$\left(ds^*\right)^2 - \left(ds\right)^2 = 2\gamma_{ij} d\alpha_i d\alpha_j$$

Ultimately, this can be expressed in terms of $2\gamma_{ij}$:

$$2\gamma_{ij} = G_{ij} - g_{ij} = A_i u_{,j} + A_j u_{,i} + u_{,i} u_{,j}$$

The physical strain ε_{ij} can be represented as: $\varepsilon_{ij} = \dfrac{\gamma_{ij}}{h_i h_j}$

where $h_i^2 = A_i^2$, h_i is equivalent to A_i (A_1, A_2, A_3) which are the lame's parameters. In this way, strains can be evaluated.

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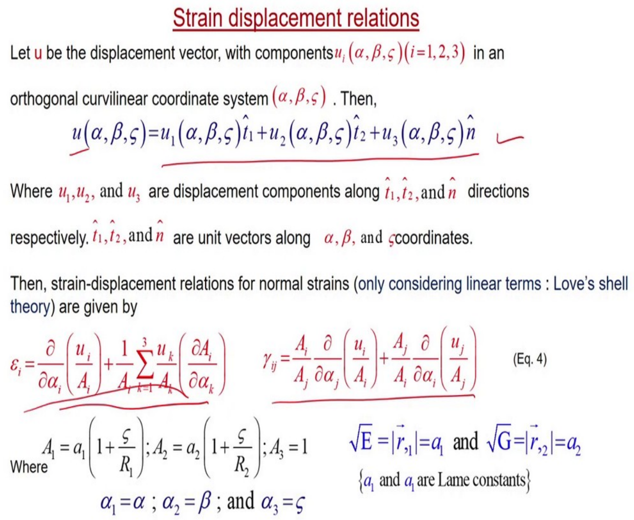

Ultimately, $u(\alpha,\beta,\varsigma)$ can be expressed as:

$$u(\alpha,\beta,\varsigma)=u_1(\alpha,\beta,\varsigma)\hat{t}_1+u_2(\alpha,\beta,\varsigma)\hat{t}_2+u_3(\alpha,\beta,\varsigma)\hat{t}_3$$

If you substitute two in that equation gives you strain components. So, in this way we derive that strain components in a curvilinear system, though I have to move briefly, for sake of completeness, I have given you the basic steps. Later on, for details, one can try the derivations.

The linear part of the strain; normal and shear strain can be expressed like equation (4)

$$\varepsilon_i=\frac{\partial}{\partial\alpha_i}\left(\frac{u_1}{A_i}\right)+\frac{1}{A_1}\sum_{k=1}^{3}\frac{u_k}{A_k}\left(\frac{\partial A_1}{\partial\alpha_k}\right).$$

This type of expression is given in most of the books based on linear shell theories. If you are interested to develop buckling analysis of shell, then you should consider some geometrical non-linearity.

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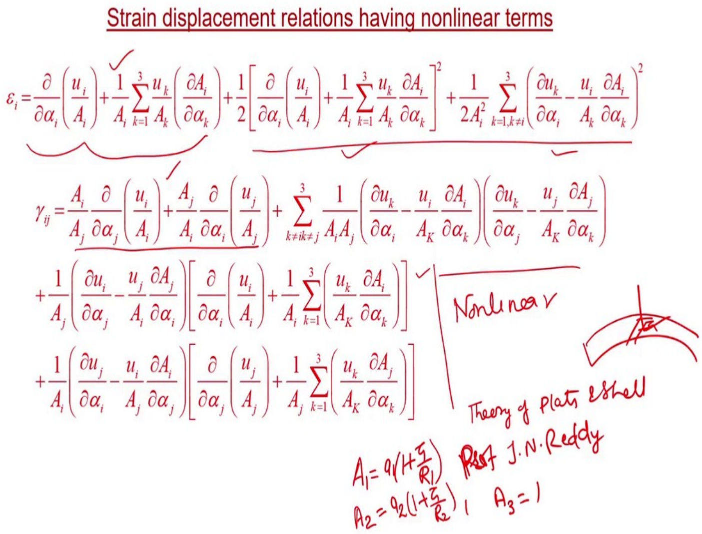

For that purpose, I have given the complete expression of strains having non-linear terms also.

$$\varepsilon_i = \frac{\partial}{\partial \alpha_i}\left(\frac{u_1}{A_i}\right) + \frac{1}{A_1}\sum_{k=1}^{3}\frac{u_k}{A_k}\left(\frac{\partial A_1}{\partial \alpha_k}\right) + \frac{1}{2}\left[\frac{\partial}{\partial \alpha_i}\left(\frac{u_1}{A_i}\right) + \frac{1}{A_1}\sum_{k=1}^{3}\frac{u_k}{A_k}\frac{\partial A_1}{\partial \alpha_k}\right] + \frac{1}{2A_1^2}\sum_{k=1,k\neq i}^{3}\left(\frac{\partial u_k}{\partial \alpha_i} - \frac{u_i}{A_k}\frac{\partial A_i}{\partial \alpha_k}\right)^2$$

$\varepsilon_i = \frac{\partial}{\partial \alpha_i}\left(\frac{u_1}{A_i}\right) + \frac{1}{A_1}\sum_{k=1}^{3}\frac{u_k}{A_k}\left(\frac{\partial A_1}{\partial \alpha_k}\right)$ is the linear part and this

$+\frac{1}{2}\left[\frac{\partial}{\partial \alpha_i}\left(\frac{u_1}{A_i}\right) + \frac{1}{A_1}\sum_{k=1}^{3}\frac{u_k}{A_k}\frac{\partial A_1}{\partial \alpha_k}\right] + \frac{1}{2A_1^2}\sum_{k=1,k\neq i}^{3}\left(\frac{\partial u_k}{\partial \alpha_i} - \frac{u_i}{A_k}\frac{\partial A_i}{\partial \alpha_k}\right)^2$ is the non-linear part

And below is the expression for γ_{ij}:

$$\gamma_{ij} = \frac{A_i}{A_j}\frac{\partial}{\partial \alpha_j}\left(\frac{u_i}{A_i}\right) + \frac{A_j}{A_i}\frac{\partial}{\partial \alpha_i}\left(\frac{u_j}{A_j}\right) + \sum_{k\neq i k\neq j}^{3}\frac{1}{A_iA_j}\left(\frac{\partial u_k}{\partial \alpha_i} - \frac{u_i}{A_k}\frac{\partial A_i}{\partial \alpha_k}\right)\left(\frac{\partial u_k}{\partial \alpha_j} - \frac{u_j}{A_k}\frac{\partial A_j}{\partial \alpha_k}\right)$$
$$+\frac{1}{A_j}\left(\frac{\partial u_i}{\partial \alpha_j} - \frac{u_j}{A_i}\frac{\partial A_j}{\partial \alpha_i}\right)\left[\frac{\partial}{\partial \alpha_i}\left(\frac{u_i}{A_i}\right) + \frac{1}{A_i}\sum_{k=1}^{3}\left(\frac{u_k}{A_k}\frac{\partial A_i}{\partial \alpha_k}\right)\right] + \frac{1}{A_i}\left(\frac{\partial u_j}{\partial \alpha_i} - \frac{u_i}{A_j}\frac{\partial A_i}{\partial \alpha_j}\right)\left[\frac{\partial}{\partial \alpha_j}\left(\frac{u_j}{A_j}\right) + \frac{1}{A_j}\sum_{k=1}^{3}\left(\frac{u_k}{A_k}\frac{\partial A_j}{\partial \alpha_k}\right)\right]$$

$\frac{A_i}{A_j}\frac{\partial}{\partial \alpha_j}\left(\frac{u_i}{A_i}\right) + \frac{A_j}{A_i}\frac{\partial}{\partial \alpha_i}\left(\frac{u_j}{A_j}\right)$ this is corresponding to the linear part

Remaining is the non-linear part.

An expression like this is given in the book of Theory of Plates and Shells by J N Reddy. There are many books published on Shell Theories. The books on linear shell theories discuss only the linear part of the strains. The books which are developing the non-linear shell theories consider the non-linear part. For the sake of completeness, I have given the linear and non-linear relations of the strain component and displacement.

But still, we need to work on this we cannot take as it is. Because for the shell case the

$A_1 = a_1\left(1+\frac{\varsigma}{R_1}\right)$ and $A_2 = a_2\left(1+\frac{\varsigma}{R_2}\right)$.

In lecture 02, I derived those relations for a reference element and $A_3 = 1$ for a doubly curved shell. If you remember, we have taken that ς coordinate R + dr. A_1, A_2, A_3 comes like this.

We have to substitute those values here to get the final form of ε_i or γ_{ij}. Afterward, we can use it for our purpose.

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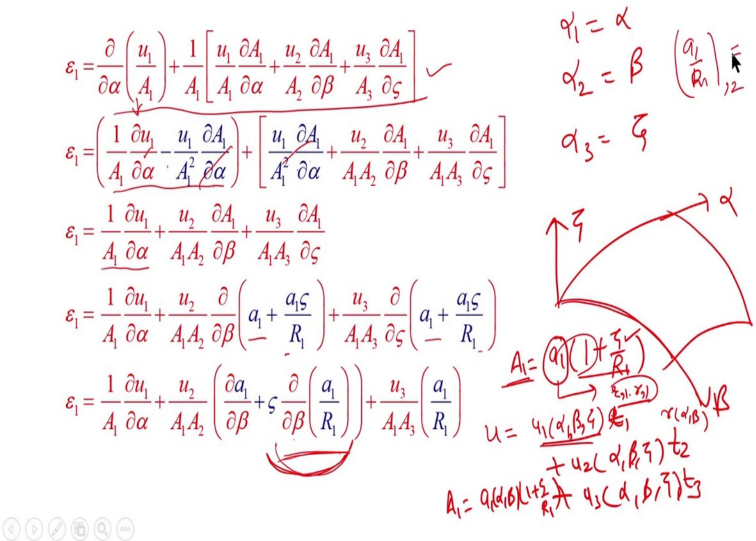

First, I will explain the linear relation, if we go back to the previous slide, this first equation tells ε_i means ε_1.

$$\varepsilon_1 = \frac{\partial}{\partial\alpha}\left(\frac{u_1}{A_1}\right) + \frac{1}{A_1}\left[\frac{u_1}{A_1}\frac{\partial A_1}{\partial\alpha} + \frac{u_2}{A_2}\frac{\partial A_1}{\partial\beta} + \frac{u_3}{A_3}\frac{\partial A_1}{\partial\varsigma}\right]$$

This can be open up. Right now, this is in index form, we have to open all these terms and then use the concept.

I will explain for the linear case, for non-linear case one can derive. For the linear case; ε_1 can be open up like this where $\alpha_1 = \alpha$, $\alpha_2 = \beta$ and $\alpha_3 = \varsigma$. Because we have taken a system. Now, these are α, β, and ς coordinates. $\frac{u_1}{A_1}$ derivative with respect to α.

The point to be noted here is that $u_1(\alpha,\beta,\varsigma)$. Because we have expressed that

$$u(\alpha,\beta,\varsigma) = u_1(\alpha,\beta,\varsigma)\hat{t}_1 + u_2(\alpha,\beta,\varsigma)\hat{t}_2 + u_3(\alpha,\beta,\varsigma)\hat{t}_3$$

We can take the derivative with respect to α A_1 and $A_1 = a_1\left(1+\frac{\varsigma}{R_1}\right)$.

a_1 is the lame's parameter, which is $r_{,1}.r_{,1}$; where r (α, β), A_1 (α, β, ς). Through this relation a_1 also becomes a function of ς.

This expression can be expressed as the first function as it is the differentiation of the second function. Then, the second function as it is the differentiation of the first function.

$$\varepsilon_1 = \left(\frac{1}{A_1}\frac{\partial u}{\partial\alpha} - \frac{u_1}{A_1^2}\frac{\partial A_1}{\partial\alpha}\right) + \left[\frac{u_1}{A_1^2}\frac{\partial A_1}{\partial\alpha} + \frac{u_2}{A_1A_2}\frac{\partial A_1}{\partial\beta} + \frac{u_3}{A_1A_3}\frac{\partial A_1}{\partial\varsigma}\right].$$

This is $-\frac{u_1}{A_1^2}\frac{\partial A_1}{\partial\alpha}$ and $\frac{u_1}{A_1^2}\frac{\partial A_1}{\partial\alpha}$ will get cancel. In this way, we will get three terms.

$$\varepsilon_1 = \frac{1}{A_1}\frac{\partial u_1}{\partial\alpha} + \frac{u_2}{A_1A_2}\frac{\partial A_1}{\partial\beta} + \frac{u_3}{A_1A_3}\frac{\partial A_1}{\partial\varsigma}$$

Again A_1 is also a function of ς.

We have to express this A_1 with respect to the derivative with β and ς. A_1 can be expressed; if you take the derivative with respect to β and if you ultimately open it, it becomes:

$$\frac{1}{A_1}\frac{\partial u_1}{\partial \alpha}+\frac{u_2}{A_1 A_2}\frac{\partial A_1}{\partial \beta}\left(a_1+\frac{a_1\varsigma}{R_1}\right)+\frac{u_3}{A_1 A_3}\frac{\partial A_1}{\partial \varsigma}\left(a_1+\frac{a_1\varsigma}{R_1}\right)$$

We have to open this, then it will be easy to work on it.

From here we can express it as:

$$\varepsilon_1=\frac{1}{A_1}\frac{\partial u_1}{\partial \alpha}+\frac{u_2}{A_1 A_2}\left[\frac{\partial a_1}{\partial \beta}+\varsigma\frac{\partial}{\partial \beta}\left(\frac{a_1}{R_1}\right)\right]+\frac{u_3}{A_1 A_3}\left(\frac{a_1}{R_1}\right)$$

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Use Codazzi condition $\left(\frac{a_1}{R_1}\right)_{,2}=\frac{a_{1,2}}{R_2}\Rightarrow\frac{\partial}{\partial \beta}\left(\frac{a_1}{R_1}\right)=\frac{1}{R_2}\frac{\partial a_1}{\partial \beta}$

Hence, $\varepsilon_1=\frac{1}{A_1}\frac{\partial u_1}{\partial \alpha}+\frac{u_2}{A_1 A_2}\left(\frac{\partial a_1}{\partial \beta}+\frac{\varsigma}{R_2}\frac{\partial a_1}{\partial \beta}\right)+\frac{u_3}{A_1 A_3}\left(\frac{a_1}{R_1}\right)$

$$\varepsilon_1=\frac{1}{A_1}\frac{\partial u_1}{\partial \alpha}+\frac{u_2}{A_1 A_2}\left(1+\frac{\varsigma}{R_2}\right)\frac{\partial a_1}{\partial \beta}+\frac{u_3}{A_1 A_3}\left(\frac{a_1}{R_1}\right)$$

$$\varepsilon_1=\frac{1}{A_1}\left[\frac{\partial u_1}{\partial \alpha}+\frac{u_2}{A_2}\left(1+\frac{\varsigma}{R_2}\right)\frac{\partial a_1}{\partial \beta}+\frac{u_3}{A_3}\left(\frac{a_1}{R_1}\right)\right]$$

$$\varepsilon_1=\frac{1}{A_1}\left[\frac{\partial u_1}{\partial \alpha}+\frac{u_2}{a_2}\frac{\partial a_1}{\partial \beta}+u_3\frac{a_1}{R_1}\right] \quad \text{(Eq. 5)} \quad \left\{A_2=a_2\left[1+\frac{\varsigma}{R_2}\right];\ \text{and } A_3=1\right\}$$

Linear strain-displacement

If you remember the theorem of Rodrigues:

$$\left(\frac{a_1}{R_1}\right)_{,2}=\frac{a_{1,2}}{R_2}=\frac{\partial}{\partial \beta}\left(\frac{a_1}{R_1}\right)=\frac{1}{R_2}\frac{\partial a_1}{\partial \beta}$$

Hence, $\varepsilon_1 = \frac{1}{A_1}\frac{\partial u_1}{\partial \alpha} + \frac{u_2}{A_1 A_2}\left(\frac{\partial a_1}{\partial \beta} + \frac{\varsigma}{R_2}\frac{\partial a_1}{\partial \beta}\right) + \frac{u_3}{A_1 A_3}\left(\frac{a_1}{R_1}\right)$

$$\varepsilon_1 = \frac{1}{A_1}\frac{\partial u_1}{\partial \alpha} + \frac{u_2}{A_1 A_2}\left(1 + \frac{\varsigma}{R_2}\right)\frac{\partial a_1}{\partial \beta} + \frac{u_3}{A_1 A_3}\left(\frac{a_1}{R_1}\right) \Rightarrow \frac{1}{A_1}\left[\frac{\partial u_1}{\partial \alpha} + \frac{u_2}{A_2}\left(1 + \frac{\varsigma}{R_2}\right)\frac{\partial a_1}{\partial \beta} + \frac{u_3}{A_3}\left(\frac{a_1}{R_1}\right)\right]$$

Ultimately, ε_1 will be:

$$\varepsilon_1 = \frac{1}{A_1}\left[\frac{\partial u_1}{\partial \alpha} + \frac{u_2}{a_2}\frac{\partial a_1}{\partial \beta} + u_3\frac{a_1}{R_1}\right] \text{ - equation (5)}$$

We have derived the relation for the theorem of Rodrigues, Weingarten formulas, or Gauss Codazzi equations, we are going to use this. Here ε_1 is find out in this form. So, this equation (5) is our final form which we are going to use for developing the shell theory, this is the linear strain displacement relation.

Wherever A_3 comes is represented by 1. This expression becomes like this.

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$$\varepsilon_1 = \frac{1}{A_1}\left[\frac{\partial u_1}{\partial \alpha} + \frac{u_2}{a_2}\frac{\partial a_1}{\partial \beta} + u_3\frac{a_1}{R_1}\right] + \frac{1}{2A_1^2}\left[\left(\frac{\partial u_1}{\partial \alpha} + \frac{u_2}{a_2}\frac{\partial a_1}{\partial \beta} + u_3\frac{a_1}{R_1}\right)^2 + \left(\frac{\partial u_2}{\partial \alpha} - \frac{u_1}{a_2}\frac{\partial a_1}{\partial \beta}\right)^2 + \left(\frac{\partial u_3}{\partial \alpha} - u_1\frac{a_1}{R_1}\right)^2\right]$$

– Prof J.N Reddy

$$\varepsilon_2 = \frac{\partial}{\partial \beta}\left(\frac{u_2}{A_2}\right) + \frac{1}{A_2}\left[\frac{u_1}{A_1}\frac{\partial A_2}{\partial \alpha} + \frac{u_2}{A_2}\frac{\partial A_2}{\partial \beta} + \frac{u_3}{A_3}\frac{\partial A_2}{\partial \varsigma}\right]$$

$$\varepsilon_2 = \frac{1}{A_2}\left[\frac{\partial u_2}{\partial \beta} + \frac{u_1}{a_1}\frac{\partial a_2}{\partial \alpha} + u_3\frac{a_2}{R_2}\right]$$

Linear ε_{22}

$$+ \frac{1}{2A_2^2}\left[\left(\frac{\partial u_2}{\partial \beta} + \frac{u_1}{a_1}\frac{\partial a_2}{\partial \alpha} + u_3\frac{a_2}{R_2}\right)^2 + \left(\frac{\partial u_1}{\partial \beta} - \frac{u_2}{a_1}\frac{\partial a_2}{\partial \alpha}\right)^2 + \left(\frac{\partial u_3}{\partial \beta} - u_2\frac{a_2}{R_2}\right)^2\right]$$

Now, for the sake of completeness that $\left(\frac{a_1}{R_1}\right)_{,2} = \frac{a_{1,2}}{R_2}$ is the Codazzi's equation. This

expression: $+\frac{1}{2A_1^2}\left[\left(\frac{\partial u_1}{\partial \alpha}+\frac{u_2}{A_2}\frac{\partial a_1}{\partial \beta}+u_3\frac{a_1}{R_1}\right)^2+\left(\frac{\partial u_2}{\partial \alpha}+\frac{u_1}{a_2}\frac{\partial a_1}{\partial \beta}\right)^2+\left(\frac{\partial u_3}{\partial \alpha}-u_1\frac{a_1}{R_1}\right)^2\right]$

is taken from the non-linear terms. Non-linear terms are very big these can be reduced to this form by using the suitable process. One can derive this also. I have not done the derivation of the non-linear part, but these are given in professor J N Reddy book.

Next is ε_2; the linear expression for ε_2:

$$\frac{\partial}{\partial \beta}\left(\frac{u_2}{a_2}\right)+\frac{1}{A_2}\left(\frac{u_1}{a_1}\frac{\partial A_2}{\partial \alpha}+\frac{u_2}{a_2}\frac{\partial A_2}{\partial \beta}+\frac{u_3}{a_3}\frac{\partial A_2}{\partial \varsigma}\right).$$

Using the same concept, you will get the expression ε_2. This is the beauty of these expressions that once you get for one then you don't need to derive for others.

Let us say for ε_1 if you derive strain in one direction then you need not derive in the second direction, just by using the symmetry one can find out the complete relation for the second case. ε_2 will be:

$$\frac{1}{A_2}\left(\frac{\partial u_2}{\partial \beta}+\frac{u_1}{a_1}\frac{\partial a_2}{\partial \alpha}+u_3\frac{a_2}{R_2}\right)$$

This will be the linear expression in ε_2 the strain in the second direction, and this term:

$$+\frac{1}{2A_1^2}\left[\left(\frac{\partial u_2}{\partial \beta}+\frac{u_1}{a_1}\frac{\partial a_2}{\partial \alpha}+u_3\frac{a_2}{R_2}\right)^2+\left(\frac{\partial u_1}{\partial \beta}+\frac{u_2}{a_1}\frac{\partial a_2}{\partial \alpha}\right)^2+\left(\frac{\partial u_3}{\partial \beta}-u_2\frac{a_2}{R_2}\right)^2\right]$$

It is corresponding to a non-linear form. If you have derived for ε_1, then you need not derive for ε_2 based on symmetry you can write.

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$$\varepsilon_3 = \frac{\partial}{\partial \varsigma}\left(\frac{u_3}{A_3}\right) + \frac{1}{A_3}\left[\frac{u_1}{A_1}\frac{\partial A_3}{\partial \alpha} + \frac{u_2}{A_2}\frac{\partial A_3}{\partial \beta} + \frac{u_3}{A_3}\frac{\partial A_3}{\partial \varsigma}\right]$$

$$A_3 = 1 \Rightarrow \frac{\partial A_3}{\partial \alpha} = \frac{\partial A_3}{\partial \beta} = \frac{\partial A_3}{\partial \varsigma} = 0$$

$$\varepsilon_3 = \frac{\partial u_3}{\partial \varsigma} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial \varsigma}\right)^2 + \left(\frac{\partial u_2}{\partial \varsigma}\right)^2 + \left(\frac{\partial u_3}{\partial \varsigma}\right)^2\right]$$

$$\gamma_{ij} = \frac{A_i}{A_j}\frac{\partial}{\partial \alpha_j}\left(\frac{u_i}{A_i}\right) + \frac{A_j}{A_i}\frac{\partial}{\partial \alpha_i}\left(\frac{u_j}{A_j}\right) + \sum_{k\neq i k\neq j}^{3}\frac{1}{A_iA_j}\left(\frac{\partial u_k}{\partial \alpha_i} - \frac{u_i}{A_K}\frac{\partial A_i}{\partial \alpha_k}\right)\left(\frac{\partial u_k}{\partial \alpha_j} - \frac{u_j}{A_K}\frac{\partial A_j}{\partial \alpha_k}\right)$$

$$+\frac{1}{A_j}\left(\frac{\partial u_i}{\partial \alpha_j} - \frac{u_j}{A_i}\frac{\partial A_j}{\partial \alpha_i}\right)\left[\frac{\partial}{\partial \alpha_i}\left(\frac{u_i}{A_i}\right) + \frac{1}{A_i}\sum_{k=1}^{3}\left(\frac{u_k}{A_K}\frac{\partial A_i}{\partial \alpha_k}\right)\right]$$

$$+\frac{1}{A_i}\left(\frac{\partial u_j}{\partial \alpha_i} - \frac{u_i}{A_j}\frac{\partial A_i}{\partial \alpha_j}\right)\left[\frac{\partial}{\partial \alpha_j}\left(\frac{u_j}{A_j}\right) + \frac{1}{A_j}\sum_{k=1}^{3}\left(\frac{u_k}{A_K}\frac{\partial A_j}{\partial \alpha_k}\right)\right]$$

Next, the expression for ε_3 is expressed like this:

$$\frac{\partial}{\partial \varsigma}\left(\frac{u_3}{A_3}\right) + \frac{1}{A_3}\left(\frac{u_1}{A_1}\frac{\partial A_3}{\partial \alpha} + \frac{u_2}{A_2}\frac{\partial A_3}{\partial \beta} + \frac{u_3}{A_3}\frac{\partial A_3}{\partial \varsigma}\right) \text{ for a linear one.}$$

So, here you see $\frac{\partial A_3}{\partial \alpha}$, $\frac{\partial A_3}{\partial \beta}$, and $\frac{\partial A_3}{\partial \varsigma}$ will give you nothing because A_3 is our constant it is not a function of α, β, ς.

These will not contribute only from a linear this term will come up: $\frac{\partial u_3}{\partial \varsigma}$

Because, $A_3 = 1 \Rightarrow \frac{\partial A_3}{\partial \alpha} = \frac{\partial A_3}{\partial \beta} = \frac{\partial A_3}{\partial \varsigma} = 0$

And from the non-linear one, this term will come:

$$+\frac{1}{2}\left(\frac{\partial u_1}{\partial \varsigma}\right)^2 + \left(\frac{\partial u_2}{\partial \varsigma}\right)^2 + \left(\frac{\partial u_3}{\partial \varsigma}\right)^2$$

Now, we have to similarly find the expression of γ_{ij} means the shear strain components, which are again for the simplicity or remembrance that I have written.

I would like to mention it here, please do not get afraid of these big equations all these equations are given in the books, one need not remember these equations. Only you have to know the basic concept of how to do a differentiation if it is two functional or the use of Gauss Codazzi's equation or any other form. All these things are given in the books.

Based on your requirement you can convert it into your form. Sometimes postgraduate students or doctoral students, just by looking at the equation they think that it is very complex or they will not understand. It is not like that these equations look big, but they are not complex, one can understand and derive easily.

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$$\gamma_{23} = \frac{A_2}{A_3}\frac{\partial}{\partial \varsigma}\left(\frac{u_2}{A_2}\right) + \frac{A_3}{A_2}\frac{\partial}{\partial \beta}\left(\frac{u_3}{A_3}\right) + nonlinear$$

$$\gamma_{23} = \frac{1}{A_2}\frac{\partial u_3}{\partial \beta} + A_2\frac{\partial}{\partial \varsigma}\left(\frac{u_2}{A_2}\right)$$

$$+\frac{1}{A_2}\left[\frac{\partial u_2}{\partial \varsigma}\left(\frac{\partial u_2}{\partial \beta} + \frac{u_1}{a_1}\frac{\partial a_2}{\partial \alpha} + \frac{a_2}{R_2}u_3\right) + \frac{\partial u_1}{\partial \varsigma}\left(\frac{\partial u_1}{\partial \beta} - \frac{u_2}{a_1}\frac{\partial a_2}{\partial \alpha}\right) + \frac{\partial u_3}{\partial \varsigma}\left(\frac{\partial u_3}{\partial \beta} - \frac{a_2}{R_2}u_2\right)\right]$$

$$\gamma_{13} = \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha} + A_1\frac{\partial}{\partial \varsigma}\left(\frac{u_1}{A_1}\right)$$

$$+\frac{1}{A_1}\left[\frac{\partial u_1}{\partial \varsigma}\left(\frac{\partial u_1}{\partial \alpha} + \frac{u_2}{a_2}\frac{\partial a_1}{\partial \beta} + \frac{a_1}{R_1}u_3\right) + \frac{\partial u_2}{\partial \varsigma}\left(\frac{\partial u_2}{\partial \alpha} - \frac{u_1}{a_2}\frac{\partial a_1}{\partial \beta}\right) + \frac{\partial u_3}{\partial \varsigma}\left(\frac{\partial u_3}{\partial \alpha} - \frac{a_1}{R_1}u_1\right)\right]$$

$$\gamma_{12} = \frac{A_1}{A_2}\frac{\partial}{\partial \alpha}\left(\frac{u_2}{A_2}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \beta}\left(\frac{u_1}{A_1}\right) + \frac{1}{A_1A_2}\left[\left(\frac{\partial u_1}{\partial \beta} - \frac{u_2}{a_1}\frac{\partial a_2}{\partial \alpha}\right)\left(\frac{\partial u_1}{\partial \alpha} + \frac{u_2}{a_2}\frac{\partial a_1}{\partial \beta} + \frac{a_1}{R_1}u_3\right)\right]$$

$$+\frac{1}{A_1A_2}\left[\left(\frac{\partial u_2}{\partial \alpha} - \frac{u_1}{a_2}\frac{\partial a_1}{\partial \beta}\right)\left(\frac{\partial u_2}{\partial \beta} + \frac{u_1}{a_1}\frac{\partial a_2}{\partial \alpha} + \frac{a_2}{R_2}u_3\right) + \left(\frac{\partial u_3}{\partial \alpha} - \frac{a_1}{R_1}u_1\right)\left(\frac{\partial u_3}{\partial \beta} - \frac{a_2}{R_2}u_2\right)\right]$$

One should not get afraid of these big equations. Here again γ_{23} will be:

$$\gamma_{23} = \frac{A_2}{A_3}\frac{\partial}{\partial \varsigma}\left(\frac{u_2}{A_2}\right) + \frac{A_3}{A_2}\frac{\partial}{\partial \beta}\left(\frac{u_3}{A_3}\right) + nonlinear$$

The linear expression plus some non-linear expression. This expression with respect to ς and β; we can say that A_3 is not a function of β. A_3 will come out and get canceled. γ_{23} will be:

$$\frac{1}{A_2}\frac{\partial u_3}{\partial \beta} + A_2\frac{\partial}{\partial \varsigma}\left(\frac{u_2}{A_2}\right)$$

Next is the non-linear terms contribution:

$$\frac{1}{A_2}\left[\frac{\partial u_2}{\partial \varsigma}\left(\frac{\partial u_2}{\partial \beta} + \frac{u_1}{a_1}\frac{\partial a_2}{\partial \alpha} + \frac{a_2}{R_2}u_3\right) + \frac{\partial u_1}{\partial \varsigma}\left(\frac{\partial u_1}{\partial \beta} - \frac{u_2}{a_1}\frac{\partial a_2}{\partial \alpha}\right) + \frac{\partial u_3}{\partial \varsigma}\left(\frac{\partial u_3}{\partial \beta} - \frac{a_2}{R_2}u_2\right)\right]$$

We need to just work on it because A_2 is a function of ς and u_2 is also a function of ς so, we have to find it further. Similarly, γ_{13} and γ_{12} can be expressed using the basic concepts.

$$\gamma_{13} = \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha} + A_1\frac{\partial}{\partial \varsigma}\left(\frac{u_1}{A_1}\right)$$
$$+\frac{1}{A_1}\left[\frac{\partial u_1}{\partial \varsigma}\left(\frac{\partial u_1}{\partial \alpha} + \frac{u_2}{a_2}\frac{\partial a_1}{\partial \beta} + \frac{a_1}{R_1}u_3\right) + \frac{\partial u_2}{\partial \varsigma}\left(\frac{\partial u_2}{\partial \alpha} - \frac{u_1}{a_2}\frac{\partial a_1}{\partial \beta}\right) + \frac{\partial u_3}{\partial \varsigma}\left(\frac{\partial u_3}{\partial \alpha} - \frac{a_1}{R_1}u_1\right)\right]$$

$$\gamma_{12} = \frac{A_1}{A_2}\frac{\partial}{\partial \alpha}\left(\frac{u_2}{A_2}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \beta}\left(\frac{u_1}{A_1}\right) + \frac{1}{A_1A_2}\left[\left(\frac{\partial u_1}{\partial \beta} - \frac{u_2}{a_1}\frac{\partial a_2}{\partial \alpha}\right)\left(\frac{\partial u_1}{\partial \alpha} + \frac{u_2}{a_2}\frac{\partial a_1}{\partial \beta} + \frac{a_1}{R_1}u_3\right)\right]$$
$$+\frac{1}{A_1A_2}\left[\left(\frac{\partial u_2}{\partial \alpha} - \frac{u_1}{a_2}\frac{\partial a_1}{\partial \beta}\right)\left(\frac{\partial u_2}{\partial \beta} + \frac{u_1}{a_1}\frac{\partial a_2}{\partial \alpha} + \frac{a_2}{R_2}u_3\right) + \left(\frac{\partial u_2}{\partial \alpha} - \frac{a_1}{R_1}u_1\right)\left(\frac{\partial u_3}{\partial \beta} - \frac{a_2}{R_2}u_2\right)\right]$$

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$$\gamma_{13} = \frac{\partial u_1}{\partial \varsigma} - \frac{u_1}{A_1}\frac{\partial A_1}{\partial \varsigma} + \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha}$$

$$\gamma_{13} = \frac{\partial u_1}{\partial \varsigma} - \frac{u_1}{A_1}\frac{\partial}{\partial \varsigma}\left(a_1 + \frac{a_1 \varsigma}{R_1}\right) + \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha}$$

$$\gamma_{13} = \frac{\partial u_1}{\partial \varsigma} - \frac{u_1}{A_1}\left(\frac{a_1}{R_1}\right) + \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha} \quad \text{(Eq. 8)}$$

Similarly, $$\gamma_{23} = \frac{A_2}{A_3}\frac{\partial}{\partial \varsigma}\left(\frac{u_2}{A_2}\right) + \frac{A_3}{A_2}\frac{\partial}{\partial \beta}\left(\frac{u_3}{A_3}\right)$$

Which gives $$\gamma_{23} = \frac{\partial u_2}{\partial \varsigma} - \frac{u_2}{A_2}\left(\frac{a_2}{R_2}\right) + \frac{1}{A_2}\frac{\partial u_3}{\partial \beta} \quad \text{(Eq. 9)}$$

And $$\gamma_{12} = \frac{A_1}{A_2}\frac{\partial}{\partial \beta}\left(\frac{u_1}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha}\left(\frac{u_2}{A_2}\right) \quad \text{(Eq. 10)}$$

Later on, you see that $\frac{\partial}{\partial \alpha}$, $\frac{u_2}{A_2}$ based on the time we will evaluate it further and the same way $\frac{u_1}{A_1}$ we can evaluate. We have to find this term γ_{23} and similarly γ_{13}.

γ_{13} can be expressed as:

$$\gamma_{13} = \frac{\partial u_1}{\partial \varsigma} - \frac{u_1}{A_1}\frac{\partial A_1}{\partial \varsigma} + \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha} \Rightarrow \frac{\partial u_1}{\partial \varsigma} - \frac{u_1}{A_1}\frac{\partial}{\partial \varsigma}\left(a_1 + \frac{a_1 \varsigma}{R_1}\right) + \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha}$$

Ultimately γ_{13} is expressed as:

$$\gamma_{13} = \frac{\partial u_1}{\partial \varsigma} - \frac{u_1}{A_1}\left(\frac{a_1}{R_1}\right) + \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha}$$

And γ_{23} will be:

$$\frac{\partial u_2}{\partial \varsigma} - \frac{u_2}{A_2}\left(\frac{a_2}{R_2}\right)\frac{1}{A_2}\frac{\partial u_3}{\partial \beta}$$ using Gauss Codazzi's equations.

And, γ_{12} will be: $\frac{A_1}{A_2}\frac{\partial}{\partial\alpha}\left(\frac{u_1}{A_1}\right)+\frac{A_2}{A_1}\frac{\partial}{\partial\beta}\left(\frac{u_2}{A_2}\right)$.

Only the linear part I have written for simplicity, later on, one can derive for the non-linear expression also.

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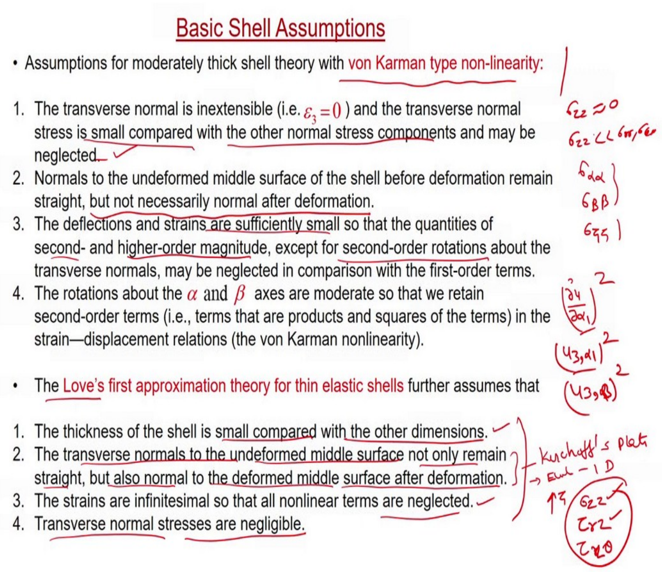

We have derived the basic strain displacement relations, in terms of A_1, A_2 and A_3. There are many shell theories. The very first shell theories are applicable for a thin elastic shell, developed by Love and it is known as Love's and Kirchhoff Shell theory. It is the extension of the 2-dimensional plate theory to the shell.

The assumptions are similar to the case of plate theory that the thickness of the shell is small as compared with the other dimensions. The transverse normal to the undeformed middle surface not only remains straight but also normal to the deformed middle surface after deformation. This assumption is the same as Kirchhoff's plate theory if you go for an Euler Bernoulli beam for 1-D case.

The strains are infinitesimal so that all non-linear terms are neglected, these have taken only the linear contribution of the strains. And transfer normal stresses are also negligible means, if this is the thickness direction then σ_{zz} ,σ_{rz} , or $\sigma_{z\theta}$ are going to be neglected for the thin elastic shell.

If we talk about a moderately thick theory and consideration of von Karman type non-linearity, then the transverse normal is inextensible. The first assumption is taken and transverse normal stress is small as compared with the other normal stress component and may be neglected means, σ_{zz} can be equal to 0. Because, σ_{zz} is very less as compared to others σ_{rr} or $\sigma_{\theta\theta}$, I am talking about in terms of cylindrical coordinate system.

If you talk about any variable α , β , and ς , instead of that you will write $\sigma_{\alpha\alpha}$, $\sigma_{\beta\beta}$ and $\sigma_{\varsigma\varsigma}$ in terms of curvilinear parameters. Then, it says that normal to the undeformed middle surface of the shell before deformation remain straight, but not necessarily normal after the deformation.

It tells you that the transverse shear strain exists because we are interested to solve thick shell theories. The deflection and strains are sufficiently small so, that quantities of second and higher-order magnitude can be neglected except for the second-order rotations. This means, we can say $\frac{\partial u}{\partial \alpha_1}$, second-order derivatives may be neglected, but the first order will be considered.

The rotation about the α and β axis is moderate, we can retain the second-order term for example, generally $\left(u_{3,\alpha 1}\right)^2 = \left(u_{3,\beta}\right)^2$ will be retained by following these assumptions.

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According to Love's first approximation theory for thin elastic shells :

$$\varepsilon_3 = 0\ ;\ \gamma_{13} = 0\ ;\ \gamma_{23} = 0$$

$$\varepsilon_3 = \frac{\partial u_3}{\partial \varsigma} = 0 \Rightarrow u_3 = \text{constant} = w_0 \,(\text{say}) \qquad \text{(Eq. 11)}$$

Now $$\gamma_{13} = \frac{\partial u_1}{\partial \varsigma} - \frac{u_1}{A_1}\left(\frac{a_1}{R_1}\right) + \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha} = 0 \Rightarrow \frac{\partial u_1}{\partial \varsigma} = -\frac{1}{A_1}\frac{\partial w_0}{\partial \alpha} + \frac{u_1}{A_1}\left(\frac{a_1}{R_1}\right)$$

at $\varsigma = 0$ ($u_1 = u_{10}$ and $A_1 = a_1$) $$\Rightarrow \left(\frac{\partial u_1}{\partial \varsigma}\right)_{\varsigma=0} = -\frac{1}{a_1}\frac{\partial w_0}{\partial \alpha} + \frac{u_{10}}{a_1}\left(\frac{a_1}{R_1}\right)$$

$$\Rightarrow \left(\frac{\partial u_1}{\partial \varsigma}\right)_{\varsigma=0} = \left(-\frac{1}{a_1}\frac{\partial w_0}{\partial \alpha} + \frac{u_{10}}{R_1}\right) \qquad \text{(Eq. 12)}$$

Integrating equation 12 with respect to ς

$$u_1 = \left(-\frac{1}{a_1}\frac{\partial w_0}{\partial \alpha} + \frac{u_{10}}{R_1}\right)\varsigma + C_1(\alpha, \beta)$$

In the case of love's Kirchhoff's first approximation the assumption of displacement field; one way is that you assume displacement field just by following the Taylor series expansion. Based on the assumptions we can get mathematically the expression of the displacement field. As it said that $\varepsilon_3 = 0$, $\gamma_{13} = 0$, and $\gamma_{23} = 0$ for thin elastic shells. $\varepsilon_3 = \partial u_3$ by $\partial \varsigma = 0$.

We are not taking the non-linear part. u_3 becomes constant if you integrate with respect to ς and find in terms of mid surfaces then $u_3 = w_0$ constant.

Then, γ_{13} expression will be:

$$\frac{\partial u_1}{\partial \varsigma} - \frac{u_1}{A_1}\left(\frac{a_1}{R_1}\right) + \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha}$$

If you write down properly, you take left-hand side terms to the right-hand side. And, then for a middle reference surface; convert into a reference surface where $\varsigma = 0$, $u_1 = u_{10}$, and $A_1 = a_1$.

This expression becomes:

$$\left(\frac{\partial u_1}{\partial \varsigma}\right)_{\varsigma=0} = -\frac{1}{a_1}\frac{\partial w_0}{\partial \alpha} + \frac{u_{10}}{a_1}\left(\frac{a_1}{R_1}\right) = -\frac{1}{a_1}\frac{\partial w_0}{\partial \alpha} + \frac{u_{10}}{R_1}$$

If you integrate with respect to ς, multiply with the ς then this is the differentiation:

$$u_1 = \left(-\frac{1}{a_1}\frac{\partial w_0}{\partial \alpha} + \frac{u_{10}}{R_1}\right)\varsigma + C_1(\alpha,\beta)$$

And the integrating constant will be a function of α and β. Now, we have to evaluate this integrating function $C_1(\alpha,\beta)$.

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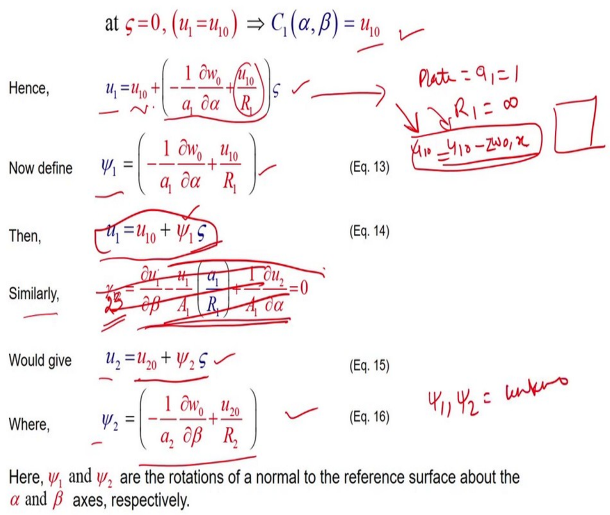

This integrating function says:

$$at\,\varsigma = 0, (u_1 = u_{10}) \Rightarrow C_1(\alpha,\beta) = u_{10}$$

From that expression u_1 can be expressed as:

$$u_1 = u_{10} + \left(-\frac{1}{a_1}\frac{\partial w_0}{\partial \alpha} + \frac{u_{10}}{R_1}\right)\varsigma$$

This is the expression for displacement, in the first direction and based on the assumptions loves Kirchhoff's theory.

Now, I would like to point out here that if we want to verify it to be a special case of a plate; for the case of a plate, what is a_1 and R_1? $a_1 = 1$ and $R_1 = \infty$. This expression will go away and it will reduce to u_{10} - ς $w_{0,x}$. Anybody can express the classical plate theory or Kirchhoff's plate theory; it reduces to that if $R_1 = \infty$ and $a_1 = 1$.

That is why I said that if you can develop equations for a shell, definitely plate and beam will be the special cases of classical theory. If you are developing a classical theory, it reduces to a classical form. If you are developing a higher-order it will reduce to a higher-order form.

This term here we denote $u_1 = u_{10} + \psi_1 \varsigma$, then similarly here gamma γ_{23} expression will be: $\gamma_{23} = \frac{\partial u_2}{\partial \varsigma} - \frac{u_2}{A_2}\left(\frac{a_2}{R_2}\right) + \frac{1}{A_2}\frac{\partial u_3}{\partial \beta} = 0$

Then, $\frac{\partial u_2}{\partial \varsigma}$ expression will give you that $u_2 = u_{20} + \psi_2 \varsigma$.

For the case of love's Kirchhoff shell theory ψ_1 and ψ_2 will be expressed like this:

$$\psi_1 = \left(-\frac{1}{a_1}\frac{\partial w_0}{\partial \alpha} + \frac{u_{10}}{R_1}\right)$$

$$\psi_2 = \left(-\frac{1}{a_2}\frac{\partial w_0}{\partial \beta} + \frac{u_{20}}{R_2}\right).$$

But if you talk about a higher order; the first-order specifically of shell theory then ψ_1 and ψ_2 will be unknown to you.

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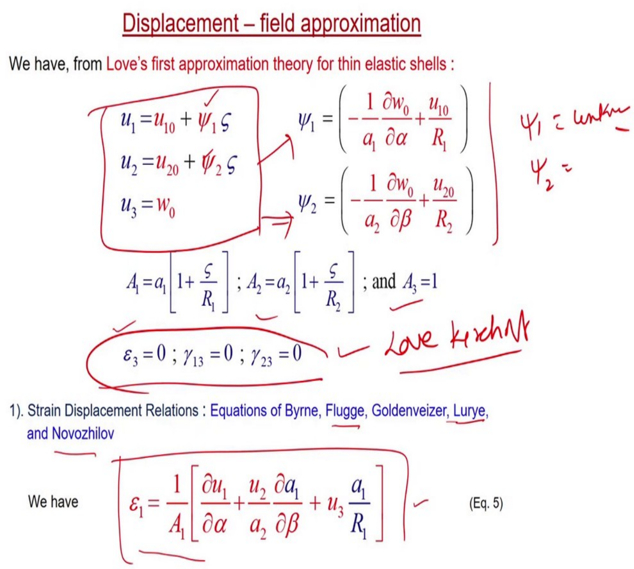

We can express finally, the displacement field like this:

$u_1 = u_{10} + \psi_1 \varsigma$, $u_2 = u_{20} + \psi_2 \varsigma$ and $u_3 = w_0$.

For the case of love's Kirchhoff shell theory ψ_1 and ψ_2 are expressed. But for the FSDT ψ_1 and ψ_2 will be unknown to us. A_1, A_2, and A_3 are expressed and these components $\varepsilon_3, \gamma_{13}$, and $\gamma_{23} = 0$ for Love Kirchhoff shell theory. From here that most of the theories start differentiating, based on the strain displacement relations.

I would like to say that almost more than 20 different types of shell theories are presented in the literature, and out of which some are very popular like Flugge shell theory, Lurye cell theory, and Novozhilov shell theories. Retaining some terms or deleting some terms gives you an entirely different shell theory.

The purpose is different, sometimes we are interested to solve and develop a governing equation for a membrane type, for a thin shell, for a thick shell, or a shallow cell.

Based on the requirement the terms can be retained or deleted. The general expression that we have for a linear case strain ε_1 in equation (5) will be retained in the case of Byrne, Flugge, Goldenveizer, Lurie, and Novozhilov. This expression is taken as it is in the strain displacement relations.

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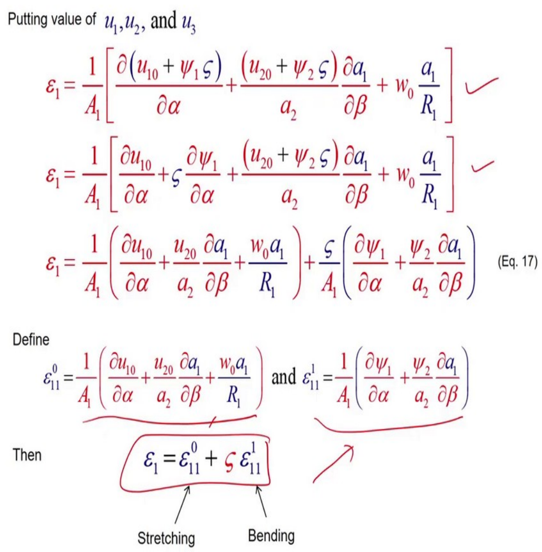

If we use the value of u_1, u_2 and u_3, then this expression becomes like this:

$$\varepsilon_1 = \frac{1}{A_1}\left[\frac{\partial\left(u_{10} + \psi_1\varsigma\right)}{\partial\alpha} + \frac{u_{20} + \psi_2\varsigma}{a_2}\frac{\partial a_1}{\partial\beta} + w_0\frac{a_1}{R_1}\right]$$

$$\varepsilon_1 = \frac{1}{A_1}\left[\frac{\partial u_{10}}{\partial\alpha} + \varsigma\frac{\partial\psi_1}{\partial\alpha} + \frac{u_{20} + \psi_2\varsigma}{a_2}\frac{\partial a_1}{\partial\beta} + w_0\frac{a_1}{R_1}\right]$$

$$\varepsilon_1 = \frac{1}{A_1}\left(\frac{\partial u_{10}}{\partial\alpha} + \frac{u_{20}}{a_2}\frac{\partial a_1}{\partial\beta} + \frac{w_0 a_1}{R_1}\right) + \frac{\varsigma}{A_1}\left(\frac{\partial\psi_1}{\partial\alpha} + \frac{\psi_2}{a_2}\frac{\partial a_1}{\partial\beta}\right) \quad \text{equation (17)}$$

And ultimately the terms related to the stretching are clubbed together, and terms relating to the bending are clubbed together.

$$\varepsilon_{11}^{0} = \frac{1}{A_1}\left(\frac{\partial u_{10}}{\partial\alpha} + \frac{u_{20}}{a_2}\frac{\partial a_1}{\partial\beta} + \frac{w_0 a_1}{R_1}\right) \qquad \varepsilon_{11}^{1} = \frac{1}{A_1}\left(\frac{\partial\psi_1}{\partial\alpha} + \frac{\psi_2}{a_2}\frac{\partial a_1}{\partial\beta}\right)$$

And finally, ε_1 can be written as ε_{11}^{0} plus $\varsigma\varepsilon_{11}^{1}$. This is the way we write the expression of strain.

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Similarly, we can show

$$\varepsilon_2 = \frac{1}{A_2}\left(\frac{\partial u_{20}}{\partial \beta} + \frac{u_{10}}{a_1}\frac{\partial a_2}{\partial \alpha} + \frac{w_0 a_2}{R_2}\right) + \frac{\varsigma}{A_2}\left(\frac{\partial \psi_2}{\partial \beta} + \frac{\psi_1}{a_1}\frac{\partial a_2}{\partial \alpha}\right) \quad \text{(Eq. 18)}$$

Define

$$\varepsilon_{22}^0 = \frac{1}{A_2}\left(\frac{\partial u_{20}}{\partial \beta} + \frac{u_{10}}{a_1}\frac{\partial a_2}{\partial \alpha} + \frac{w_0 a_2}{R_2}\right) \text{ and } \varepsilon_{22}^1 = \frac{1}{A_2}\left(\frac{\partial \psi_2}{\partial \beta} + \frac{\psi_1}{a_1}\frac{\partial a_2}{\partial \alpha}\right)$$

Then

$$\varepsilon_2 = \varepsilon_{22}^0 + \varsigma\, \varepsilon_{22}^1$$

Stretching (ε_{22}^0) Bending (ε_{22}^1)

And

$$\gamma_{12} = \frac{A_1}{A_2}\frac{\partial}{\partial \beta}\left(\frac{u_1}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha}\left(\frac{u_2}{A_2}\right) \quad \text{(Eq. 10)}$$

Putting value of u_1 and u_2

Similarly, ε_2: substituting the value of u_1, u_2 and u_3 then arranging the stretching and bending terms.

$$\varepsilon_2 = \frac{1}{A_2}\left(\frac{\partial u_{20}}{\partial \beta} + \frac{u_{10}}{a_1}\frac{\partial a_2}{\partial \alpha} + \frac{w_0 a_2}{R_2}\right) + \frac{\varsigma}{A_2}\left(\frac{\partial \psi_2}{\partial \beta} + \frac{\psi_1}{a_1}\frac{\partial a_2}{\partial \alpha}\right) \quad \text{equation (18)}$$

$$\varepsilon_{22}^0 = \frac{1}{A_2}\left(\frac{\partial u_{20}}{\partial \beta} + \frac{u_{10}}{a_1}\frac{\partial a_2}{\partial \alpha} + \frac{w_0 a_2}{R_2}\right) \qquad \varepsilon_{22}^1 = \frac{1}{A_2}\left(\frac{\partial \psi_2}{\partial \beta} + \frac{\psi_1}{a_1}\frac{\partial a_2}{\partial \alpha}\right)$$

$$\varepsilon_2 = \varepsilon_{22}^0 + \varsigma \varepsilon_{22}^1$$

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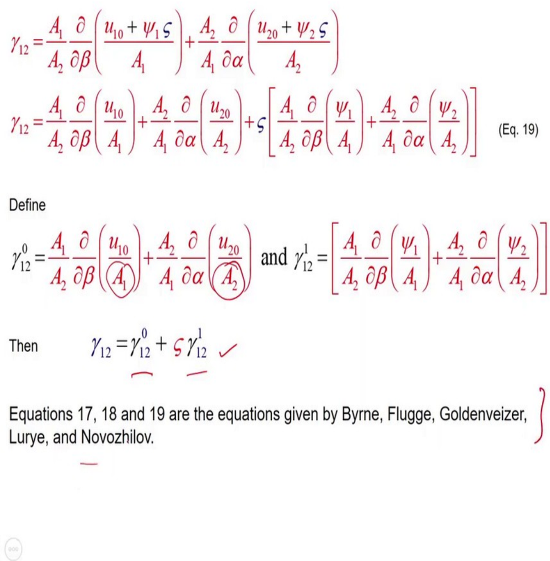

For γ_{12} expression;

$$\gamma_{12} = \frac{A_1}{A_2}\frac{\partial}{\partial \beta}\left(\frac{u_{10}}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha}\left(\frac{u_{20}}{A_2}\right) + \varsigma\left[\frac{A_1}{A_2}\frac{\partial}{\partial \beta}\left(\frac{\psi_1}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha}\left(\frac{\psi_2}{A_2}\right)\right] \quad \text{equation (19)}$$

Substitute those expressions properly. A_1 and A_2 are retained here, but in the end, we will have to convert it to back.

$$\gamma_{12}^0 = \frac{A_1}{A_2}\frac{\partial}{\partial \beta}\left(\frac{u_{10}}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha}\left(\frac{u_{20}}{A_2}\right) \qquad \gamma_{12}^1 = \left[\frac{A_1}{A_2}\frac{\partial}{\partial \beta}\left(\frac{\psi_1}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha}\left(\frac{\psi_2}{A_2}\right)\right]$$

$$\gamma_{12} = \gamma_{12}^0 + \varsigma\gamma_{12}^1$$

These equations (17), (18), and (19) are given in the following shell theories.

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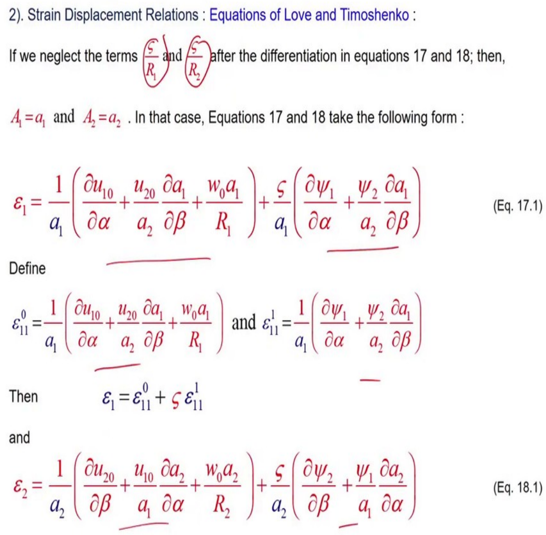

In Love's and Timoshenko's shell theories; they have neglected the expression of $\frac{\varsigma}{R_1}$ and $\frac{\varsigma}{R_2}$ in (17) and (18) expression. If we neglect those terms after the differentiation, then these expressions will be slightly different like this

$$\varepsilon_1 = \frac{1}{a_1}\left(\frac{\partial u_{10}}{\partial\alpha} + \frac{u_{20}}{a_2}\frac{\partial a_1}{\partial\beta} + \frac{w_0 a_1}{R_1}\right) + \frac{\varsigma}{a_1}\left(\frac{\partial\psi_1}{\partial\alpha} + \frac{\psi_2}{a_2}\frac{\partial a_1}{\partial\beta}\right) \text{ equation (17.1)}$$

$$\varepsilon_2 = \frac{1}{a_2}\left(\frac{\partial u_{20}}{\partial\beta} + \frac{u_{10}}{a_1}\frac{\partial a_2}{\partial\alpha} + \frac{w_0 a_2}{R_2}\right) + \frac{\varsigma}{a_2}\left(\frac{\partial\psi_2}{\partial\beta} + \frac{\psi_1}{a_1}\frac{\partial a_2}{\partial\alpha}\right) \text{ equation (18.1).}$$

$$\gamma_{12} = \frac{A_1}{A_2}\left(\frac{1}{A_1}\frac{\partial u_{10}}{\partial \beta} - \frac{u_{10}}{A_1^2}\frac{\partial A_1}{\partial \beta}\right) + \frac{A_2}{A_1}\left(\frac{1}{A_2}\frac{\partial u_{20}}{\partial \alpha} - \frac{u_{20}}{A_2^2}\frac{\partial A_2}{\partial \alpha}\right) \quad \text{equation (19.1)}$$

$$+\varsigma\frac{A_1}{A_2}\left[\left(\frac{1}{A_1}\frac{\partial \psi_1}{\partial \beta} - \frac{\psi_1}{A_1^2}\frac{\partial A_1}{\partial \beta}\right) + \frac{A_2}{A_1}\left(\frac{1}{A_2}\frac{\partial \psi_2}{\partial \alpha} - \frac{\psi_2}{A_2^2}\frac{\partial A_2}{\partial \alpha}\right)\right]$$

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Define

$$\varepsilon_{22}^0 = \frac{1}{a_2}\left(\frac{\partial u_{20}}{\partial \beta} + \frac{u_{10}}{a_1}\frac{\partial a_2}{\partial \alpha} + \frac{w_0 a_2}{R_2}\right) \text{ and } \varepsilon_{22}^1 = \frac{1}{a_2}\left(\frac{\partial \psi_2}{\partial \beta} + \frac{\psi_1}{a_1}\frac{\partial a_2}{\partial \alpha}\right)$$

Then $\varepsilon_2 = \varepsilon_{22}^0 + \varsigma\,\varepsilon_{22}^1$

and for equation 19

$$\gamma_{12} = \frac{A_1}{A_2}\frac{\partial}{\partial \beta}\left(\frac{u_{10}}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha}\left(\frac{u_{20}}{A_2}\right) + \varsigma\left[\frac{A_1}{A_2}\frac{\partial}{\partial \beta}\left(\frac{\psi_1}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha}\left(\frac{\psi_2}{A_2}\right)\right]$$

$$\gamma_{12} = \frac{A_1}{A_2}\left(\frac{1}{A_1}\frac{\partial u_{10}}{\partial \beta} - \frac{u_{10}}{A_1^2}\frac{\partial A_1}{\partial \beta}\right) + \frac{A_2}{A_1}\left(\frac{1}{A_2}\frac{\partial u_{20}}{\partial \alpha} - \frac{u_{20}}{A_2^2}\frac{\partial A_2}{\partial \alpha}\right)$$

$$+\varsigma\left[\frac{A_1}{A_2}\left(\frac{1}{A_1}\frac{\partial \psi_1}{\partial \beta} - \frac{\psi_1}{A_1^2}\frac{\partial A_1}{\partial \beta}\right) + \frac{A_2}{A_1}\left(\frac{1}{A_2}\frac{\partial \psi_2}{\partial \alpha} - \frac{\psi_2}{A_2^2}\frac{\partial A_2}{\partial \alpha}\right)\right]$$

There is another set of equations, these are the equation of Reissner and Naghdi, where they have neglected before the derivation. This shell theory is for a shallow shell theory. (Refer Slide Time: 45:28)

Equation 17.1, 18.1 and 19.1 are the equations given by Love and Timoshenko.

3). Strain Displacement Relations : Equations of Reissner and Naghdi :

In this case terms $\frac{\varsigma}{R_1}$ and $\frac{\varsigma}{R_2}$ are neglected before the derivation. Then equations 17, 18 and 19 take the following form.

$$\varepsilon_1 = \frac{1}{a_1}\left(\frac{\partial u_{10}}{\partial \alpha} + \frac{u_{20}}{a_2}\frac{\partial a_1}{\partial \beta} + \frac{w_0 a_1}{R_1}\right) + \frac{\varsigma}{a_1}\left(\frac{\partial \psi_1}{\partial \alpha} + \frac{\psi_2}{a_2}\frac{\partial a_1}{\partial \beta}\right) \quad \text{(Eq. 17.2)}$$

$$\varepsilon_2 = \frac{1}{a_2}\left(\frac{\partial u_{20}}{\partial \beta} + \frac{u_{10}}{a_1}\frac{\partial a_2}{\partial \alpha} + \frac{w_0 a_2}{R_2}\right) + \frac{\varsigma}{a_2}\left(\frac{\partial \psi_2}{\partial \beta} + \frac{\psi_1}{a_1}\frac{\partial a_2}{\partial \alpha}\right) \quad \text{(Eq. 18.2)}$$

$$\gamma_{12} = \frac{a_1}{a_2}\frac{\partial}{\partial \beta}\left(\frac{u_{10}}{a_1}\right) + \frac{a_2}{a_1}\frac{\partial}{\partial \alpha}\left(\frac{u_{20}}{a_2}\right) + \varsigma\left[\frac{a_1}{a_2}\frac{\partial}{\partial \beta}\left(\frac{\psi_1}{a_1}\right) + \frac{a_2}{a_1}\frac{\partial}{\partial \alpha}\left(\frac{\psi_2}{a_2}\right)\right] \quad \text{(Eq. 19.2)}$$

The expression will be slightly different, very few terms will be there. The curvature is used, but $\frac{\varsigma}{R_1}$ and $\frac{\varsigma}{R_2}$ are neglected, it is saying that the shell is very thin and we have to take care of these terms. So, in this way by retaining some terms or deleting some terms will give you an entirely different shell theory.

If you want to develop a shell theory based on Reissner and Naghdi, then you have to consider, the strain displacement relations derived in equations (17.2), (18.2), and (19.2).

$$\varepsilon_1 = \frac{1}{a_1}\left(\frac{\partial u_{10}}{\partial \alpha} + \frac{u_{20}}{a_2}\frac{\partial a_1}{\partial \beta} + \frac{w_0 a_1}{R_1}\right) + \frac{\varsigma}{a_1}\left(\frac{\partial \psi_1}{\partial \alpha} + \frac{\psi_2}{a_2}\frac{\partial a_1}{\partial \beta}\right) \quad \text{equation (17.2)}$$

$$\varepsilon_2 = \frac{1}{a_2}\left(\frac{\partial u_{20}}{\partial \beta} + \frac{u_{10}}{a_1}\frac{\partial a_2}{\partial \alpha} + \frac{w_0 a_2}{R_2}\right) + \frac{\varsigma}{a_2}\left(\frac{\partial \psi_2}{\partial \beta} + \frac{\psi_1}{a_1}\frac{\partial a_2}{\partial \alpha}\right) \quad \text{equation (18.2)}$$

$$\gamma_{12} = \frac{a_1}{a_2}\frac{\partial}{\partial \beta}\left(\frac{u_{10}}{a_1}\right) + \frac{a_2}{a_1}\frac{\partial}{\partial \alpha}\left(\frac{u_{20}}{a_2}\right) + \varsigma\left[\frac{a_1}{a_2}\frac{\partial}{\partial \beta}\left(\frac{\psi_1}{a_1}\right) + \frac{a_2}{a_1}\frac{\partial}{\partial \alpha}\left(\frac{\psi_2}{a_2}\right)\right] \quad \text{equation (19.2)}$$

And if you want to do it for love and Timoshenko, then you have to consider the strain displacement relations used in equations (17.1), (18.1), and (19.1). But in our present case, we are going to use the general expression, which is used by Flugge cell theory, and we are going to develop governing equation using strain displacement relations.

In the next lecture, I will explain the stress resultant and start developing the governing equation of shell using the variational principle; the principle of Hamilton.

Thank you very much.