**Theory of Composite Shells**

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**Week - 08**

**Lecture - 27**

**Advanced Material**

Dear learners welcome to week- 08, lecture- 04. In this lecture, I shall explain the Advanced Material. Till now I completed the shell theories and covered 2-dimensional solutions as well as 3-dimensional solutions for Composite Shells.

But these days if you design shell structures for the aerospace application you may have heard about smart structures which means when composite laminated structures are integrated with smart materials like piezoelectric materials, magnetostrictive materials, or shape memory alloys then these structures are known as smart structures.

Besides their original structure-function, they also help to control the vibration, or sometimes we want to know the deflection or any desired function can be done with the help of this smart material. In this lecture, I shall explain to develop these mathematical models, let us say a composite shell having some piezoelectric layer over it.

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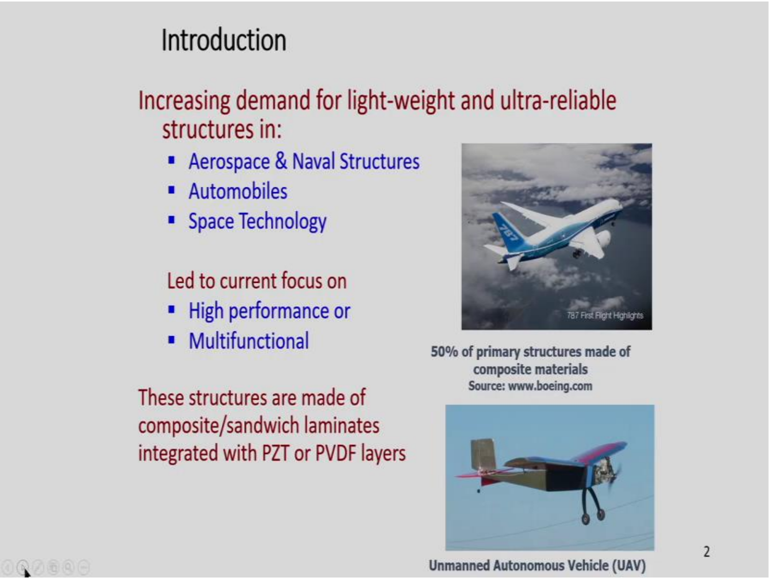

Before that, I will explain the application of these smart structures. Smart structures are extensively used in lightweight and ultra-reliable structures in aerospace & naval structures, automobiles, and space technology. And we need high performance or multifunctional as I said at the beginning of this lecture that we need extra function.

Composites are very good because they give tailorable properties and we can design a lightweight structure. Besides the structural performance, they can do one more task like control or sense, and that is known as multifunctional. When these structures are integrated with some smart materials then these are known as smart structures.

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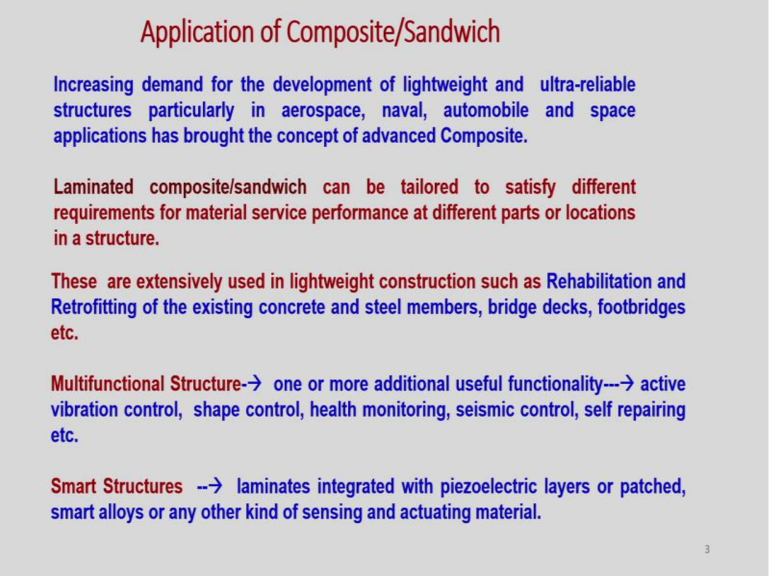


Multifunctional structures have active vibration control, shape control, health monitoring, seismic control, and self-repairing ability. When the structure has any of these functions or may have all the functions depending upon the requirement, then these structures need to be analyzed or mathematically modeled.

The composite material is an orthotropic material. Similarly, the piezoelectric material or the magneto strictive materials are also orthotropic but the behavior becomes more complex because now we have electrical and mechanical field coupling.

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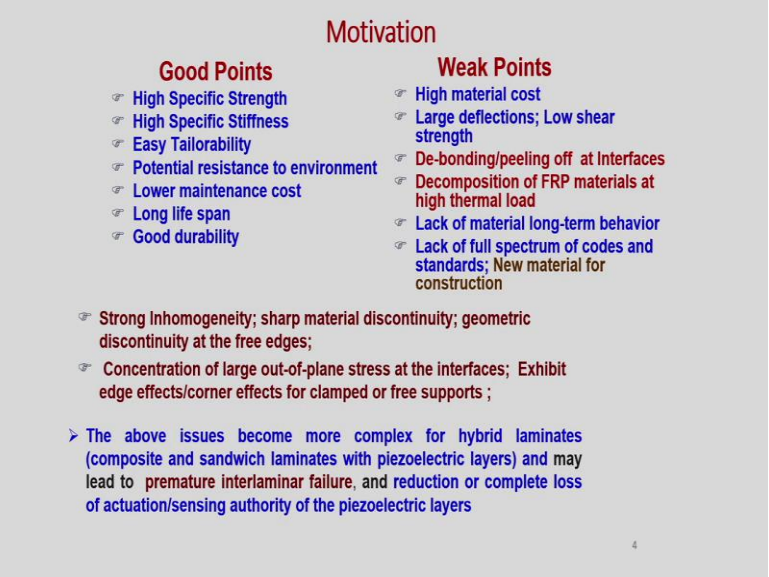


The advantages of these smart structures are high specific strength, stiffness, easy tolerability, low maintenance cost, long life spans, and good durability.

The weak points are the high material cost, de-bonding or peel off takes place at the interfaces which is the most important reason why their complete application is not allowed in different fields and the lack of material long-term behavior like fatigue fracture.

At the interfaces, there is a strong inhomogeneity that occurs like sharp material discontinuity, and geometric discontinuities at the free edges. This causes large out-of-plane stresses at the interfaces and also exhibits the edge effects sometimes it is also known as corner effects for different support conditions and these issues become more complex for a hybrid. The term hybrid is used when we integrate composite laminates with the piezoelectric layer.

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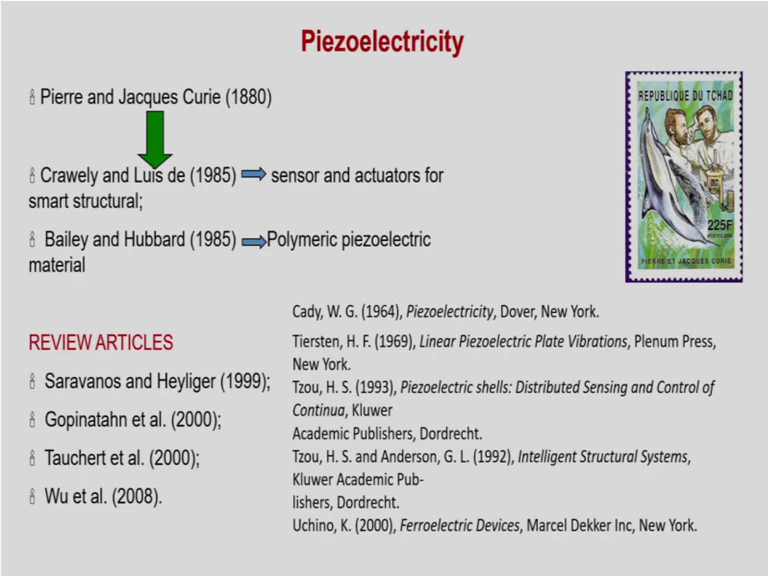


Sometimes we called it a smart structure or a hybrid laminate and these may lead to premature interlaminar failures or in the terms of sensing and actuation these may lead to complete loss of actuation or sensing authority of the piezoelectric layers. First of all, we must know what is the piezoelectricity? The word here piezo means pressure, therefore piezoelectricity means the electricity produced due to pressure.

This is the effect that was discovered by Pierre and Jacques Curie in 1880 and discovered in quartz and crystal. If you apply pressure across the thickness then electrical voltage or current is generated. And there is a vice versa also if you applied an electrical field then the material gets expand or contract.

Initially, it was discovered for some natural materials, then there were some other materials like sugar cane, quartz, crystal, and Rochelle salt. All of them have discovered that this effect exists and most surprisingly even human bones or bones have this piezoelectricity effect. The real application of this effect in structural applications has started in 1985 the first work was by Crawley and Luis de, they developed ultrasonic sensors and sonar devices for underwater applications, and later on, man-made polymeric piezoelectric materials were developed. In this field from the 2000 A.D the review articles are presented and, in these articles, various works done by different researchers have been cited. There are several books available on this topic. The very first book was by Caddy and then by Tiersten.

The book by Tiersten is very famous, when are you going to develop a mathematical model as I developed for a shell that book was devoted to Linear Piezoelectric Plate Vibrations. Then is Tzou, piezoelectric shell, in that book the mathematical modeling of the piezoelectric shell is explained.

Recently Professor Kenji Uchino from Penn State University has developed many devices using the piezoelectric or their effects and has written more than 10 books on Ferroelectric Devices and their manufacturing behavior recently also in 2020 he has published several lectures. If we talk about mathematical modeling then Wu et al 2008 paper is very much important in this field.

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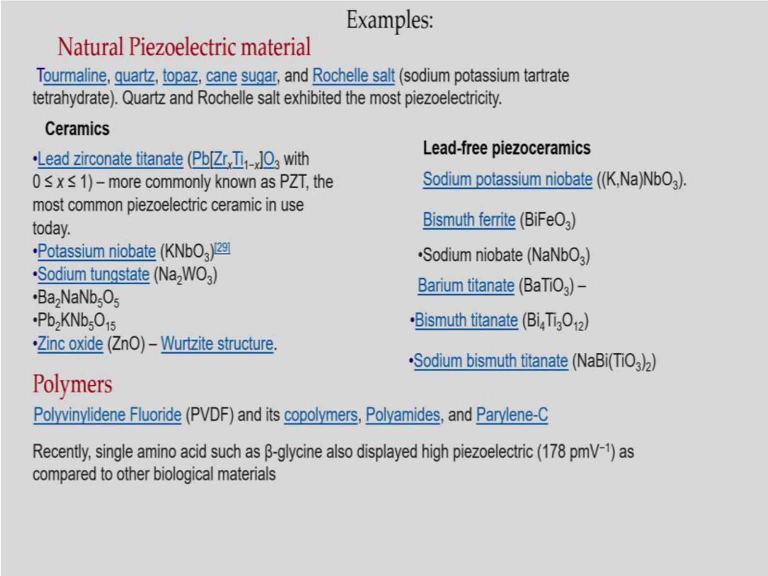


The natural piezoelectric materials are tourmaline, quartz, topaz, cane sugar, and Rochelle salt which shows the effect. We have ceramic materials the most famous are lead zirconate titanate, potassium niobate, sodium tungstate, zinc oxide, etc, these are lead-based piezoelectric materials.

Lead causes environmental pollution and is not good for health, so the concept of lead-free piezo ceramics comes into the picture and the researchers developed different materials like sodium potassium niobate, bismuth ferrite, sodium niobate, barium titanate, bismuth titanate, and sodium bismuth titanate.

Because the ceramics are brittle, therefore, for a flexible application you need a flexible membrane for that case polymers like polyvinylidene fluoride and its copolymers, polyamides, and parylene-C showing the piezoelectric effects are used. In this way, you can see that we have 4 categories of piezoelectric materials: natural, lead-based ceramics, lead-free ceramics, and polymers.

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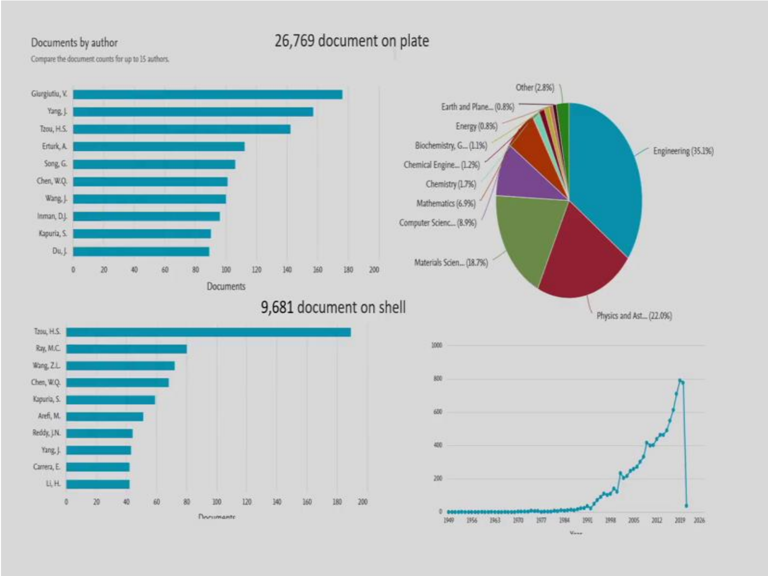


If we look at the published paper in this field, we will see that about 26000 articles are published on the plate and the top authors who published in this field of piezoelectrics are Yang, Tzou, Song, Wang J, and Professor Kapuria. Professor Kapuria from IIT Delhi has developed and worked in this field and a lot of articles have been published by him.

We can see that as far as Chinese, American, and our Indian authors have worked in this field and most of the applications are engineering applications, and more than 27000 articles are published only in mathematical modeling excluding manufacturing and fabrication.

Similarly, if you talk about piezoelectric shells and shell application Tzou H S, Professor M C Ray from IIT Kharagpur, Professor J N Reddy, Yang, and Carrera all developed mathematical models for the shell case and around 10000 articles are available.

These are some analyses that I have taken from a Scopus. These are the number of papers published in this field and the top authors who published in this area. The motto behind showing this slide is just to know that this field is very active and you can see on the graph it is increasing. It has started in 1980 and it is going up and up. In 2019 because of COVID it suddenly dropped, but obviously, it will go up.

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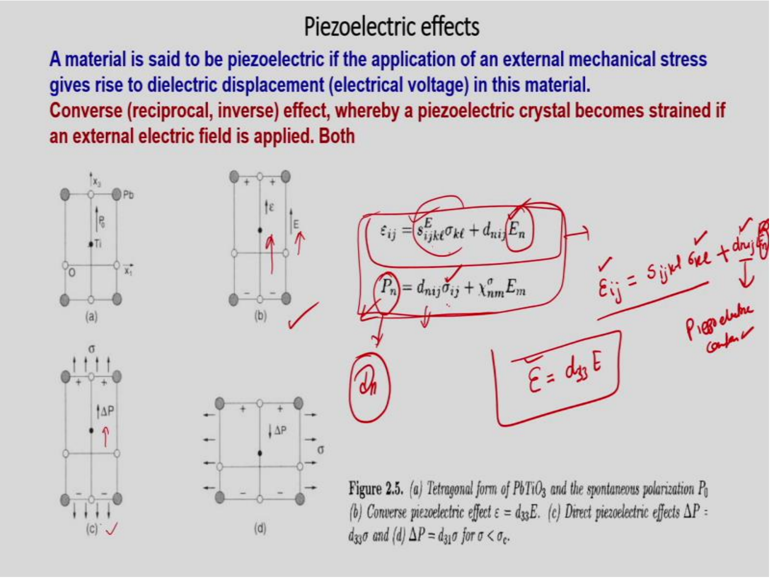


Now, how has this material had this effect? A material is said to be piezoelectric if the application of external mechanical stress gives rise to dielectric displacement in this or it causes an electrical voltage. If you apply a voltmeter to an electrical voltage then there will be a converse effect sometimes it is known as a reciprocal effect or inverse effect.

If you apply an electrical field then there will be a strain. For that purpose, we will have a constitutive relation. If you talk about a lead titanium oxide initially this happens due to that and this is not symmetric it is a slightly stretched position and it is not in the center. When you apply a pulling or stress, dipole moment changes, and due to the net effect is that some charges or electrical voltage can be detected.

Similarly, when you apply an electrical field, then again, a strain takes place in the crystal, this effect is explained here. The constitutive relations just to correlate if you talk about a pure mechanical material let us say steel or composite material:

$$\varepsilon_{ij} = s^{E}_{ijkl}\sigma_{kl} + d_{nij}E_n \,.$$

$s^{E}_{ijkl}\sigma_{kl}$ is we know because stress strains are generated.

If it is a purely elastic material then it will have some piezoelectric effect. Therefore, plus $d_{nij}E_n$.

Where, $E_n$ is the electrical field and $d_{nij}$ is the piezoelectric constant. If you apply an electrical field and this constant, then the strains will be developed. Let us say, there will be no stresses then

$$\varepsilon = d_{33}E$$

If you talk about only a single dimension this converse effect will be there.

Similarly, there is another equation:

$$P_n = d_{nij}\sigma_{ij} + X^0_{nm}E_m \, .$$

Sometimes, it is denoted by $P_n$ or electrical displacement or by electrical flux density. In this way, if you say that some stress is applied then an electrical field will be generated that $\sigma_{ij}$ is stress here $d_{nij}$ is the piezoelectric constant. In this way, our electrical field is coupled with the mechanical field. We have to solve these 2 equations together.

We cannot solve this just by this; we have to take care because the coupling takes place here. You can say that this is a corresponding term, but it will be electrical flux density.

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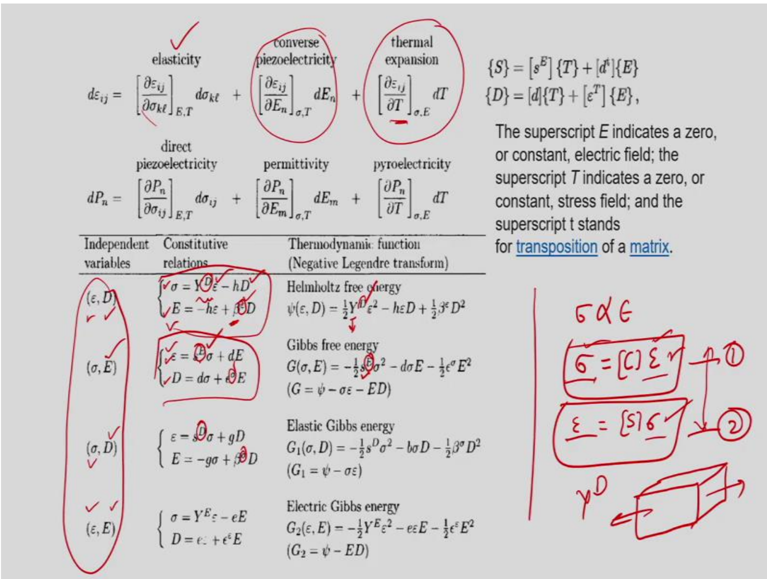


Then again in a general sense here I would like to tell you that these variables can be written in different ways. For example, if you talk about an elastic material sometimes, we say stress is directly proportional to strain.

If you talk about a generalized Hooke's law:

$\sigma = [C]\varepsilon$

And converse of $\varepsilon = [S]\sigma$.

This is for only 2 possibilities available for the case of an elastic material. But now we have 4 variables; first, we say that the stresses are expressed in terms of strain, electrical displacement, and then in the electrical field. Now, we have 2 sets of equations and 4 variables:

$$(\varepsilon, D) \quad \begin{cases} \sigma = Y^D \varepsilon - hD \\ E = -h\varepsilon + \beta^0 D \end{cases}$$

Either you express stresses in terms of strain or strains in terms of stresses here, if strain and electrical displacements are independent variables then stresses and electric fields are expressed like this. If you are saying stress and the electric field is your variable then strain and electrical displacement are expressed like this:

$$(\sigma, E) \quad \begin{cases} \varepsilon = s^E \sigma + dE \\ D = d\sigma + e^0 E \end{cases}$$

$$(\sigma, D) \quad \begin{cases} \varepsilon = s^D \sigma + gD \\ E = -g\sigma + \beta^0 D \end{cases}$$

In this way, 4 combinations are available, depending upon the requirement we may use any one of them. Generally, the small change in strain with respect to the stresses, electrical field, and temperature. Temperature also comes into the picture that effect is also there known as the pyroelectric effect i.e., if you rise a temperature then there will be stress or if you are having some electrical field then some stress will be generated.

$$d_{\varepsilon ij} = \left[\frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}}\right]_{E,T} d\sigma_{kl} + \left[\frac{\partial \varepsilon_{ij}}{\partial E_n}\right]_{\sigma,T} dE_n + \left[\frac{\partial \varepsilon_{ij}}{\partial \mathrm{T}}\right]_{\sigma,E} d\mathrm{T}$$

Where, $\left[\frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}}\right]_{E,T} d\sigma_{kl}$ is the mechanical coupling,

$\left[\frac{\partial \varepsilon_{ij}}{\partial E_n}\right]_{\sigma,T} dE_n$ is the converse piezoelectric coupling, and

$\left[\dfrac{\partial \varepsilon_{ij}}{\partial \mathrm{T}}\right]_{\sigma,E} d\mathrm{T}$ is the thermal extension.

Here, you see that $Y^D$, $s^E$; what is the meaning of that? It means with the constant electrical displacement or with the constant electrical field or with the constant stress and sometimes with the constant strain.

When you study electric or mechanical properties, these mechanical properties changes with their electrical field. If I say that with a constant strain this electrical field, this property or $\beta$ will be obtained having the constant strain or if you say that this Young's modulus is obtained when we subject a sample under constant electrical displacement.

If you come up here the E is the constant electrical field, when a sample is subjected under a constant electrical field and then you apply extension then Young's modulus will be slightly different. It is dependent upon D and E or $\sigma$ and $\varepsilon$. Based on that when you talk about an $\varepsilon$ then that will be dependent on either E or D and E and D are dependent upon either $\sigma$ or strain.

These are the two important things that these properties either you talk about young's modulus or you talk about the dielectric constant are dependent on stress or the electrical field. This means if you change the electric field then you will get a slightly different value.

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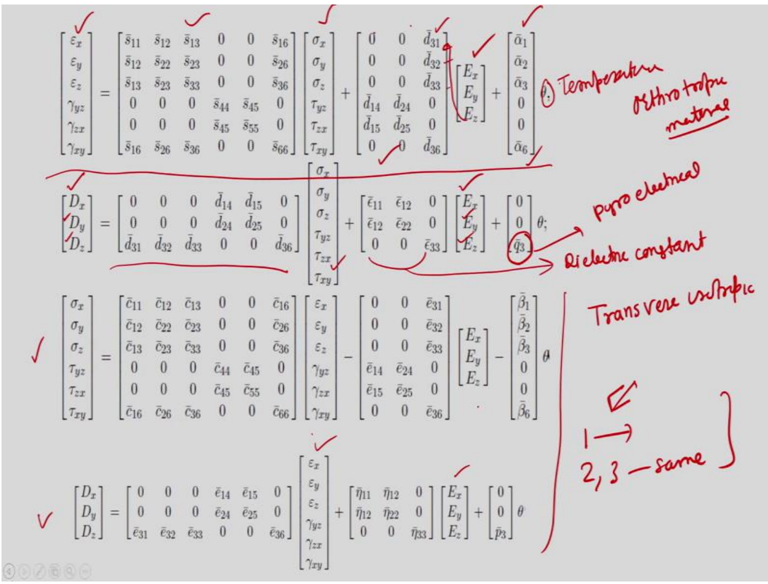

The detailed 3-dimensional constitutive relations are expressed here:

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{zx}\\ \gamma_{xy}\end{bmatrix}=\begin{bmatrix}\bar{s}_{11} & \bar{s}_{12} & \bar{s}_{13} & 0 & 0 & \bar{s}_{16}\\ \bar{s}_{12} & \bar{s}_{22} & \bar{s}_{23} & 0 & 0 & \bar{s}_{26}\\ \bar{s}_{13} & \bar{s}_{23} & \bar{s}_{33} & 0 & 0 & \bar{s}_{36}\\ 0 & 0 & 0 & \bar{s}_{44} & \bar{s}_{45} & 0\\ 0 & 0 & 0 & \bar{s}_{45} & \bar{s}_{55} & 0\\ \bar{s}_{16} & \bar{s}_{26} & \bar{s}_{36} & 0 & 0 & \bar{s}_{66}\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{bmatrix}+\begin{bmatrix}0 & 0 & \bar{d}_{31}\\ 0 & 0 & \bar{d}_{32}\\ 0 & 0 & \bar{d}_{33}\\ \bar{d}_{14} & \bar{d}_{24} & 0\\ \bar{d}_{15} & \bar{d}_{25} & 0\\ 0 & 0 & \bar{d}_{36}\end{bmatrix}\begin{bmatrix}E_x\\ E_y\\ E_z\end{bmatrix}+\begin{bmatrix}\bar{\alpha}_1\\ \bar{\alpha}_2\\ \bar{\alpha}_3\\ 0\\ 0\\ \bar{\alpha}_6\end{bmatrix}\theta$$

You say that these are the strains, compliance matrix, stresses, piezoelectric constants, electrical field, and the coefficient thermal expansion and $\theta$ is the temperature.

Similarly, electrical displacement can be expressed as:

$$\begin{bmatrix}D_x\\ D_y\\ D_z\end{bmatrix}=\begin{bmatrix}0 & 0 & 0 & \bar{d}_{14} & \bar{d}_{15} & 0\\ 0 & 0 & 0 & \bar{d}_{24} & \bar{d}_{25} & 0\\ \bar{d}_{31} & \bar{d}_{32} & \bar{d}_{33} & 0 & 0 & \bar{d}_{36}\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{bmatrix}+\begin{bmatrix}\bar{\in}_{11} & \bar{\in}_{12} & 0\\ \bar{\in}_{12} & \bar{\in}_{22} & 0\\ 0 & 0 & \bar{\in}_{33}\end{bmatrix}\begin{bmatrix}E_x\\ E_y\\ E_z\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \bar{q}_3\end{bmatrix}\theta$$

These are piezoelectric constants, stresses, relative permittivity or dielectric constants, and electrical fields. Now, this $\bar{q}_3$ is known as the pyroelectric constant. Vice versa if you want to know the stresses in terms of strains, and electrical field:

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{bmatrix}=\begin{bmatrix}\bar{c}_{11} & \bar{c}_{12} & \bar{c}_{13} & 0 & 0 & \bar{c}_{16}\\ \bar{c}_{12} & \bar{c}_{22} & \bar{c}_{23} & 0 & 0 & \bar{c}_{26}\\ \bar{c}_{13} & \bar{c}_{23} & \bar{c}_{33} & 0 & 0 & \bar{c}_{36}\\ 0 & 0 & 0 & \bar{c}_{44} & \bar{c}_{45} & 0\\ 0 & 0 & 0 & \bar{c}_{45} & \bar{c}_{55} & 0\\ \bar{c}_{16} & \bar{c}_{26} & \bar{c}_{36} & 0 & 0 & \bar{c}_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{zx}\\ \gamma_{xy}\end{bmatrix}+\begin{bmatrix}0 & 0 & \bar{e}_{31}\\ 0 & 0 & \bar{e}_{32}\\ 0 & 0 & \bar{e}_{33}\\ \bar{e}_{14} & \bar{e}_{24} & 0\\ \bar{e}_{15} & \bar{e}_{25} & 0\\ 0 & 0 & \bar{e}_{36}\end{bmatrix}\begin{bmatrix}E_x\\ E_y\\ E_z\end{bmatrix}-\begin{bmatrix}\bar{\beta}_1\\ \bar{\beta}_2\\ \bar{\beta}_2\\ 0\\ 0\\ \bar{\beta}_6\end{bmatrix}\theta$$

Electrical displacement in terms of strains and electrical field:

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \bar{e}_{14} & \bar{e}_{15} & 0 \\ 0 & 0 & 0 & \bar{e}_{24} & \bar{e}_{25} & 0 \\ \bar{e}_{31} & \bar{e}_{32} & \bar{e}_{33} & 0 & 0 & \bar{e}_{36} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} + \begin{bmatrix} \bar{\eta}_{11} & \bar{\eta}_{12} & 0 \\ \bar{\eta}_{12} & \bar{\eta}_{22} & 0 \\ 0 & 0 & \bar{\eta}_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \bar{p}_3 \end{bmatrix} \theta$$

Here, the relation can be found i.e., it will be as inverse or then some more terms will be there you can find all these things. These matrices are valid only for orthotropic materials, i.e., we assume that the material properties are orthotropic. But if you say that it may be transversely isotropic or it may be some unisotropic then wherever there is 0, some more terms will come up.

But till now whatever we have used during the modeling even though piezoelectric materials are transverse isotropic which means 1 and 3 directions properties remain the same and only the second direction is different; sometimes 1 is different than 2 and 3 are same like the composite material. piezoelectric properties are also following the same manner.

We have now defined the constitutive relations for a piezoelectric material where elastic piezoelectric and thermal are expressed.

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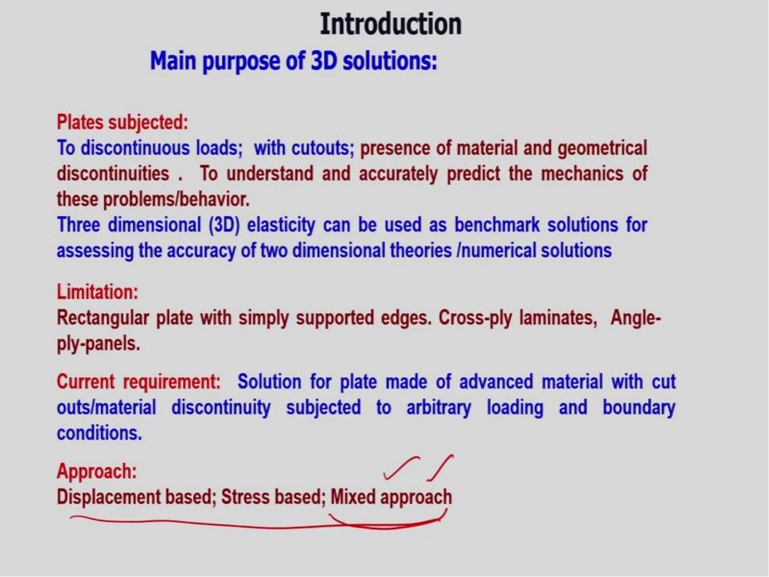


In week 8, I developed the 3-dimensional solutions. Here also I will explain to develop a

3-dimensional solution for a piezoelectric shell, then I will explain the basics of 2-dimensional solutions.

Already we have explained the displacement base approach, stress base approach, and mixed base approach. Out of which Professor Kapuria has developed a mixed base approach. I am going to follow this approach as this is a far better approach as compared to the other two approaches because the accuracy of stresses, as well as displacements, are at the same level.

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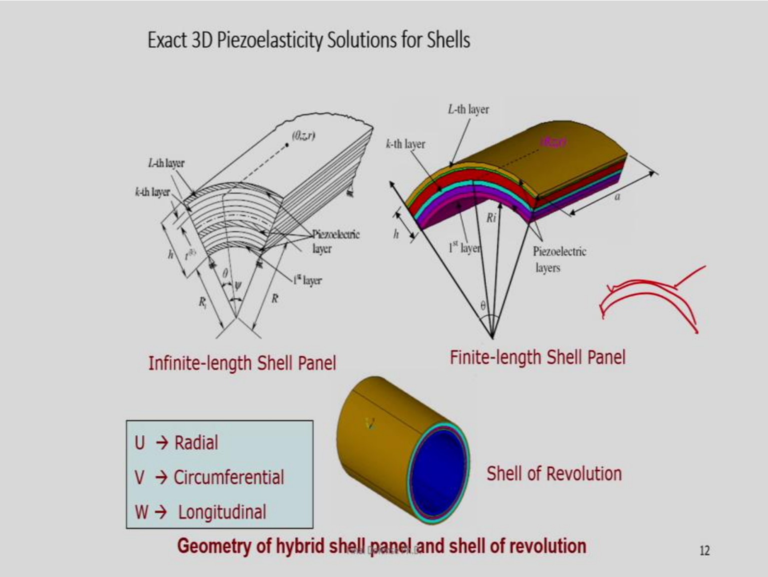


Now, let us assume a cylindrical shell in which some of the layers are piezoelectric. Our concept is the hybrid shell which means that some layers are made of composite materials and some layers are made of piezoelectric materials. In most cases generally, the top layer or bottom layer is a piezoelectric layer, but it may be in between.

This formulation is very generalized you may consider in between piezoelectric layers. U is the radial displacement, V is a circumferential displacement, and W is the longitudinal displacement. The total thickness is h and $\theta$, z, r is the coordinate system and this is the finite length shell. Already you are having some idea about infinite shell and finite shell, this is the geometry of an infinite length shell panel.

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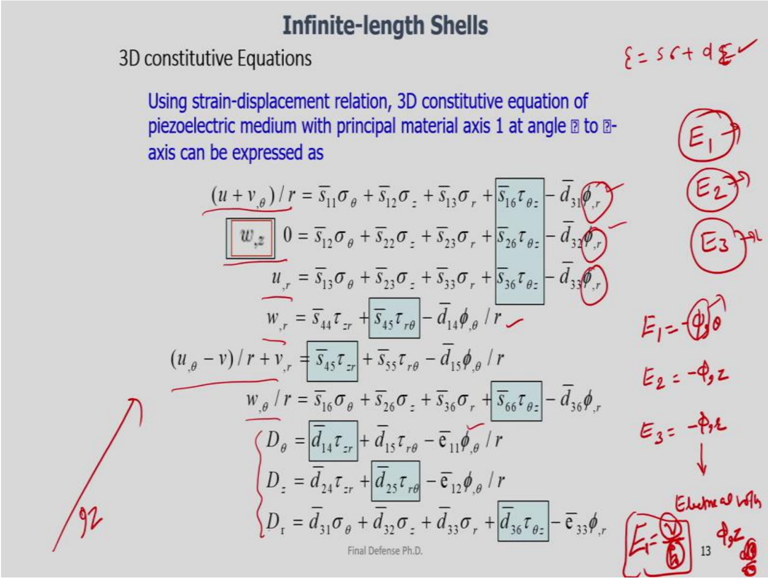


Infinite length panel: if you say that $\varepsilon$ is equal to s times of $\sigma$ plus d times of E, that I will explain later. The strains in the cylindrical coordinate system are expressed here:

$$
\begin{aligned}
&\left(u+v_{,\theta}\right)/r=\bar{s}_{11}\sigma_{\theta}+\bar{s}_{12}\sigma_{z}+\bar{s}_{13}\sigma_{r}+\bar{s}_{16}\tau_{\theta z}-\bar{d}_{31}\phi_{,r}\\
&w,_{z}=\bar{s}_{12}\sigma_{\theta}+\bar{s}_{22}\sigma_{z}+\bar{s}_{23}\sigma_{r}+\bar{s}_{26}\tau_{\theta z}-\bar{d}_{32}\phi_{,r}\\
&u,_{r}=\bar{s}_{13}\sigma_{\theta}+\bar{s}_{23}\sigma_{z}+\bar{s}_{33}\sigma_{r}+\bar{s}_{36}\tau_{\theta z}-\bar{d}_{33}\phi_{,r}\\
&w,_{x}=\bar{s}_{44}\tau_{zr}+\bar{s}_{45}\tau_{r\theta}-\bar{d}_{14}\phi_{,\theta}/r\\
&\left(u_{,\theta}-v\right)/r+v_{,r}=\bar{s}_{45}\tau_{zr}+\bar{s}_{55}\tau_{r\theta}-\bar{d}_{15}\phi_{,\theta}/r\\
&w,_{\theta}=\bar{s}_{16}\sigma_{\theta}+\bar{s}_{26}\sigma_{z}+\bar{s}_{36}\sigma_{r}+\bar{s}_{66}\tau_{\theta z}-\bar{d}_{36}\phi_{,r}
\end{aligned}
$$

Following are the new terms which I have not explained till now

$$
\begin{aligned}
&D_{\theta}=\bar{d}_{14}\tau_{zr}+\bar{d}_{15}\tau_{r\theta}-\bar{e}_{11}\phi_{,\theta}/r\\
&D_{z}=\bar{d}_{24}\tau_{zr}+\bar{d}_{25}\tau_{r\theta}-\bar{e}_{12}\phi_{,\theta}/r\\
&D_{r}=\bar{d}_{31}\sigma_{\theta}+\bar{d}_{32}\sigma_{z}+\bar{d}_{33}\sigma_{r}+\bar{d}_{36}\tau_{\theta z}-\bar{e}_{33}\phi_{,\theta}
\end{aligned}
$$

You see a new variable $\phi$ before that it was $E_1$, $E_2$, and $E_3$. $E_1$ was the electrical field along one direction, $E_2$ is the electrical field along the second direction, and $E_3$ electrical field along the third direction. Let us say:

$E_1 = -\phi,_{\theta}$, $E_2 = -\phi,_{z}$, and $E_3 = -\phi,_{r}$.

Here $\phi$ is known as an electrical voltage.

If we talk in a discrete sense because we have written in a vector form it will be $\frac{v}{h}$, means the voltage across the thickness $-\phi,_z$.

If we write $\phi,_z$ in terms of $\frac{d\phi}{dz}$ ; a small change in voltage upon a small change in thickness. Change in voltage across the thickness if we do so then it gives you an electrical field in that direction.

Here, we are talking about $d_{31}, d_{32}, d_{33}$, if you see slide at 20:32; $E_3$ comes into the picture. $E_3 = -\phi,_r$ and $E_2 = -\phi,_z$.

For the case of an infinite shell panel the derivative with respect to z, the longitudinal axis is eliminated here, you do not have to find dz here.

Therefore, $w,_z = 0$.

Here, all the entities are independent of z and their derivatives are neglected.

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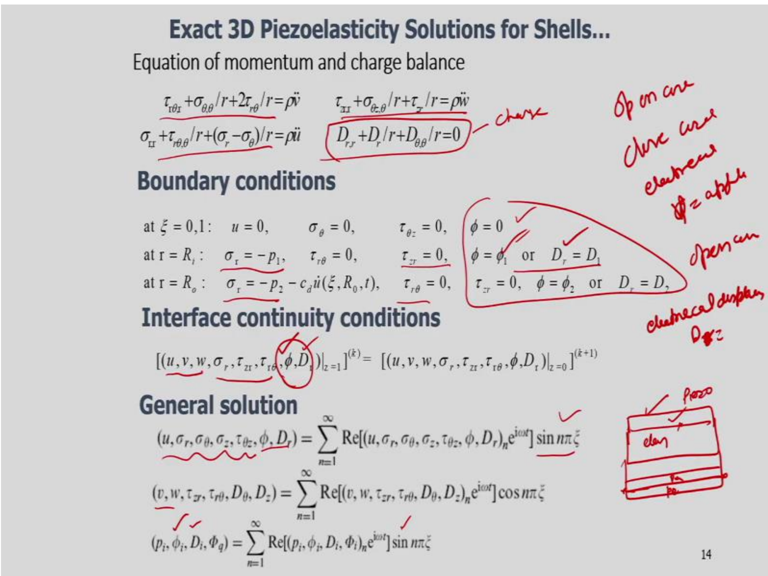


Then we use the concept of the equation of momentum:

$$\tau_{r\theta,r} + \sigma_{\theta,\theta} / r + 2\tau_{r\theta} / r = p\ddot{v}$$
$$\tau_{zr,r} + \sigma_{\theta z,\theta} / r + \tau_{zr} / r = p\ddot{w}$$
$$\sigma_{r,r} + \tau_{r\theta,\theta} / r + \left(\sigma_r - \sigma_\theta\right) / r = p\ddot{u}$$

These equations you are aware of in the case of a cylindrical coordinate system.

$D_{r,r} + D_r / r + D_{\theta,\theta} / r$ is our new equation which is the charge balance equation.

I am here going to explain the new variables

$$\sigma_r = -p_1$$

$$\sigma_r = -p_2 - c_d \dot{u}\left(\xi, R_0, t\right)$$

$\phi = 0;\ \phi = \phi_1\ or\ D_r = D_1;\ \tau_{zr} = 0;\ \phi = \phi_2\ or\ D_r = D_2$ are the new variables that at the top or bottom we can apply electrical field, electrical voltage, or electrical displacements any one of this which means here the concept of open circuit and closed circuit.

When we say closed circuit, it means the electrical voltage applied maybe 0 or non-zero and $\phi$ is applied. When we say open circuit, it means you cannot apply $\phi$, which means their electrical displacement need to be specified that is taken as 0, $D_r$, or $D_z$.

For 3-dimensional solutions we need to satisfy the inter-phase continuity condition:

$$\left[\left(u, v, w, \sigma_r, \tau_{\theta z}, \tau_{r\theta}, \phi, D_1\right)\Big|_{z=1}\right]^{(k)} = \left[\left(u, v, w, \sigma_r, \tau_{\theta z}, \tau_{r\theta}, \phi, D_r\right)\Big|_{z=0}\right]^{(k+1)}$$

$u, v, w$ are the displacements and $\sigma_r, \tau_{\theta z}, \tau_{r\theta}$ are the stresses.

Now, the two variables further come into the picture which is $\phi$ (the electrical field) and $D_1$ the electrical displacement that needs to be continuous at the interfaces. These 2 variables come when we have an interface of a piezo layer, a piezo layer maybe in between or at the interfaces.

A general solution can be written for a simply supported case and we have developed one for a free vibration also. In that case:

$$\left(u, \sigma_r, \sigma_\theta, \sigma_z, \tau_{\theta z}, \phi, D_r\right) = \sum_{n=1}^{\infty} \mathrm{Re}\left[\left(u, \sigma_r, \sigma_\theta, \sigma_z, \tau_{\theta z}, \phi, D_r\right)_n e^{i\omega t}\right] \sin n\pi\xi \ ;$$

$\xi$ is the non-dimensionless coordinate along the $\theta$ direction. Similarly,

$$\left(v, w, \tau_{zr}, \tau_{r\theta}, D_\theta, D_z\right) = \sum_{n=1}^{\infty} \operatorname{Re}\left[\left(v, w, \tau_{zr}, \tau_{r\theta}, D_\theta, D_z\right)_n e^{i\omega t}\right] \cos n\pi\xi \text{ and}$$

$$\left(p_i, \phi_i, D_i, \Phi_q\right) = \sum_{n=1}^{\infty} \operatorname{Re}\left[\left(p_i, \phi_i, D_i, \Phi_q\right)_n e^{i\omega t}\right] \sin n\pi\xi \ .$$

This formulation is a coupled formulation, where you are taking electrical voltage as a variable.

But there are some formulations where electrical voltage is not taken as a variable it is treated as a loading vector, like in the thermal case, we do not solve that, let us say the temperature is known to you and as a loading variable in that case $\phi$ and D will not be a variable.

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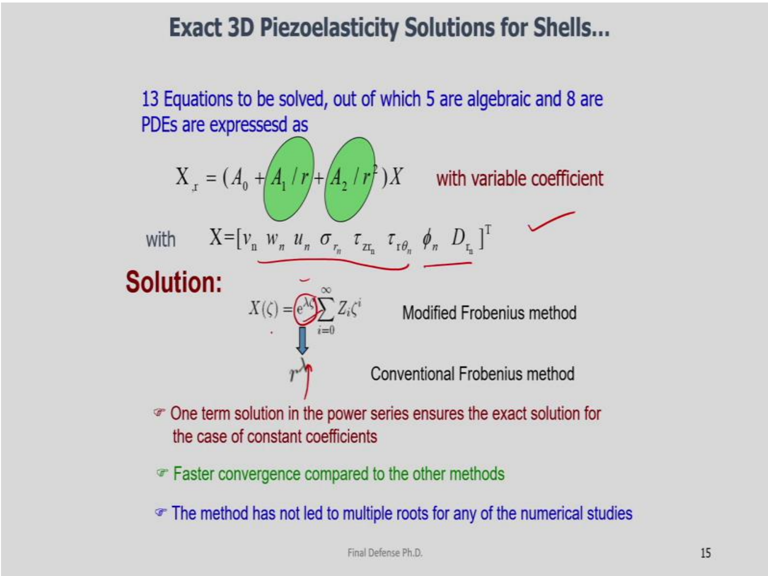


And then you know that governing equations can be expressed in differential equations with varying coefficients:

$$X_{,r} = \left(A_0 + A_1/r + A_2/r^2\right)X$$

Previously, we have 6 variables now we have 8 variables:

$$X = \left[v_n\ w_n\ u_n\ \sigma_{r_n}\ \tau_{zr_n}\ \tau_{r\theta_n}\ \phi_n\ D_{r_n}\right]^T .$$

We can explain the homogeneous solutions using the modified Frobenius series:

$$X(\zeta) = e^{\lambda\zeta} \sum_{i=0}^{\infty} Z_i \zeta_i$$

In the conventional Frobenius series instead of $e^{\lambda\zeta}$ we have $r^{\lambda}$. Presently it is expressed as $e^{\lambda\zeta}$ and if we substitute it into this equation, it becomes an eigenvalue equation.

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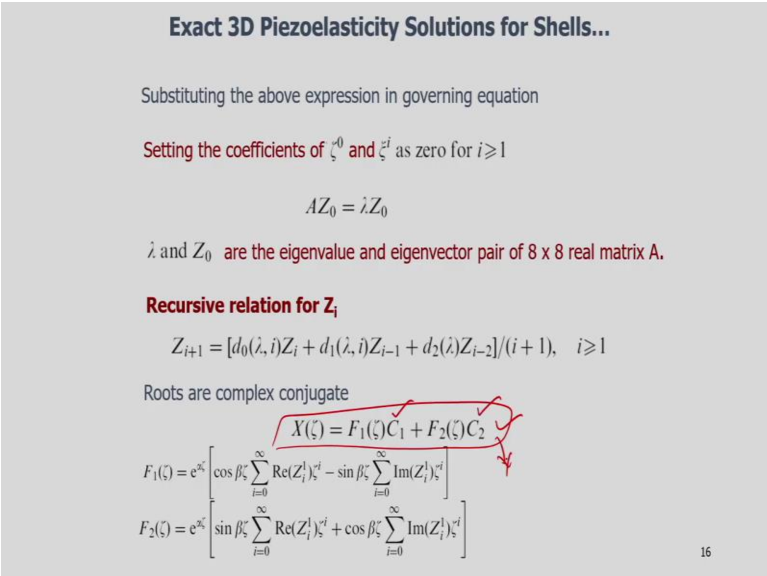


And we solve in a recursive manner:

$$Z_{i+1} = \left[d_0(\lambda, i) Z_i + d_1(\lambda, i) Z_{i-1} + d_2(\lambda, i) Z_{i-2}\right] / (i+1), \; i \geq 1$$

$Z_{i+2}$, $Z_{i+3}$, $Z_{i+4}$, $Z_{i+5}$ can be solved, and depending upon the accuracy or depending upon the required convergence we can say how many terms we have to consider. We have obtained not more than 20 terms in this kind of analysis.

Below the 20 terms the solution converges, therefore the convergence rate is very high and the solution can be written. In most cases, for the case of piezoelectric shells the roots are complex conjugate or real and the final solutions can be represented as:

$$X(\zeta) = F_1 \zeta C_1 + F_2 \zeta C_2$$

Here, $C_1$ and $C_2$ are the arbitrary constants that can be found by satisfying the boundary conditions at the top and bottom.

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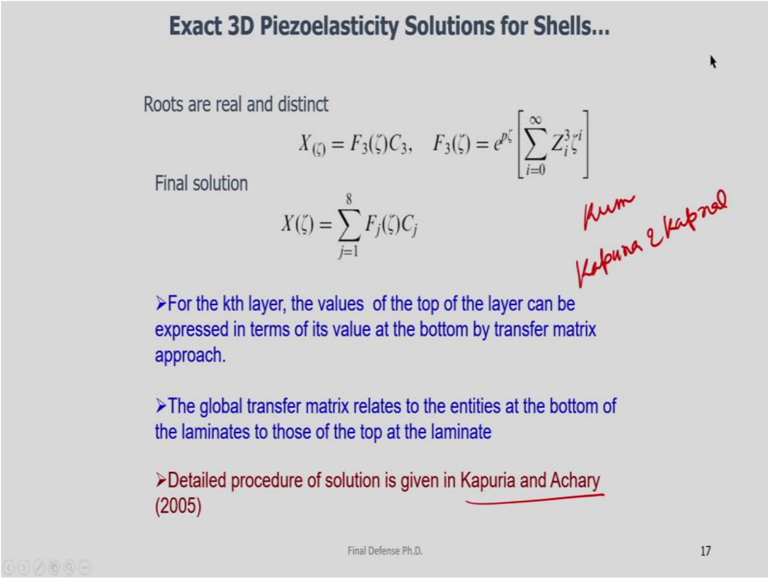


The detailed solution procedures you can find out in Kapuria and Achary's paper and Kapuria and Kumari's paper also, where we have solved a 3-dimensional solution of an infinite-length shell.

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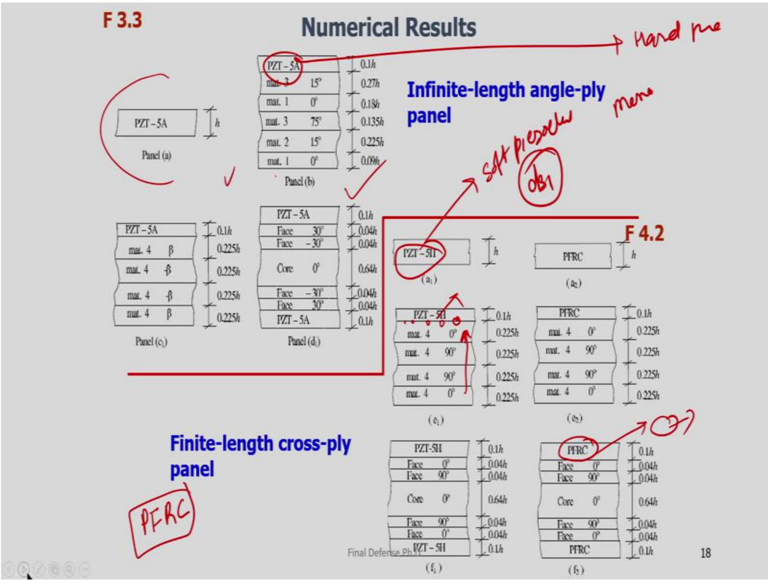


And I would like to say that some results are presented here and these are the configurations.

Let us say the complete single layer of a piezoelectric shell, we may have a composite laminar having any kind of angle and then symmetric layoff, the sandwich in which the core is thicker very less density than face, and PZT-5A symmetric panels. Similarly, we have a concept of PFRC means the Piezo Fiber Reinforced Composites. I would like to explain here that the term is a PZT-5H is a material that is known as soft piezoelectric material having high very high d31 and PZT-5A is known as a hard piezoelectric material.

The monolithic layers, if we attach these layers to these composite laminates, are ceramic in nature, and they crack very easily. The concept of piezo fiber-reinforced composite is where we mix some reinforcing matrix material and prepare it but this is not commercially available and very less work is done in this area.

PFRC is still a concept that Piezo Fiber Reinforced Composite where we assume that like glass fiber, we have a piezoelectric fiber and epoxy matrix so that we will have some flexible laminate so that there will be no cracking. But the concept is that in that only the actuation is less because the dielectric constant becomes low, these days researchers are working in this direction to get a high piezoelectric constant so that the same level of actuations can be obtained like PZT-5A or PZT-5H.

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## 3.2 Material Properties

| Material | $Y_1$ | $Y_2$ | $Y_3$ | $G_{23}$ | $G_{13}$ | $G_{12}$ | $\nu_{12}$ | $\nu_{13}$ | $\nu_{23}$ |
|---|---|---|---|---|---|---|---|---|---|
| Mat. 1* | 6.9 | 6.9 | 6.9 | 2.76 | 2.76 | 2.76 | 0.25 | 0.25 | 0.25 |
| Mat. 2* | 224.25 | 6.9 | 6.9 | 1.38 | 56.58 | 56.58 | 0.25 | 0.25 | 0.25 |
| Mat. 3* | 172.5 | 6.9 | 6.9 | 1.38 | 3.45 | 3.45 | 0.25 | 0.25 | 0.25 |
| Mat. 4† | 181.0 | 10.3 | 10.3 | 2.87 | 7.17 | 7.17 | 0.28 | 0.28 | 0.33 |
| Face‡ | 131.1 | 6.9 | 6.9 | 2.3322 | 3.588 | 3.588 | 0.32 | 0.32 | 0.49 |
| Core‡ | $2.208\times10^{-4}$ | $2.001\times10^{-4}$ | 2.76 | 0.4554 | 0.5451 | 0.01656 | 0.99 | $3\times10^{-5}$ | $3\times10^{-5}$ |
| PZT-5A† | 61.0 | 61.0 | 53.2 | 21.1 | 21.1 | 22.6 | 0.35 | 0.38 | 0.38 |
| | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $k_1$ | $k_2$ | $k_3$ | $\rho$ | | |
| Mat. 1* | 35.6 | 35.6 | 35.6 | 0.12 | 0.12 | 0.12 | 1578 | | |
| Mat. 2* | 0.25 | 35.6 | 35.6 | 7.2 | 1.44 | 1.44 | 1578 | | |
| Mat. 3* | 0.57 | 35.6 | 35.6 | 1.92 | 0.96 | 0.96 | 1578 | | |
| Mat. 4† | 0.02 | 22.5 | 22.5 | 1.5 | 0.5 | 0.5 | 1578 | | |
| Face‡ | 0.0225 | 22.5 | 22.5 | 1.5 | 0.5 | 0.5 | 1000 | | |
| Core‡ | 30.6 | 30.6 | 30.6 | 3.0 | 3.0 | 3.0 | 70 | | |
| PZT-5A† | 1.5 | 1.5 | 2.0 | 1.8 | 1.8 | 1.8 | 7600 | | |
| | $d_{31}$ | $d_{32}$ | $d_{33}$ | $d_{24}$ | $d_{15}$ | $\eta_{11}$ | $\eta_{22}$ | $\eta_{33}$ | $p_3$ |
| PZT-5A† | -171 | -171 | 374 | 584 | 584 | 15.3 | 15.3 | 15.0 | 0.0007 |

Relative Permittivity

$10^{-12}$ $\times 10^{-8}$

**Nondimensionalized natural frequencies and modal displacements**

$$\bar{\omega}_n = \omega_n R S_1 (\rho_0/Y_0)^{1/2}, \quad (\bar{u}, \bar{v}, \bar{w}) = (u, v, w)/\max(u, v, w)$$

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These are the material properties, here you see Young's modulus materials, core, then PZT-5A contains 64 GPA and you see that 1 and 2 material properties are the same. It means the piezoelectric 5H materials are 1, 2 means transverse isometric in 1st and 2nd direction, the third direction is different and these are Young's modulus the extra material properties are piezoelectric constants $d_{31}$, $d_{32}$, $d_{34}$, $d_{24}$, $d_{15}$.

These are in the terms of a picometer say $10^{-12}$. Then we have piezoelectric, dielectric, or relative permittivity, which is known as 15.3, it is having $10^{-8}$ and this is the piezoelectric constant having a unit $10^{-12}$ and this you know in terms of a GPA. It will have $10^{9}$.

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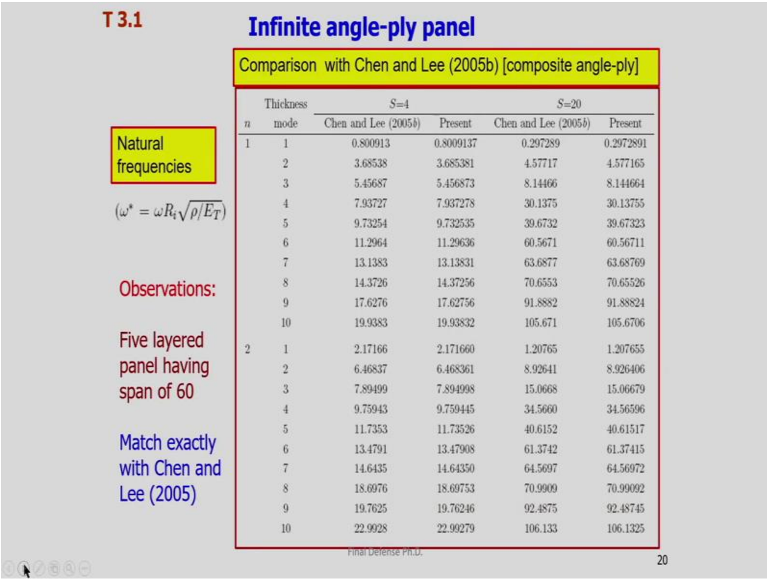


| $n$ | Thickness mode | $S$=4 Chen and Lee (2005$b$) | $S$=4 Present | $S$=20 Chen and Lee (2005$b$) | $S$=20 Present |
|---|---|---|---|---|---|
| 1 | 1 | 0.800913 | 0.8009137 | 0.297289 | 0.2972891 |
| | 2 | 3.68538 | 3.685381 | 4.57717 | 4.577165 |
| | 3 | 5.45687 | 5.456873 | 8.14466 | 8.144664 |
| | 4 | 7.93727 | 7.937278 | 30.1375 | 30.13755 |
| | 5 | 9.73254 | 9.732535 | 39.6732 | 39.67323 |
| | 6 | 11.2964 | 11.29636 | 60.5671 | 60.56711 |
| | 7 | 13.1383 | 13.13831 | 63.6877 | 63.68769 |
| | 8 | 14.3726 | 14.37256 | 70.6553 | 70.65526 |
| | 9 | 17.6276 | 17.62756 | 91.8882 | 91.88824 |
| | 10 | 19.9383 | 19.93832 | 105.671 | 105.6706 |
| 2 | 1 | 2.17166 | 2.171660 | 1.20765 | 1.207655 |
| | 2 | 6.46837 | 6.468361 | 8.92641 | 8.926406 |
| | 3 | 7.89499 | 7.894998 | 15.0668 | 15.06679 |
| | 4 | 9.75943 | 9.759445 | 34.5660 | 34.56596 |
| | 5 | 11.7353 | 11.73526 | 40.6152 | 40.61517 |
| | 6 | 13.4791 | 13.47908 | 61.3742 | 61.37415 |
| | 7 | 14.6435 | 14.64350 | 64.5697 | 64.56972 |
| | 8 | 18.6976 | 18.69753 | 70.9909 | 70.99092 |
| | 9 | 19.7625 | 19.76246 | 92.4875 | 92.48745 |
| | 10 | 22.9928 | 22.99279 | 106.133 | 106.1325 |

We developed that solution and we compared our results with the previous literature results and it was found as a good match for an angle ply shell.

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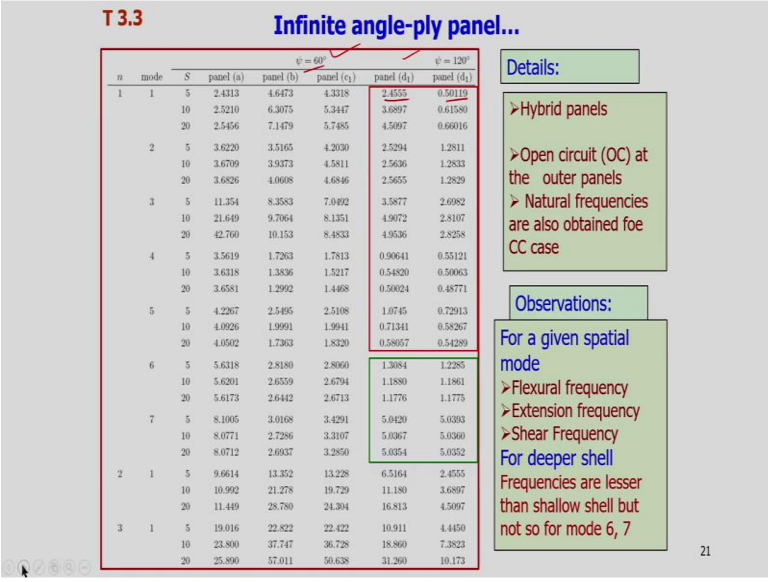


| | | | $\psi = 60^\circ$ | | | | $\psi = 120^\circ$ |
|---|---|---|---|---|---|---|---|
| $n$ | mode | $S$ | panel (a) | panel (b) | panel ($c_1$) | panel ($d_1$) | panel ($d_1$) |
| 1 | 1 | 5 | 2.4313 | 4.6473 | 4.3318 | 2.4555 | 0.50119 |
| | | 10 | 2.5210 | 6.3075 | 5.3447 | 3.6897 | 0.61580 |
| | | 20 | 2.5456 | 7.1479 | 5.7485 | 4.5097 | 0.66016 |
| | 2 | 5 | 3.6220 | 3.5165 | 4.2030 | 2.5294 | 1.2811 |
| | | 10 | 3.6709 | 3.9373 | 4.5811 | 2.5636 | 1.2833 |
| | | 20 | 3.6826 | 4.0608 | 4.6846 | 2.5655 | 1.2829 |
| | 3 | 5 | 11.354 | 8.3583 | 7.0492 | 3.5877 | 2.6982 |
| | | 10 | 21.649 | 9.7064 | 8.1351 | 4.9072 | 2.8107 |
| | | 20 | 42.760 | 10.153 | 8.4833 | 4.9536 | 2.8258 |
| | 4 | 5 | 3.5619 | 1.7263 | 1.7813 | 0.90641 | 0.55121 |
| | | 10 | 3.6318 | 1.3836 | 1.5217 | 0.54820 | 0.50063 |
| | | 20 | 3.6581 | 1.2992 | 1.4468 | 0.50024 | 0.48771 |
| | 5 | 5 | 4.2267 | 2.5495 | 2.5108 | 1.0745 | 0.72913 |
| | | 10 | 4.0926 | 1.9991 | 1.9941 | 0.71341 | 0.58267 |
| | | 20 | 4.0502 | 1.7363 | 1.8320 | 0.58057 | 0.54289 |
| | 6 | 5 | 5.6318 | 2.8180 | 2.8060 | 1.3084 | 1.2285 |
| | | 10 | 5.6201 | 2.6559 | 2.6794 | 1.1880 | 1.1861 |
| | | 20 | 5.6173 | 2.6442 | 2.6713 | 1.1776 | 1.1775 |
| | 7 | 5 | 8.1005 | 3.0168 | 3.4291 | 5.0420 | 5.0393 |
| | | 10 | 8.0771 | 2.7286 | 3.3107 | 5.0367 | 5.0360 |
| | | 20 | 8.0712 | 2.6937 | 3.2850 | 5.0354 | 5.0352 |
| 2 | 1 | 5 | 9.6614 | 13.352 | 13.228 | 6.5164 | 2.4555 |
| | | 10 | 10.992 | 21.278 | 19.729 | 11.180 | 3.6897 |
| | | 20 | 11.449 | 28.780 | 24.304 | 16.813 | 4.5097 |
| 3 | 1 | 5 | 19.016 | 22.822 | 22.422 | 10.911 | 4.4450 |
| | | 10 | 23.800 | 37.747 | 36.728 | 18.860 | 7.3823 |
| | | 20 | 25.890 | 57.011 | 50.638 | 31.260 | 10.173 |

And then we developed a result for a piezoelectric shell in which an open circuit at the outer panel and natural frequencies. When you have a closed circuit and an open circuit the frequencies changes. Having a piezoelectric layer and if you change the electrical circuit condition its natural frequencies change. You can see in $d_1$ panel in open circuit and close circuit $\phi = 60^\circ$ and $\phi = 120^\circ$.

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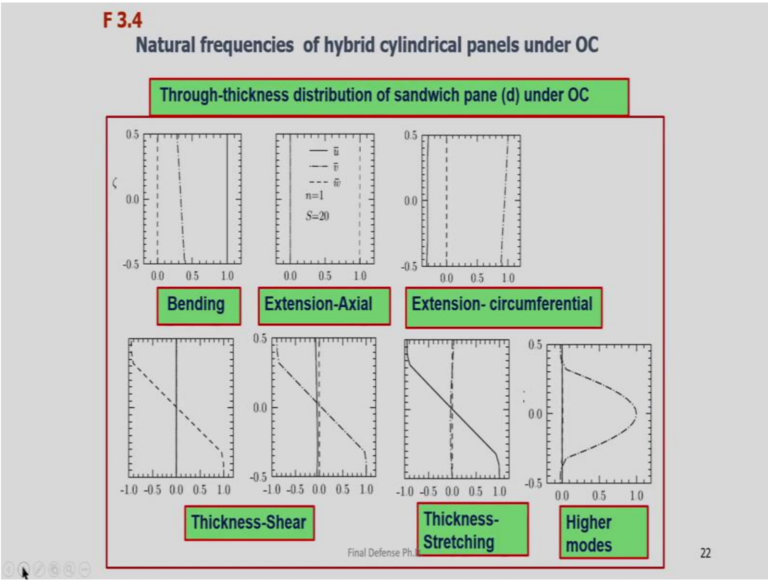

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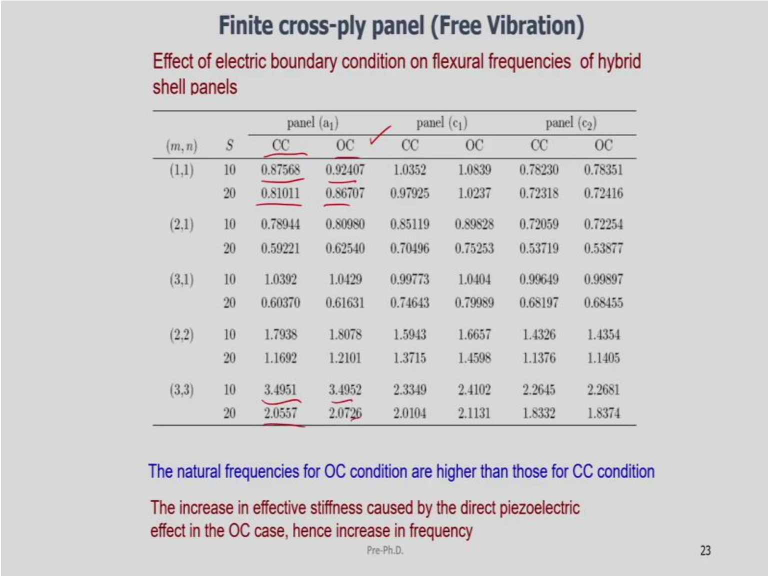
Finite cross-ply panel (Free Vibration)

Effect of electric boundary condition on flexural frequencies of hybrid shell panels

| | | panel ($a_1$) | | panel ($c_1$) | | panel ($c_2$) | |
|---|---|---|---|---|---|---|---|
| $(m, n)$ | $S$ | CC | OC | CC | OC | CC | OC |
| (1,1) | 10 | 0.87568 | 0.92407 | 1.0352 | 1.0839 | 0.78230 | 0.78351 |
| | 20 | 0.81011 | 0.86707 | 0.97925 | 1.0237 | 0.72318 | 0.72416 |
| (2,1) | 10 | 0.78944 | 0.80980 | 0.85119 | 0.89828 | 0.72059 | 0.72254 |
| | 20 | 0.59221 | 0.62540 | 0.70496 | 0.75253 | 0.53719 | 0.53877 |
| (3,1) | 10 | 1.0392 | 1.0429 | 0.99773 | 1.0404 | 0.99649 | 0.99897 |
| | 20 | 0.60370 | 0.61631 | 0.74643 | 0.79989 | 0.68197 | 0.68455 |
| (2,2) | 10 | 1.7938 | 1.8078 | 1.5943 | 1.6657 | 1.4326 | 1.4354 |
| | 20 | 1.1692 | 1.2101 | 1.3715 | 1.4598 | 1.1376 | 1.1405 |
| (3,3) | 10 | 3.4951 | 3.4952 | 2.3349 | 2.4102 | 2.2645 | 2.2681 |
| | 20 | 2.0557 | 2.0726 | 2.0104 | 2.1131 | 1.8332 | 1.8374 |

The natural frequencies for OC condition are higher than those for CC condition

The increase in effective stiffness caused by the direct piezoelectric effect in the OC case, hence increase in frequency

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If you see it from the engineering point of view you say that is change is not much than the 10%, but it may affect the very sophisticated instrument. If you design an instrument that is used to find a very accurate measurement then they play a major role in designing the equipment for sophisticated applications. But in general, if you say 0.87, 0.92 non-dimensionalized natural frequency, there is not too much difference. Initially when you go for a higher mode then you will find there will be not much difference.

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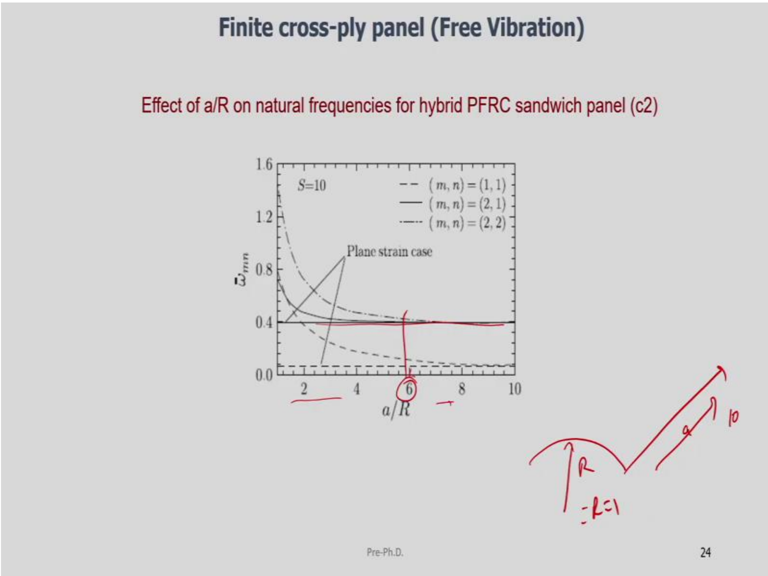

Then the effect of say length, if $\frac{a}{R}$ is increasing then this effect of changes for the plane strain assumption. After 6, your length is along this direction and the radius. If this $\frac{a}{R}$ is coming 6 aspect ratio; that means, it is good you can apply a plane strain assumptions there will be not much difference between the natural frequencies. Either you do this through a very accurate 3-dimensional formulation or by using the plane strain assumptions.

When it is a qualitative like infinite length along the z-direction, let us say if R = 1 and length = 10R, then it will be treated as a plane strain. Infinity does not mean very fine 100 or 20 for the case of a shell, because if you have less means more than 6 it comes into that category. At 10 it is perfectly the same for the case of frequency.

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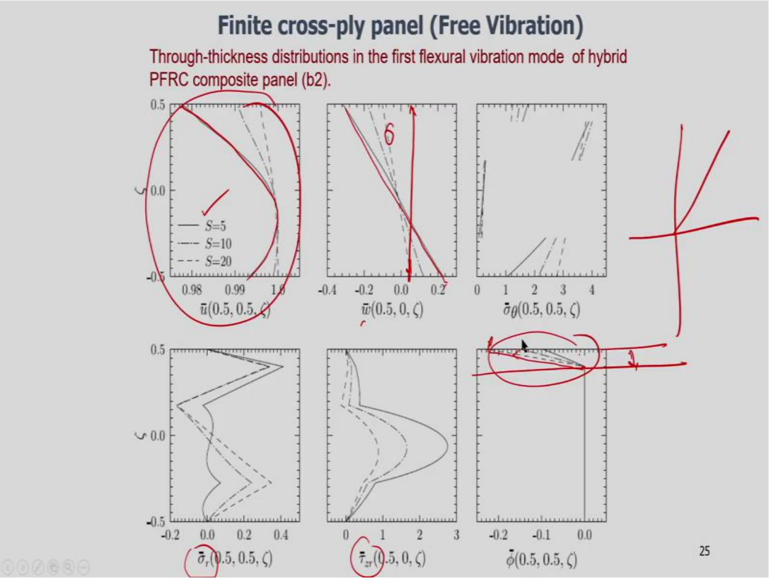


Similarly, the values across the thickness how this displacement varies along the thickness. If you remember that in the case of a first order shear deformation theory we assume that our displacements u varies across the thick linearly. But here you see that in the case of a piezoelectric shell it depending upon the thickness.

Similarly, the values across the thickness mean how this displacement varies along the thickness. If you remember that in the case of a first-order shear deformation theory, we assume that our displacement u varies across the thickness linearly. But here you see that in the case of a piezoelectric shell it is depending upon the thickness.

If it is thick, you see that how it is varying is completely non-linear or quadratic in nature. It is not linearly varying, this 3-dimensional solution helps to tell you that for accurate production we have to go for more terms. And similarly, w is linearly varying but we take a constant for the case of 2-dimensional shell theories. If you assume a 2-dimensional theory even for the case of a thin shell which is the dotted one has 20 still it is not straight.

It is having some variation so by taking more terms in u, v, and w, we can get more accurate solutions, and then you see the behavior of transverse shear stresses and the electrical displacement.

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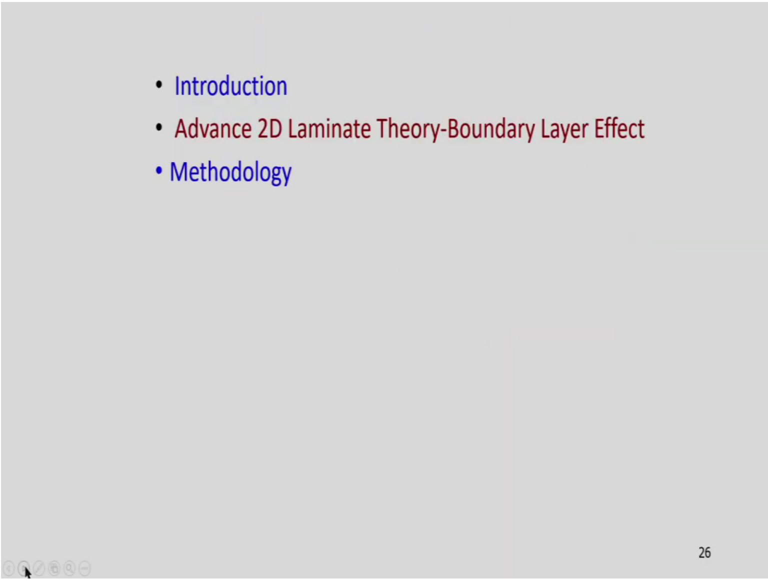


Now we talk about 2-dimensional shell theories. The 2-dimensional theories are divided into two main categories one is coupled another is uncoupled.

If you see in the literature most of the uncoupled theories are available. In the uncoupled theories $\phi$ is not taken as a variable.

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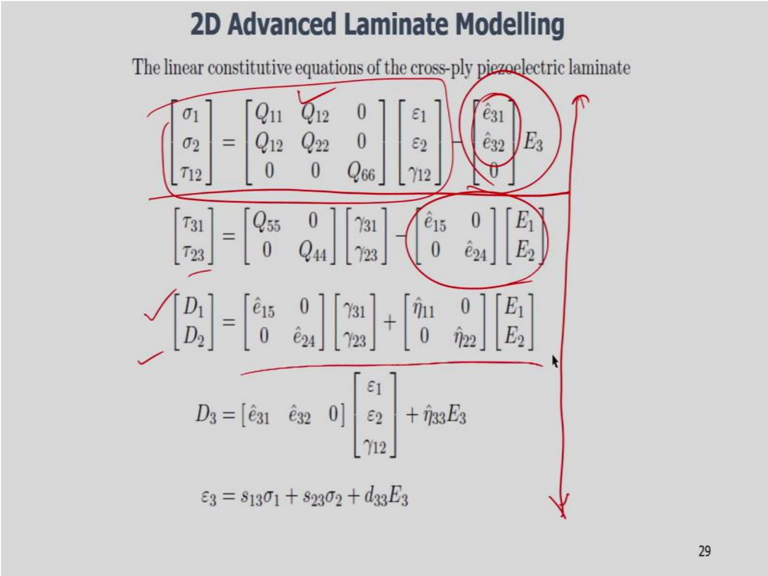


For the case of plane stress:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}$$ is the reduced stiffness matrix

$$\begin{bmatrix} \hat{e}_{31} \\ \hat{e}_{32} \\ 0 \end{bmatrix} E_3$$ is the electrical piezoelectric constant.

Following are the governing equations:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} - \begin{bmatrix} \hat{e}_{31} \\ \hat{e}_{32} \\ 0 \end{bmatrix} E_3$$

$$\begin{bmatrix} \tau_{31} \\ \tau_{23} \end{bmatrix} = \begin{bmatrix} Q_{55} & 0 \\ 0 & Q_{44} \end{bmatrix} \begin{bmatrix} \gamma_{31} \\ \gamma_{23} \end{bmatrix} - \begin{bmatrix} \hat{e}_{15} & 0 \\ 0 & \hat{e}_{24} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} \hat{e}_{15} & 0 \\ 0 & \hat{e}_{24} \end{bmatrix} \begin{bmatrix} \gamma_{31} \\ \gamma_{23} \end{bmatrix} + \begin{bmatrix} \hat{\eta}_{11} & 0 \\ 0 & \hat{\eta}_{22} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

$$D_3 = \begin{bmatrix} \hat{e}_{31} & \hat{e}_{32} & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} + \hat{\eta}_{33} E_3 \quad 10^{-12}$$

$$\varepsilon_3 = s_{13}\sigma_1 + s_{23}\sigma_2 + d_{33}E_3$$

Initially, if you remember the case of an elastic shell I have discussed up to here, but if you want to analyze the piezoelectric then you have to consider these terms $\hat{e}_{31}$, $\hat{e}_{32}$. Similarly, in the transverse shear stresses and displacements, this comes into the picture.

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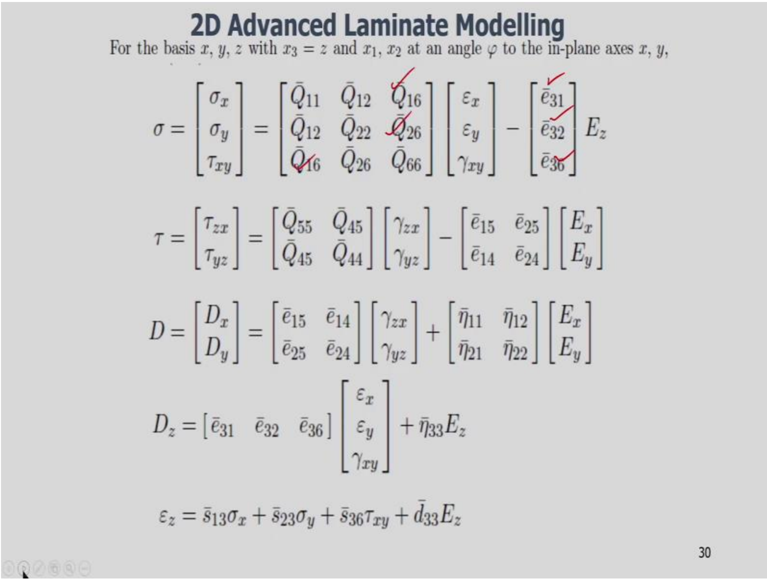


In the case of a full orthotropic, all terms come into the picture:

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} - \begin{bmatrix} \hat{e}_{31} \\ \hat{e}_{32} \\ \hat{e}_{36} \end{bmatrix} E_z$$

$$\tau = \begin{bmatrix} \tau_{zx} \\ \tau_{yz} \end{bmatrix} = \begin{bmatrix} Q_{55} & Q_{45} \\ Q_{45} & Q_{44} \end{bmatrix} \begin{bmatrix} \gamma_{zx} \\ \gamma_{yz} \end{bmatrix} - \begin{bmatrix} \hat{e}_{15} & \hat{e}_{25} \\ \hat{e}_{14} & \hat{e}_{24} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

$$D = \begin{bmatrix} D_x \\ D_y \end{bmatrix} = \begin{bmatrix} \hat{e}_{15} & \hat{e}_{14} \\ \hat{e}_{25} & \hat{e}_{24} \end{bmatrix} \begin{bmatrix} \gamma_{zx} \\ \gamma_{yz} \end{bmatrix} + \begin{bmatrix} \hat{\eta}_{11} & \hat{\eta}_{12} \\ \hat{\eta}_{21} & \hat{\eta}_{22} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

$$D_z = \begin{bmatrix} \hat{e}_{31} & \hat{e}_{32} & \hat{e}_{36} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} + \hat{\eta}_{33} E_z$$

$$\varepsilon_z = \bar{s}_{13}\sigma_x + \bar{s}_{23}\sigma_y + \bar{s}_{36}\tau_{xy} + \bar{d}_{33}E_z$$

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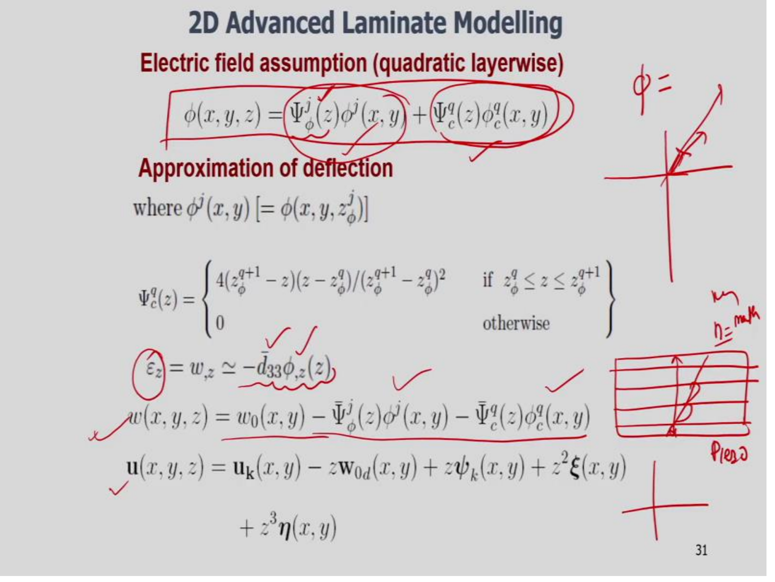


Here, I would like to explain Professor Kapuria has developed electrical field assumption:

$$\phi(x,y,z) = \Psi_\phi^j(z)\phi^j(x,y) + \Psi_c^q(z)\phi_c^q(x,y).$$

If you see the slide at 39:09, this upper layer is our piezoelectric layer. It is varying, it is going linearly let us say 0 to some value. This variation we need to capture, then if we assume a constant, it will not going to solve our purpose. Initially, some different variables were expressed, and after that Professor Kulkarni and Professor Kapuria developed a quadratic piece. Initially, it was a piece-wise linear, let us say in a layer it is linear and next layer not a constant function is taken linear function.

This function is $(z)\phi^j$, it changes from layer to layer. A piezoelectric layer is divided into some n mathematical layers. It is one layer of a piezo, but mathematically it is divided into n sub-layers and in each layer, the variation goes like this piecewise linear, so that we can capture the effect. It has been found that if we consider only this variation $\Psi_\phi^j(z)\phi^j(x,y)$, it does not give an accurate estimation.

Therefore, we need to divide each piezoelectric layer into 4 sub-parts or 5 sub-parts. Later on, Professor Kapuria has given this concept of quadratic consideration can be taken as:

$$\phi(x,y,z) = \Psi_\phi^j(z)\phi^j(x,y) + \Psi_c^q(z)\phi_c^q(x,y)\ .$$

With the help of this concept, we need not mathematically divide it into sub-layers it will be a single layer and by this concept, we can get an accurate estimation in a single layer.

In this way, $w(x,y,z) = w_0(x,y) - \bar{\Psi}_\phi^j(z)\phi^j(x,y) + \bar{\Psi}_c^q(z)\phi_c^q(x,y)$,

This makes a major difference in the accuracy of piezoelectric plates or a shell because of these terms we can get an accurate deflection as well as the other terms. It also reflects because of this it also comes into the displacement field also

$$u(x,y,z) = u_k(x,y) - z w_{0d}(x,y) + z\Psi_k(x,y) + z^2\xi(x,y) + z^3\eta(x,y)\,.$$

We assumed that the strain $\varepsilon_z$ in a piezoelectric shell or panels or plates heavily depends on:

$$w_{,z} \approx -\bar{d}_{33}\phi_{,z}(z)\,.$$

Previously, we have taken $w_{,z}$ = 0, but now we assumed that

$$w_{,z} \neq 0,$$

$$w_{,z} = -\bar{d}_{33}\phi_{,z}(z)$$

And it is substantially giving a significant contribution and because of this some new terms have come up and the rest of the procedure is the same.

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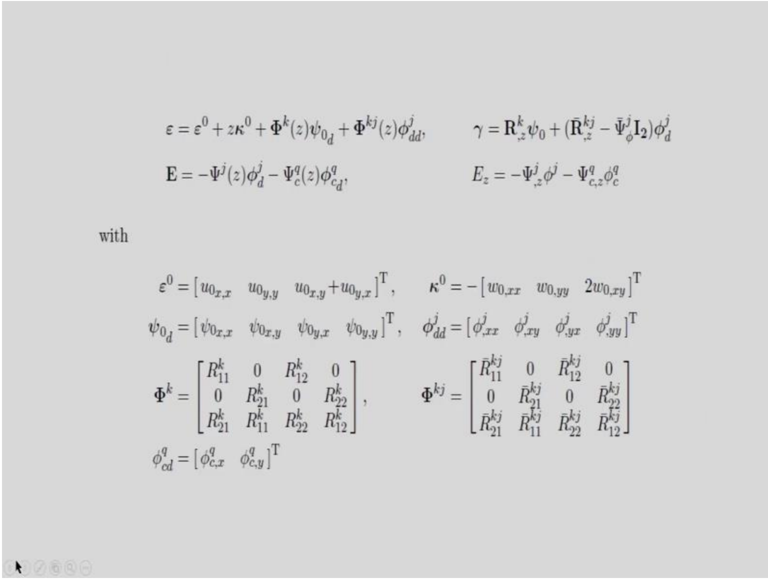


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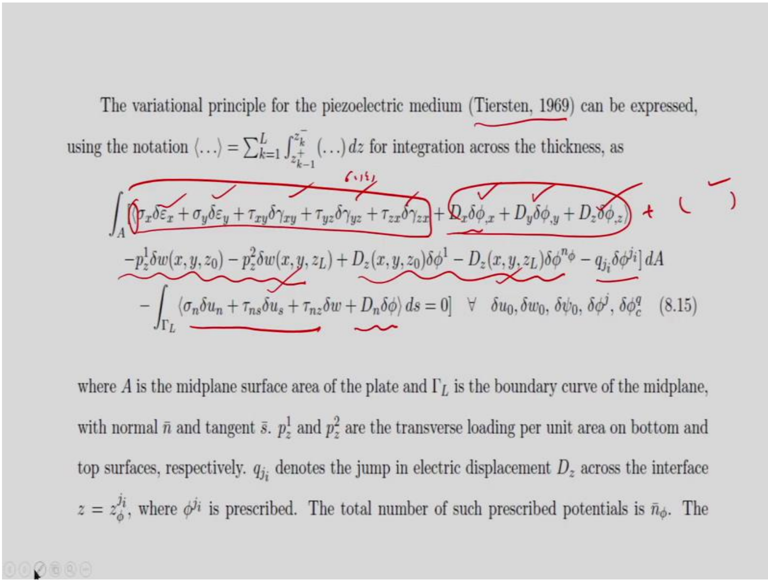


We obtained u and then strains then here is the principle of Hamilton.

In the Hamilton principle, till here $\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx}$, you are aware, these all 3-dimensional terms are there. Now, we have to add extra terms i.e., energy contribution given in Tiersten's book of piezoelectric medium for the electrical field and electrical voltage, $D_x\delta\phi_{,x} + D_y\delta\phi_{,y} + D_z\delta\phi_{,z}$, these 3 terms come into the picture. Plus the contribution due to the external work done $p^1_z\delta w(x,y,z_0) - p^2_z\delta w(x,y,z_L)$ and external work done due to the electrical field $D_z\left(x,y,z_0\right)\delta\phi^1 - D_z\left(x,y,z_L\right)\delta\phi^{n_\phi} - q_{j_i}\delta\phi^{j_i}$ .

These terms come into the picture and equations become more complex and it is known as a modified Hamilton principle for the case of a piezoelectric medium.

$$\int_A \Big[ \big\langle \sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx} + D_x \delta\phi_{,x} + D_y \delta\phi_{,y} + D_z \delta\phi_{,z} \big\rangle - p_z^1 \delta w\left(x,y,z_0\right) - p_z^2 \delta w\left(x,y,z_L\right) + D_z\left(x,y,z_0\right)\delta\phi^1 - D_z\left(x,y,z_L\right)\delta\phi^{n_\phi} - q_{j_i}\delta\phi^{j_i} \Big] dA$$

If you want to study for a piezoelectric medium you have to consider these terms. These terms are given in Professor Kapuria’s paper, you can go through that or any piezoelectric book for the modified Hamilton principle. Some electrical work done is also here:

$$-\int_{TL} \big\langle \sigma_n \delta u_n + \tau_{ns}\delta u_s + \tau_{nz}\delta w + D_n \delta\phi \big\rangle ds = 0$$

Now, I am going for a more generalized form, let us say, you want to study a magnetic plate. You will include some magnetic energy also by the same way and then it is on the area and this small angle bracket is for $\int\limits_{-h/2}^{h/2}$ , if you talk about a composite then it will be $\sum_{k=1}^{L} \int_{z_{k-1}}^{z_k} + (...)\, dz$ .

These are defined like this:

$$\int_\Omega \Big[ \delta\bar{\varepsilon}_1^T F_1 + \delta\bar{\varepsilon}_2^T F_2 + \delta\bar{\varepsilon}_3^T F_3 + \delta\bar{\varepsilon}_4^T F_4 - P_3 \delta w_0 - P_\phi^j \delta\phi^j \Big] dA - \int_{TL} \Big[ N_n \delta u_{0_n} + N_s \delta u_{0_s} - M_n \delta w_{o,n} + \left(V_n + M_{ns,s}\right)\delta w_0 + P_n \delta\psi_{0s} + \left(H_n^j - V_{\phi n}^j - S_{ns,n}^j\right)\delta\phi^j + \delta\phi_{,n}^j S_n^j + \left(\tilde{H}_n^q - \tilde{V}_{\phi n}^q\right)\delta\phi_c^q \Big] ds - \sum_i \Delta M_{ns}\left(s_i\right)\delta w_0\left(s_i\right) - \Delta S_{ns}^j\left(s_i\right)\delta\phi^j\left(s_i\right) = 0$$

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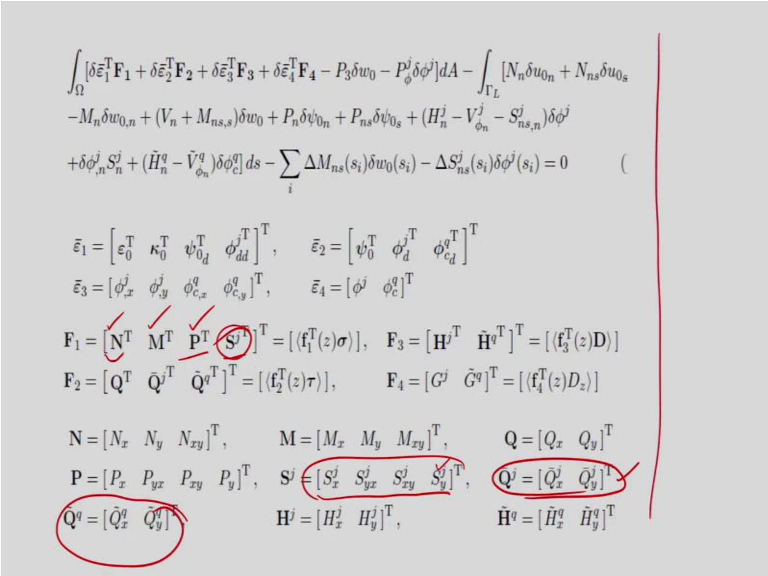

$F_1$, $F_2$, $F_3$ and $F_4$ are the stress resultants.

$$F_1 = \left[ N^T \quad M^T \quad P^T \quad S^{jT} \right]^T = \left[ \left\langle f_1^T(z)\sigma \right\rangle \right]$$

$$F_2 = \left[ Q^T \quad \bar{Q}^{jT} \quad \bar{Q}^{qT} \right]^T = \left[ \left\langle f_2^T(z)\tau \right\rangle \right]$$

Plane stress resultant, moment resultant, higher-order moments, and some electrical field resultants are here. In this way, some new terms need to be defined, whether it is electrical stress resultant.

And $S^j = \left[ S_x^j \quad S_{yx}^j \quad S_{xy}^j \quad S_y^j \right]^T$ , these are due to the electrical field.

Similarly,

$$\bar{Q}^j = \left[ \bar{Q}_x^j \quad \bar{Q}_y^j \right]^T \quad \tilde{Q}^q = \left[ \tilde{Q}_x^q \quad \tilde{Q}_y^q \right]^T .$$

These are some new terms that come into the picture and corresponding definitions need to find out. One can go through this kind of formulation and can develop the governing equations for the case of the shell.

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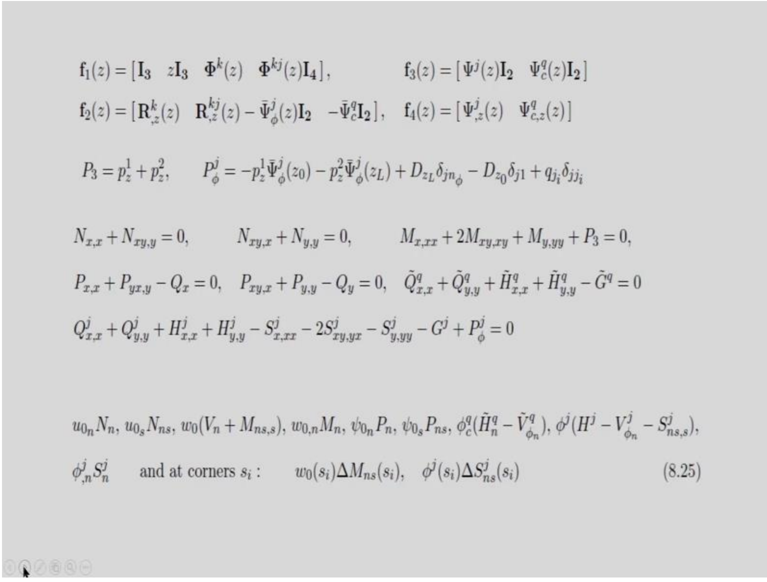


I have tried to briefly explain to develop the governing equations or what are the basics behind the development. You have to systematically add your electrical terms.

I hope that it will help you to develop for more advanced material and you can go through a number of books and papers on this piezoelectric shell modeling.

With this, I would like to say thank you very much.