Theory of Composite Shells

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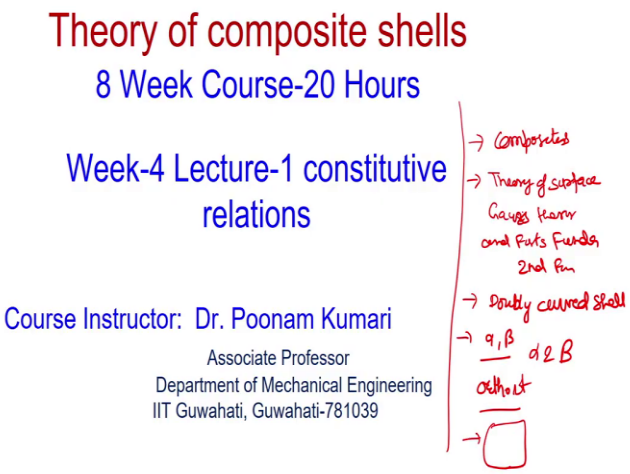
Week – 04

Lecture – 01

Constitutive relations



Dear learners welcome to week- 04, lecture- 01. In the 1st week, we covered the basic concept of composites, in the 2nd week, we developed the Theory of Surfaces in which we obtained the Gauss theorem, the first and second fundamental theorem of surfaces, and other derivatives of normal and tangent vectors.

In the 3rd week, we developed governing equations for a doubly curved shell. We derived that from the basics by assuming a first-order displacement field i.e.,

$u_1 = u_{10} + \psi_1 \varsigma$, $u_2 = u_{20} + \psi_2 \varsigma$, and $u_3 = w_0$,

Where transverse displacement is constant along the thickness.

Using that displacement field, we developed the basic governing equations of shell in curvilinear parameters α and β . These are orthogonal curvilinear parameters. And, we also discussed the special cases. From the basic shell equations can we get the equation of shells for a cylindrical shell, a spherical shell, a conical shell, a plate, or a circular plate?

In today's lecture, I will discuss proceeding further after getting the governing equations. What are the steps to be followed to get the solutions? Ultimately, we are interested to find the displacement and stresses moments of the shell subjected to loading and boundary conditions.

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$$\frac{1}{a_1a_2}\left[(N_{11}a_2)_{,\alpha}-N_{22}a_{2,\alpha}+(N_{21}a_1)_{,\beta}+N_{12}a_{1,\beta}\right]+\frac{Q_1}{R_1}+\left(\hat{N}_{11}\frac{1}{a_1R_1}\left(w_{0,\alpha}-\frac{a_1u_{10}}{R_1}\right)\right)$$
$$+\tilde{N}_{12}\frac{1}{a_2R_1}\left(w_{0,\beta}-u_{20}\frac{a_2}{R_2}\right)+q_1=\left(I_0\ddot{u}_{10}+I_1\ddot{\psi}_1\right)$$

$$\frac{1}{a_1a_2}\left[-N_{11}a_{1,\beta}+(N_{22}a_1)_{,\beta}+N_{21}a_{2,\alpha}+(N_{12}a_2)_{,\alpha}\right]+\frac{Q_2}{R_2}+\left(\hat{N}_{22}\frac{1}{a_2R_2}\left(w_{0,\beta}-\frac{a_2u_{20}}{R_2}\right)\right)$$
$$+\tilde{N}_{12}\frac{1}{a_1R_2}\left(w_{0,\alpha}-u_{10}\frac{a_1}{R_1}\right)+q_2=\left(I_0\ddot{u}_{20}+I_1\ddot{\psi}_2\right)$$

$$\frac{1}{a_1a_2}\left[-M_{22}a_{2,\alpha}+(M_{11}a_2)_{,\alpha}+(M_{21}a_1)_{,\beta}+M_{12}a_{1,\beta}\right]-Q_1=(I_1\ddot{u}_{10}+I_2\ddot{\psi}_1)$$

$$\frac{1}{a_1a_2}\left[-M_{11}a_{1,\beta}+(M_{22}a_1)_{,\beta}+M_{21}a_{2,\alpha}+(M_{12}a_2)_{,\alpha}\right]-Q_2=(I_1\ddot{u}_{20}+I_2\ddot{\psi}_2)$$

$$\frac{1}{a_1a_2}\left\{\left(\hat{N}_{11}\frac{a_2}{a_1}\left(w_{0,\alpha}-\frac{a_1u_{10}}{R_1}\right)\right)_{,\alpha}+\left(\hat{N}_{22}\frac{a_1}{a_2}\left(w_{0,\beta}-\frac{a_2u_{20}}{R_2}\right)\right)_{,\beta}+\left(\tilde{N}_{12}\left(w_{0,\beta}-\frac{a_2u_{20}}{R_2}\right)\right)_{,\alpha}+\left(\tilde{N}_{12}\left(w_{0,\alpha}-u_{10}\frac{a_1}{R_1}\right)\right)_{,\beta}\right\}$$
$$+\left[-\frac{N_{11}}{R_1}-\frac{N_{22}}{R_2}\right]+\frac{(Q_1a_2)_{,\alpha}}{a_1a_2}+\frac{(Q_2a_1)_{,\beta}}{a_1a_2}-q_3=I_0\ddot{w}_0$$

$$5\left\{\begin{array}{l} u_{10}\\ u_{20}\\ \psi_1\\ \psi_2\\ w_0 \end{array}\right.$$

Following are the basic governing equations of shell in which we have taken the non-linear terms as well as linear terms.

$$\frac{1}{a_1a_2}\left[\left(N_{11}a_2\right)_{,\alpha}-N_{22}a_{2,\alpha}+\left(N_{21}a_1\right)_{,\beta}+N_{12}a_{1,\beta}\right]+\frac{Q_1}{R_1}+\left(N_{11}\frac{1}{a_1R_1}\left(w_{0,\alpha}-\frac{a_1u_{10}}{R_1}\right)\right)$$

$$+N_{12}\frac{1}{a_2R_1}\left(w_{0,\beta}-u_{20}\frac{a_2}{R_2}\right)+q_1=\left(I_0\ddot{u}_{10}+I_1\ddot{\psi}_1\right)\quad equation(1)$$

$$\frac{1}{a_1a_2}\left[-N_{11}a_{1,\beta}+\left(N_{22}a_1\right)_{,\beta}+N_{21}a_{2,\alpha}+\left(N_{12}a_2\right)_{,\alpha}\right]+\frac{Q_2}{R_2}+\left(N_{22}\frac{1}{a_2R_2}\left(w_{0,\beta}-\frac{a_2u_{20}}{R_2}\right)\right)$$

$$+N_{12}\frac{1}{a_1R_2}\left(w_{0,\beta}-u_{10}\frac{a_1}{R_1}\right)+q_2=\left(I_0\ddot{u}_{20}+I_1\ddot{\psi}_2\right)\quad equation(2)$$

$$\frac{1}{a_1a_2}\left[-M_{22}a_{2,\alpha}+\left(M_{11}a_2\right)_{,\alpha}+\left(M_{21}a_1\right)_{,\beta}+M_{12}a_{1,\beta}\right]-Q_1=\left(I_1\ddot{u}_{10}+I_2\ddot{\psi}_1\right)\quad equation(3)$$

$$\frac{1}{a_1a_2}\left[-M_{11}a_{1,\beta}+\left(M_{22}a_1\right)_{,\beta}+M_{21}a_{2,\alpha}+\left(M_{12}a_2\right)_{,\alpha}\right]-Q_2=\left(I_1\ddot{u}_{20}+I_2\ddot{\psi}_2\right)\quad equation(4)$$

$$\frac{1}{a_1a_2}\left[\left(N_{11}\frac{a_2}{a_1}\left(w_{0,\alpha}-\frac{a_1u_{10}}{R_1}\right)\right)_{,\alpha}+\left(N_{22}\frac{a_1}{a_2}\left(w_{0,\beta}-\frac{a_2u_{20}}{R_2}\right)\right)_{,\beta}+\left(N_{12}\left(w_{0,\beta}-u_{20}\frac{a_2}{R_2}\right)\right)_{,\alpha}+\left(N_{12}\left(w_{0,\alpha}-u_{10}\frac{a_1}{R_1}\right)\right)_{,\beta}\right]$$

$$+\left(-\frac{N_{11}}{R_1}-\frac{N_{22}}{R_2}\right)+\frac{\left(Q_1a_2\right)_{,\alpha}}{a_1a_2}+\frac{\left(Q_2a_1\right)_{,\beta}}{a_1a_2}=I_0\ddot{w}_0\quad equation(5)$$

We developed these five sets of governing equations.

Can we solve this in this form? Can we get the solution of a shell subjected to boundary conditions?

Whether you talk about simply a circular cylindrical shell if you convert them to the equations, can we get the solution in the same form? No, we cannot get the solution in terms of primary displacements $u_{10},\, u_{20},\, w_0,\, \psi_1,\, and\, \psi_2$.

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Linear Shell Equations

$$\frac{1}{a_1a_2}\left[\left(N_{11}a_2\right)_{,\alpha} - N_{22}a_{2,\alpha} + \left(N_{21}a_1\right)_{,\beta} + N_{12}a_{1,\beta}\right] + \frac{Q_1}{R_1} + q_1 = \left(I_0\ddot{u}_{10} + I_1\ddot{\psi}_1\right)$$

$$\frac{1}{a_1a_2}\left[-N_{11}a_{1,\beta} + \left(N_{22}a_1\right)_{,\beta} + N_{21}a_{2,\alpha} + \left(N_{12}a_2\right)_{,\alpha}\right] + \frac{Q_2}{R_2} + q_2 = \left(I_0\ddot{u}_{20} + I_1\ddot{\psi}_2\right)$$

$$\frac{1}{a_1a_2}\left[-M_{22}a_{2_{,\alpha}} + \left(M_{11}a_2\right)_{,\alpha} + \left(M_{21}a_1\right)_{,\beta} + M_{12}a_{1,\beta}\right] - Q_1 = \left(I_1\ddot{u}_{10} + I_2\ddot{\psi}_1\right)$$

$$\frac{1}{a_1a_2}\left[-M_{11}a_{1_{,\beta}} + \left(M_{22}a_1\right)_{,\beta} + M_{21}a_{2,\alpha} + \left(M_{12}a_2\right)_{,\alpha}\right] - Q_2 = \left(I_1\ddot{u}_{20} + I_2\ddot{\psi}_2\right)$$

$$\left[-\frac{N_{11}}{R_1} - \frac{N_{22}}{R_2}\right] + \frac{\left(Q_1a_2\right)_{,\alpha}}{a_1a_2} + \frac{\left(Q_2a_1\right)_{,\beta}}{a_1a_2} - q_3 = I_0\ddot{w}_0$$

We have to convert these equations into the primary variables $u_{10}, u_{20}, w_0, \psi_1, and\, \psi_2$. For that, we use the shell constitutive relations. Here, I have given the linear shell equations, in most of the cases the linear shell equations are complex to solve and if we add non-linear terms, it will further make it more difficult to solve.

Analytically or from a lecture point of view we aim to explain the basic steps to solve a linear equation, and later on, the non-linear terms can be solved.

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Shell Constitive Equation

$\perp$ $\varepsilon_{ij} = S_{ijkl}\sigma_{KL}$ $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} + \underbrace{\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} T}$$

For moderately thick to thin shell case

$\sigma_{33} \approx 0$

3D

GyE eqn

Theo-

For that, we need the shell constitutive relations. You may be aware of the generalized Hooke's law which is:

$\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$ or vice versa $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$.

This is just the reverse relation. A purely mechanical shell means elastic shell that $\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$

Here, S is known as compliances matrix and C is known as stiffness matrix. We are aware that it is available for an orthotropic material.

Why have we chosen an orthotropic material? The shell is made of a composite shell, a sandwich shell, a graphite-epoxy shell, a glass epoxy shell, or any other kind of composite material, they are generally orthotropic. Metal properties are orthotropic.

So, we have chosen an orthotropic matrix. If we have chosen an isotropic matrix then some of the elements will be just a function of first. S_{11}, S_{22}, S_{33} will be the same for the case of isotropic material. So, we have taken an orthotropic material and this is the 3-dimensional relation available for a material where strain can be expressed in terms of stresses using this compliance matrix and we have also taken thermal loading.

Sometimes, we want to analyze a composite shell under temperature loading if we are interested in such cases then we can include the temperature effect also. For a known temperature we can find the stress is known as thermoelastic analysis of a shell.

These developed shell theories give accurate analysis to moderately thick to thin shells and the basic assumption behind is what we have developed under plane stress assumption. Under plane stress assumption, we assume that σ_{33} = 0. There is no stress in the third direction, it is the thickness direction.

As per the physics, we can say that the shell is very thin and the material is less, it cannot take stress in the third direction. By using that assumption, we can develop the shell theory. If we apply this assumption and put σ_{33} = 0, then these equations are reduced and can be expressed like this:

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{bmatrix}=\begin{bmatrix}S_{11} & S_{12} & S_{13} & o & o & o\\ S_{12} & S_{22} & S_{23} & o & o & o\\ S_{13} & S_{23} & S_{33} & o & o & o\\ o & o & o & S_{44} & o & o\\ o & o & o & o & S_{55} & o\\ o & o & o & o & o & S_{66}\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \tau_{23}\\ \tau_{13}\\ \tau_{12}\end{bmatrix}+\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ o\\ o\\ o\end{bmatrix}T$$

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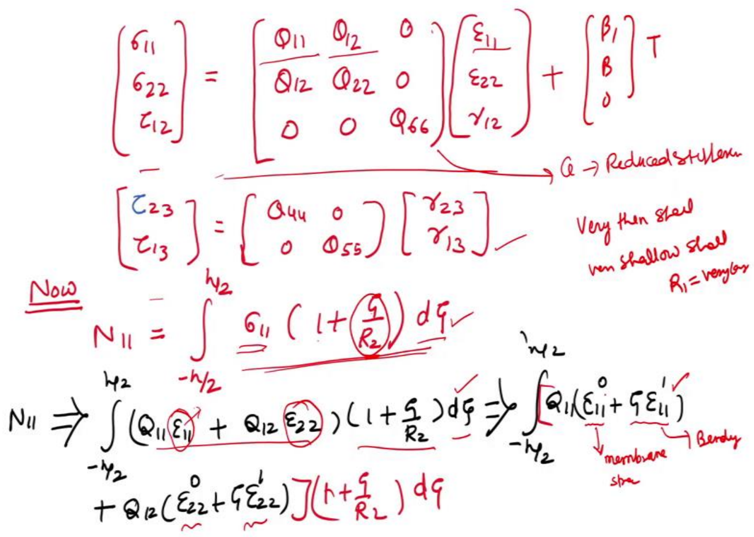

I have just expressed for the mechanical case, including the temperature also one can write like this:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \tau_{12}\end{bmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & o\\ Q_{12} & Q_{22} & o\\ o & o & Q_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{bmatrix}\quad \begin{bmatrix}\tau_{23}\\ \tau_{13}\end{bmatrix}=\begin{bmatrix}Q_{44} & o\\ o & Q_{55}\end{bmatrix}\begin{bmatrix}\gamma_{23}\\ \gamma_{13}\end{bmatrix}$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \tau_{12}\end{bmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & o\\ Q_{12} & Q_{22} & o\\ o & o & Q_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{bmatrix}+\begin{bmatrix}\beta_1\\ \beta_2\\ o\end{bmatrix}T$$

σ_{11} can be expressed in terms of a matrix Q, where matrix Q is known as a reduced stiffness matrix.

This is a very standard notation and one can find these types of notations in any book of mechanics of composites or the books on the theory of composite plates and shells. In the earlier books of shells, only the isotropic shells are discussed. These types of equations are not discussed in those books, but in the recent books after the 1990s, the composite material has been taken and research has been done.

In those papers or books, you will find this kind of matrix.

What is the definition of in-plane stress resultant N_{11}?

It is defined as:

$$N_{11} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11}\left(1+\frac{\varsigma}{R_2}\right)d\varsigma$$

Now, from here the concept of solution on the theory starts changing, if we say that we want to apply only for a very thin shell or very shallow shell, shallow shell means when you have a very large radius of curvature. When the shell is thin and it is very large, $\frac{\varsigma}{R_2}$ component can be neglected compared to 1.

But, in most of the thin shell theories book, you will find that $\frac{\varsigma}{R_2}$ is not considered and when N_{11} is obtained like this:

$$N_{11} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11}\left(1+\frac{\varsigma}{R_2}\right)d\varsigma$$ and we proceed further.

But if you want to solve a complete shell that because at any time we can reduce to. We are going to solve a generalized shell that can give us the solution for a thick, shallow as well as thin shell. We will consider $\left(1+\frac{\varsigma}{R_2}\right)$.

Here we consider, $\sigma_{11} = Q_{11}\varepsilon_{11} + Q_{12}\varepsilon_{22}$,

Therefore, N_{11} will be: $\int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11}\varepsilon_{11} + Q_{12}\varepsilon_{22}\left(1+\frac{\varsigma}{R_2}\right)d\varsigma$.

We know ε_{11} and ε_{22}, initially, when we defined the strain components, we said that it contains two parts - one will be causing the membrane stretching or the stretching in the shell, and the second part will cause bending in the shell.

Some terms are clubbed under the head ε_{11}^{0} and some terms are clubbed under the head ε_{11}^{1}. This term ε_{11}^{0} is called membrane stretching and ε_{11}^{1} is called bending or curvature the same way, ε_{22} is expressed.

N_{11} is $\int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11}\left(\varepsilon_{11}^{0}+\varsigma\varepsilon_{11}^{1}\right)+Q_{12}\left(\varepsilon_{22}^{0}+\varsigma\varepsilon_{22}^{1}\right)\left(1+\frac{\varsigma}{R_2}\right)d\varsigma$, it is not even that simplified, we further need to work on it.

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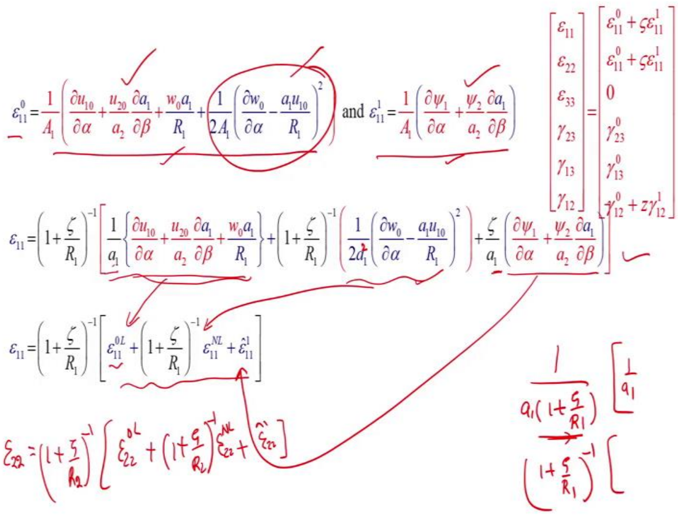

Here,

$$\varepsilon_{11}^{0}=\frac{1}{A_1}\left(\frac{\partial u_{10}}{\partial\alpha}+\frac{u_{20}}{a_2}\frac{\partial a_1}{\partial\beta}+\frac{w_0 a_1}{R_1}+\frac{1}{2A_1}\left(\frac{\partial w_0}{\partial\alpha}-\frac{a_1 u_{10}}{R_1}\right)^2\right)$$ consists of these explicit terms.

$$\varepsilon_{11}^{1}=\frac{1}{A_1}\left(\frac{\partial\psi_1}{\partial\alpha}+\frac{\psi_2}{a_2}\frac{\partial a_1}{\partial\beta}\right)$$

The bending terms and the stretching terms.

Where, these $\frac{1}{2A_1}\left(\frac{\partial w_0}{\partial\alpha}-\frac{a_1 u_{10}}{R_1}\right)^2$ are non-linear terms

And these terms $\frac{\partial u_{10}}{\partial \alpha}+\frac{u_{20}}{a_2}\frac{\partial a_1}{\partial \beta}+\frac{w_0 a_1}{R_1}$ are linear contributions.

Here it is $\frac{1}{A_1}$ is common, we can take it as $\left(1+\frac{\varsigma}{R_1}\right)^{-1}$

Because $A_1 = a_1\left(1+\frac{\varsigma}{R_1}\right)$.

We will take $\frac{1}{a_1}$ inside and $\left(1+\frac{\varsigma}{R_1}\right)^{-1}$ will be taken common. ε_{11} will be:

$$\left(1+\frac{\varsigma}{R_1}\right)^{-1}\left[\frac{1}{a_1}\left(\frac{\partial u_{10}}{\partial \alpha}+\frac{u_{20}}{a_2}\frac{\partial a_1}{\partial \beta}+\frac{w_0 a_1}{R_1}\right)+\left(1+\frac{\varsigma}{R_1}\right)^{-1}\left(\frac{1}{2a_1^{\;2}}\left(\frac{\partial w_0}{\partial \alpha}-\frac{a_1 u_{10}}{R_1}\right)^2\right)\frac{\varsigma}{a_1}\left(\frac{\partial \psi_1}{\partial \alpha}+\frac{\psi_2}{a_2}\frac{\partial a_1}{\partial \beta}\right)\right]$$

Now, we can say:

This term $\frac{1}{a_1}\left(\frac{\partial u_{10}}{\partial \alpha}+\frac{u_{20}}{a_2}\frac{\partial a_1}{\partial \beta}+\frac{w_0 a_1}{R_1}\right)$, will be ε_{11}^{0L},

This term $\left(\frac{1}{2a_1^{\;2}}\left(\frac{\partial w_0}{\partial \alpha}-\frac{a_1 u_{10}}{R_1}\right)^2\right)$ is having slight difference it is known as ε_{11}^{NL},

And this term $\frac{\varsigma}{a_1}\left(\frac{\partial \psi_1}{\partial \alpha}+\frac{\psi_2}{a_2}\frac{\partial a_1}{\partial \beta}\right)$ is known as $\hat{\varepsilon}_{11}^{1}$.

This ε_{11} has three components in which $\left(1+\frac{\varsigma}{R_1}\right)^{-1}$ is common can be written like this:

$$\varepsilon_{11}=\left(1+\frac{\varsigma}{R_1}\right)^{-1}\left[\varepsilon_{11}^{0L}+\left(1+\frac{\varsigma}{R_1}\right)^{-1}\varepsilon_{11}^{NL}+\hat{\varepsilon}_{11}^{1}\right].$$

Similarly, ε_{22} can be written as:

$$\varepsilon_{22}=\left(1+\frac{\varsigma}{R_2}\right)^{-1}\left[\varepsilon_{22}^{0L}+\left(1+\frac{\varsigma}{R_2}\right)^{-1}\varepsilon_{22}^{NL}+\hat{\varepsilon}_{22}^{1}\right]$$

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$$\varepsilon_{11} = \varepsilon_{11}^{0} + \varsigma\varepsilon_{11}^{1} \Rightarrow \frac{1}{A_1}\left[\varepsilon_{11}^{0L} + \varepsilon_{11}^{NL} + \varsigma\varepsilon_{11}^{1}\right]$$

$$\Rightarrow \left(1+\frac{\varsigma}{R_1}\right)^{-1}\left[\frac{1}{a_1}\varepsilon_{11}^{0L} + \frac{1}{a_1}\varepsilon_{11}^{NL} + \frac{1}{a_1}\varsigma\varepsilon_{11}^{1}\right]$$

Similarly

$$\varepsilon_{22} = \left(1+\frac{\varsigma}{R_2}\right)^{-1}\left[\frac{1}{a_2}\varepsilon_{22}^{0L} + \frac{1}{a_2}\varepsilon_{22}^{NL} + \frac{1}{a_2}\varsigma\varepsilon_{22}^{1}\right]$$

$$\gamma_{12} = \gamma_{12}^{0} + \varsigma\gamma_{12}^{1} \Rightarrow \frac{1}{A_1}[\ldots] + \frac{1}{A_2}[\ldots] + \frac{1}{2A_1A_2}[\ldots] + \frac{\varsigma}{A_1}[\ldots] + \frac{\varsigma}{A_2}[\ldots]$$

$$N_{11} = \int_{-h/2}^{h/2} Q_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1}\left[\varepsilon_{11}^{0L} + \left(1+\frac{\varsigma}{R_1}\right)^{-1}\varepsilon_{11}^{NL} + \varsigma\hat{\varepsilon}_{11}\right] + Q_{12}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_2}\right)^{-1}\left[\varepsilon_{22}^{0L} + \left(1+\frac{\varsigma}{R_2}\right)^{-1}\varepsilon_{22}^{NL} + \varsigma\hat{\varepsilon}_{22}\right] d\varsigma$$

If we substitute like this, N_{11} can be written as:

$$N_{11} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \begin{bmatrix} Q_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1}\left[\varepsilon_{11}^{0L} + \left(1+\frac{\varsigma}{R_1}\right)^{-1}\varepsilon_{11}^{NL} + \hat{\varepsilon}_{11}^{1}\right] + \\ Q_{12}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_2}\right)^{-1}\left[\varepsilon_{22}^{0L} + \left(1+\frac{\varsigma}{R_2}\right)^{-1}\varepsilon_{22}^{NL} + \hat{\varepsilon}_{22}^{1}\right] \end{bmatrix} d\varsigma$$

But, for the case of N_{22}:

$\left(1+\frac{\varsigma}{R_2}\right)^{-1}$ common and the same terms $\varepsilon_{22}^{0L} + \left(1+\frac{\varsigma}{R_2}\right)^{-1}\varepsilon_{22}^{NL} + \varsigma\,\hat{\varepsilon}_{22}^{1}$ multiply $d\varsigma$

will be there. We will have terms like this $Q_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1}$ and

$\left[\varepsilon_{11}^{0L} + \left(1+\frac{\varsigma}{R_1}\right)^{-1}\varepsilon_{11}^{NL} + \hat{\varepsilon}_{11}^{1}\right]$ is the strain part.

For the case of ε_{11}^{0L} this $Q_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1}$ will be the coefficient. For the non-linear, if multiply with then these three terms ε_{11}^{0L}, ε_{11}^{NL}, and $\hat{\varepsilon}_{11}^{1}$ will be the coefficient, for the case of bending $\varsigma\hat{\varepsilon}_{11}^{1}$ will be the coefficient.

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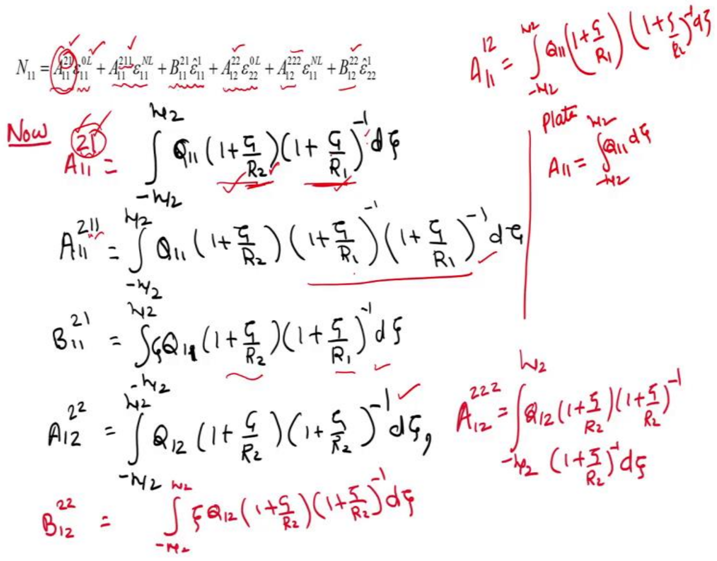

The coefficient is denoted by some terms A_{ij}. Generally, in the plate, we called it A_{ij}^{2} index.

$$N_{11} = A_{11}^{21}\varepsilon_{11}^{0L} + A_{11}^{211}\varepsilon_{11}^{NL} + B_{11}^{21}\hat{\varepsilon}_{11}^{1} + A_{12}^{22}\varepsilon_{22}^{0L} + A_{12}^{222}\varepsilon_{22}^{NL} + B_{12}^{22}\hat{\varepsilon}_{22}^{1}.$$

Here, this index is noted as:

For the case of linear it will be A_{11}^{21}, for the case of non-linear it will be A_{11}^{211}, and for the case of bending it will be B_{11}^{21}. Q_{12} contribution will be A_{12}^{22}, A_{12}^{222}, and B_{12}^{22}.

Why we have put that index 21? It tells you that 2 means $\left(1+\frac{\varsigma}{R_2}\right)$ and 1 means $\left(1+\frac{\varsigma}{R_1}\right)$.

$$A_{11}^{21} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1} d\varsigma,$$

No inverse in the first index, the second index is inverse and it also tells you the radius of curvature in that direction.

Let us say $A_{11}^{12} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11}\left(1+\frac{\varsigma}{R_1}\right)\left(1+\frac{\varsigma}{R_2}\right)^{-1} d\varsigma$.

When we talk about non-linear terms then, it will have 2 inverses.

$$A_{11}^{211} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1}\left(1+\frac{\varsigma}{R_1}\right)^{-1} d\varsigma \, .$$

$$B_{11}^{21} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \varsigma Q_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1} d\varsigma$$

$$A_{12}^{22} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{12}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_2}\right)^{-1} d\varsigma$$

$$B_{12}^{22} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \varsigma Q_{12}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_2}\right)^{-1} d\varsigma \, .$$

It is carefully designed so that we know the meaning of every index.

In the case of the plate just A_{11} is sufficient because A_{11} in the case of the plate is just $\int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11} d\varsigma$, here, we do not need the second index at the top because there are no terms, but now we have $\left(1+\frac{\varsigma}{R_1}\right)$ and $\left(1+\frac{\varsigma}{R_2}\right)$ these two terms. You will have the components of N_{22} , N_{12} , and N_{21} . We may forget which will be the inverse one, it will help us to find that properly.

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$$M_{11} = \int_{-h/2}^{h/2} \left(1+\frac{\varsigma}{R_2}\right) \sigma_{11} \varsigma \, d\varsigma$$

$$M_{11} = \int_{-h/2}^{h/2} \left(1+\frac{\varsigma}{R_2}\right) \varsigma \left\{ \left[Q_{11} \left(1+\frac{\varsigma}{R_1}\right)^{-1} \left[\varepsilon_{11}^{0L} + \left(1+\frac{\varsigma}{R_1}\right)^{-1} \varepsilon_{11}^{NL} + \varsigma \hat{\varepsilon}_{11} \right] \right] + Q_{12} \left(1+\frac{\varsigma}{R_2}\right)^{-1} \left[\varepsilon_{22}^{0L} + \left(1+\frac{\varsigma}{R_2}\right)^{-1} \varepsilon_{22}^{0L} + \varsigma \hat{\varepsilon}_{22} \right] \right\} d\varsigma$$

$$M_{11} = B_{11}^{21} \varepsilon_{11}^{0L} + B_{11}^{211} \varepsilon_{11}^{NL} + D_{11}^{21} \hat{\varepsilon}_{11} + B_{12}^{22} \varepsilon_{22}^{0L} + B_{12}^{222} \varepsilon_{22}^{NL} + D_{12}^{22} \hat{\varepsilon}_{22}$$

$$B_{11}^{21} = \int_{-h/2}^{h/2} \varsigma Q_{11} \left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1} d\varsigma, \quad B_{11}^{211} = \int_{-h/2}^{h/2} \varsigma Q_{11} \left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1}\left(1+\frac{\varsigma}{R_1}\right)^{-1} d\varsigma$$

$$D_{11}^{21} = \int_{-h/2}^{h/2} \varsigma^2 Q_{11} \left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1} d\varsigma, \quad D_{12}^{22} = \int_{-h/2}^{h/2} \varsigma^2 Q_{12} \left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_2}\right)^{-1} d\varsigma$$

Now, we have to find the definition of a moment.

The definition of the moment M_{11} is:

$$M_{11} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11} \varsigma \left(1+\frac{\varsigma}{R_2}\right) d\varsigma$$

Now, we can express σ_{11} in terms of $Q_{11}\left(1+\frac{\varsigma}{R_1}\right)^{-1}\left[\varepsilon_{11}^{0L} + \left(1+\frac{\varsigma}{R_1}\right)^{-1}\varepsilon_{11}^{NL} + \hat{\varepsilon}_{11}\right]$ and

$$Q_{12}\left(1+\frac{\varsigma}{R_2}\right)^{-1}\left[\varepsilon_{22}^{0L} + \left(1+\frac{\varsigma}{R_2}\right)^{-1}\varepsilon_{22}^{NL} + \hat{\varepsilon}_{22}\right]$$

Therefore, M_{11} will become:

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} \left(1+\frac{\varsigma}{R_2}\right) \varsigma \left\{ Q_{11}\left(1+\frac{\varsigma}{R_1}\right)^{-1}\left[\varepsilon_{11}^{0L} + \left(1+\frac{\varsigma}{R_1}\right)^{-1}\varepsilon_{11}^{NL} + \hat{\varepsilon}_{11}\right] + Q_{12}\left(1+\frac{\varsigma}{R_2}\right)^{-1}\left[\varepsilon_{22}^{0L} + \left(1+\frac{\varsigma}{R_2}\right)^{-1}\varepsilon_{22}^{NL} + \hat{\varepsilon}_{22}\right]\right\} d\varsigma$$

Already I have explained the meaning of ε_{11}^{0L}, ε_{11}^{NL} and $\hat{\varepsilon}_{11}$.

$$M_{11} = B_{11}^{21}\varepsilon_{11}^{0L} + B_{11}^{211}\varepsilon_{11}^{NL} + D_{11}^{21}\hat{\varepsilon}_{11} + B_{12}^{22}\varepsilon_{22}^{0L} + B_{12}^{222}\varepsilon_{22}^{NL} + D_{12}^{22}\hat{\varepsilon}_{22}$$

Here, $B_{11}^{21} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \varsigma Q_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1} d\varsigma$,

$$B_{11}^{211} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \varsigma Q_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1}\left(1+\frac{\varsigma}{R_1}\right)^{-1} d\varsigma \quad ,$$

and $\varsigma\hat{\varepsilon}_{11}$ will be ς^2.

Whenever ς^2 come up we denote it as D, bending stiffness.

$$D_{11}^{21} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \varsigma^2 Q_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1} d\varsigma$$

$$D_{12}^{22} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \varsigma^2 Q_{12}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_2}\right)^{-1} d\varsigma \quad .$$

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$$\gamma_{12}^0 = \frac{1}{A_1}\left[\underbrace{\frac{\partial u_{20}}{\partial \alpha} - \frac{u_{10}}{a_2}\frac{\partial a_1}{\partial \beta}}_{= \omega_1^{0L}} + \underbrace{\frac{1}{2A_2}\left(\frac{\partial w_0}{\partial \alpha} - \frac{a_1 u_{10}}{R_1}\right)\left(\frac{\partial w_0}{\partial \beta} - \frac{a_2 u_{20}}{R_2}\right)}_{= \omega_1^{NL}}\right] + \frac{1}{A_2}\left[\underbrace{\frac{\partial u_{10}}{\partial \beta} - \frac{u_{20}}{a_1}\frac{\partial a_2}{\partial \alpha}}_{= \omega_2^{0L}}\right] + \underbrace{\frac{1}{2A_1A_2}\left(\frac{\partial w_0}{\partial \alpha} - \frac{a_1 u_{10}}{R_1}\right)\left(\frac{\partial w_0}{\partial \beta} - \frac{a_2 u_{20}}{R_2}\right)}_{= \omega_2^{NL}}$$

$$\gamma_{12}^1 = \frac{1}{A_1}\left[\underbrace{\frac{\partial \psi_2}{\partial \alpha} - \frac{\psi_1}{a_2}\frac{\partial a_1}{\partial \beta}}_{= \hat{\omega}_1}\right] + \frac{1}{A_2}\left[\underbrace{\frac{\partial \psi_1}{\partial \beta} - \frac{\psi_2}{a_1}\frac{\partial a_2}{\partial \alpha}}_{= \hat{\omega}_2}\right]$$

$$N_{12} = \int_{-h/2}^{h/2} \sigma_{12}\left(1+\frac{\varsigma}{R_2}\right) d\varsigma = \int_{-h/2}^{h/2} \left(1+\frac{\varsigma}{R_2}\right) Q_{66}\left[\omega_1^{0L} + \omega_1^{NL} + \varsigma\hat{\omega}_1\right] d\varsigma$$

$$N_{21} = \int_{-h/2}^{h/2} \sigma_{21}\left(1+\frac{\varsigma}{R_1}\right) = \int\left(1+\frac{\varsigma}{R_1}\right) Q_{66}\left[\omega_2^{0L} + \omega_2^{NL} + \varsigma\hat{\omega}_2\right] d\varsigma$$

We have defined the definition of M_{11} and N_{11}, now a slightly different definition is of N_{12}. What is the definition of γ_{12}^0 and γ_{12}^1?

γ_{12}^0 have two parts:

The first part is:

$$\frac{1}{A_1}\left[\frac{\partial u_{20}}{\partial\alpha}-\frac{u_{10}}{a_2}\frac{\partial a_2}{\partial\beta}+\frac{1}{2A_2}\left(\frac{\partial w_0}{\partial\alpha}-\frac{a_1u_{10}}{R_1}\right)\left(\frac{\partial w_0}{\partial\beta}-\frac{a_2u_{20}}{R_2}\right)\right]$$

And the second part is:

$$\frac{1}{A_2}\left(\frac{\partial u_{10}}{\partial\beta}-\frac{u_{20}}{a_1}\frac{\partial a_2}{\partial\alpha}\right).$$

When we were developing the shell governing equations, the first part is corresponding to N_{12} and the second part corresponding to N_{21}.

$$N_{12} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \tau_{12}\left(1+\frac{\varsigma}{R_2}\right)d\varsigma .$$

$$\tau_{12} = Q_{66}\ \gamma_{12}$$

γ_{12} have three contributions $\omega_1^{0L}\ plus\ \omega_1^{NL}\ plus\ \varsigma\hat{\omega}_1$.

The same way N_{21} will be:

$$\int_{\frac{-h}{2}}^{\frac{h}{2}}\left(1+\frac{\varsigma}{R_1}\right)Q_{66}\left(\omega_2^{0L}+\omega_2^{NL}+\varsigma\hat{\omega}_2\right)d\varsigma .$$

Now, we can say, $\frac{1}{A_1}$ is common, so, $\left(1+\frac{\varsigma}{R_1}\right)^{-1}$ come here.

N_{12} will be:

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{66}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1}\left[\begin{array}{l}\frac{1}{a_1}\left(u_{20},_{\alpha}-\frac{u_{10}}{a_2}a_1,_{\beta}\right)+\frac{1}{2a_1^{2}}\left(1+\frac{\varsigma}{R_1}\right)^{-1}\left(w_0,_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\\ \left(w_0,_{\beta}-\frac{a_2\partial u_{20}}{R_2}\right)+\frac{\varsigma}{a_1}\left(\psi_2,_{\alpha}-\frac{\psi_1}{a_2}a_1,_{\beta}\right)\end{array}\right]d\varsigma$$

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$$N_{12} = \int_{-h/2}^{h/2} Q_{66}\left(1+\frac{\zeta}{R_2}\right)\left[\left(1+\frac{\zeta}{R_1}\right)^{-1}\left[\frac{1}{a_1}\left(u_{20,\alpha} - \frac{u_{10}}{a_2}a_{1,\beta}\right) + \frac{1}{2a_1}\left(1+\frac{\zeta}{R_2}\right)^{-1}\left(w_{0,\beta} - \frac{a_1 u_{10}}{R_1}\right)\left(w_{0,\beta} - \frac{a_2 u_{20}}{R_2}\right) + \frac{\zeta}{a_1}\left(\psi_{2,\alpha} - \frac{\psi_1}{a_2}a_{1,\beta}\right)\right]\right] d\zeta$$

$$N_{21} = \int_{-h/2}^{h/2} Q_{66}\left(1+\frac{\zeta}{R_1}\right)\left[\left(1+\frac{\zeta}{R_2}\right)^{2}\left[\frac{1}{a_2}\left(u_{10,\beta} - \frac{u_{20}}{a_1}a_{2,\alpha}\right) + \frac{1}{2a_1}\left(1+\frac{\zeta}{R_1}\right)^{-1}\left(w_{0,\alpha} - \frac{a_1 u_{10}}{R_1}\right)\left(w_{0,\beta} - \frac{a_2 u_{20}}{R_2}\right) + \frac{\zeta}{a_2}\left[\psi_{1,\beta} - \frac{\psi_2}{a_1}a_{2,\alpha}\right]\right]\right] d\zeta$$

$$N_{12} = A_{66}^{21}\omega_1^{0L} + \frac{1}{2}A_{66}^{212}\omega_1^{NL} + B_{66}^{21}\hat{\omega}_1$$

$$N_{21} = A_{66}^{12}\omega_2^{0L} + \frac{1}{2}A_{66}^{121}\omega_2^{NL} + B_{66}^{12}\hat{\omega}_2$$

The first term $Q_{66}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1}$ will become the definition of A_{66}^{21}, the bending term will be denoted by B_{66}^{21}.

N_{12} will become:

$$A_{66}^{21}\omega_1^{0L} + \frac{1}{2}A_{66}^{212}\omega_1^{NL} + B_{66}^{21}\hat{\omega}_1.$$

Similarly, N_{21} will be:

$$A_{66}^{12}\omega_2^{0L} + \frac{1}{2}A_{66}^{121}\omega_2^{NL} + B_{66}^{12}\hat{\omega}_2.$$

Sometimes 1/2 is taken inside the definition because it is common in all the non-linear terms. In some books of shell theories, 1/2 is taken inside the definition of ω_1^{NL} and ω_2^{NL}, but in some books, it is kept outside.

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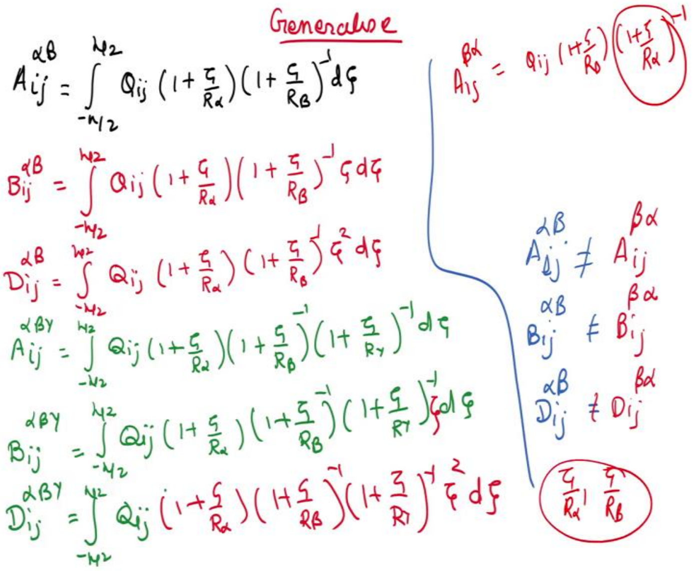

We can write or generalizes in an index form.

The very first definition is $A_{ij}^{\alpha\beta}$ will be:

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{ij}\left(1+\frac{\varsigma}{R_\alpha}\right)\left(1+\frac{\varsigma}{R_\beta}\right)^{-1} d\varsigma \,.$$

The definition of $B_{ij}^{\alpha\beta}$ will be:

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{ij}\left(1+\frac{\varsigma}{R_\alpha}\right)\left(1+\frac{\varsigma}{R_\beta}\right)^{-1} \varsigma d\varsigma \,.$$

What is the definition of D? The definition of $D_{ij}^{\alpha\beta}$ will be:

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{ij}\left(1+\frac{\varsigma}{R_\alpha}\right)\left(1+\frac{\varsigma}{R_\beta}\right)^{-1} \varsigma^2 d\varsigma \,.$$

The definition of non-linear terms $A_{ij}^{\alpha\beta\gamma}$ will be:

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{ij}\left(1+\frac{\varsigma}{R_\alpha}\right)\left(1+\frac{\varsigma}{R_\beta}\right)^{-1}\left(1+\frac{\varsigma}{R_\gamma}\right)^{-1} d\varsigma \ .$$

For the case of non-linear terms, we have three indexes. If you are not considering the non-linear terms then do not worry, but if you consider the non-linear terms then it will have three indexes and the last two indexes $\left(1+\frac{\varsigma}{R_\beta}\right)^{-1}\left(1+\frac{\varsigma}{R_\gamma}\right)^{-1}$ will have inverse terms.

Similarly,

$$B_{ij}^{\alpha\beta\gamma} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{ij}\left(1+\frac{\varsigma}{R_\alpha}\right)\left(1+\frac{\varsigma}{R_\beta}\right)^{-1}\left(1+\frac{\varsigma}{R_\gamma}\right)^{-1} \varsigma d\varsigma$$

and $D_{ij}^{\alpha\beta\gamma}$ will be:

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{ij}\left(1+\frac{\varsigma}{R_\alpha}\right)\left(1+\frac{\varsigma}{R_\beta}\right)^{-1}\left(1+\frac{\varsigma}{R_\gamma}\right)^{-1} \varsigma^2 d\varsigma \ .$$

These are the generalized definition for A_{ij}, B_{ij}, and D_{ij}. The point to be noted here

$A_{ij}^{\alpha\beta} \neq A_{ij}^{\beta\alpha}$, because $A_{ij}^{\beta\alpha} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{ij}\left(1+\frac{\varsigma}{R_\beta}\right)\left(1+\frac{\varsigma}{R_\alpha}\right)^{-1} d\varsigma$.

Here, the term $\left(1+\frac{\varsigma}{R_\alpha}\right)^{-1}$ is inverse but in the case of $A_{ij}^{\alpha\beta}$, the term $\left(1+\frac{\varsigma}{R_\beta}\right)^{-1}$ is inverse. They will be equal only when you say that your shell is very thin and you are going to neglect these terms R_α and R_β, then they will be the same.

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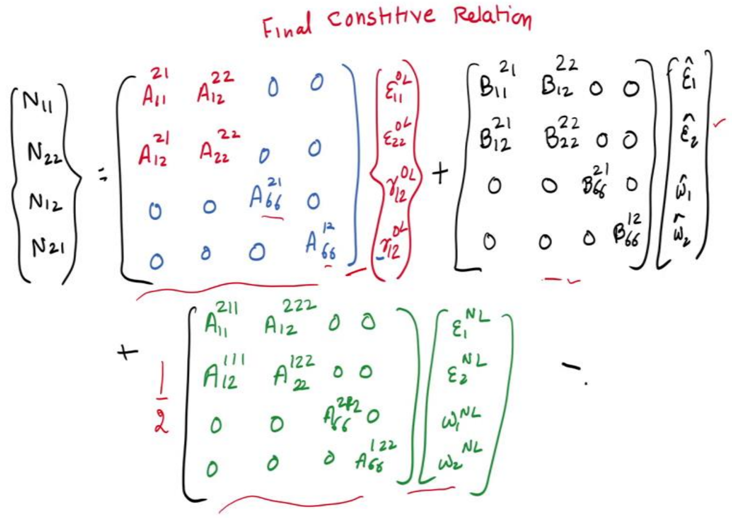

Now, we are clubbing into a matrix form that N_{11}, N_{22}, N_{12}, and N_{21}.

$$\begin{bmatrix} N_{11} \\ N_{22} \\ N_{12} \\ N_{21} \end{bmatrix} = \begin{bmatrix} A_{11}^{21} & A_{12}^{22} & o & o \\ A_{12}^{21} & A_{22}^{22} & o & o \\ o & o & A_{66}^{21} & o \\ o & o & o & A_{66}^{12} \end{bmatrix} \begin{bmatrix} \varepsilon_{11}^{0L} \\ \varepsilon_{22}^{0L} \\ \gamma_{12}^{0L} \\ \gamma_{-12}^{0L} \end{bmatrix} + \begin{bmatrix} B_{11}^{21} & B_{12}^{22} & o & o \\ B_{12}^{21} & B_{22}^{22} & o & o \\ o & o & B_{66}^{21} & o \\ o & o & o & B_{66}^{12} \end{bmatrix} \begin{bmatrix} \hat{\varepsilon}_1 \\ \hat{\varepsilon}_2 \\ \hat{\omega}_1 \\ \hat{\omega}_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} A_{11}^{211} & A_{12}^{222} & o & o \\ A_{12}^{111} & A_{22}^{122} & o & o \\ o & o & A_{66}^{212} & o \\ o & o & o & A_{66}^{122} \end{bmatrix} \begin{bmatrix} \varepsilon_1^{NL} \\ \varepsilon_2^{NL} \\ \omega_1^{NL} \\ \omega_2^{NL} \end{bmatrix}$$

There will be a matrix in which the non-zero components are placed like this. The first component are A_{11}^{21}, A_{12}^{22}, A_{12}^{21} , A_{22}^{22}, A_{66}^{21}, and A_{66}^{12}, these will contain the linear combination, the stretching part, and the component B_{11}^{21}, B_{12}^{21} will have the bending part, plus the green terms are non-linear contribution.

Depending upon the definition in some of the books 1 by 2 is kept outside, sometimes, it is kept inside the definition. So, it is just a matter of choice. The point to be noted here is that if it is a plate, then the terms B_{11}^{21} B_{12}^{21} may not exist, but for the case of shell, all the terms will exist.

Similarly, we can represent the couple constitutive relation.

$$\begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \\ M_{21} \end{bmatrix} = \begin{bmatrix} B_{11}^{21} & B_{12}^{22} & o & o \\ B_{12}^{21} & B_{22}^{22} & o & o \\ o & o & B_{66}^{21} & o \\ o & o & o & B_{66}^{12} \end{bmatrix} \begin{bmatrix} \varepsilon_{11}^{0L} \\ \varepsilon_{22}^{0L} \\ \gamma_{12}^{0L} \\ \gamma_{-12}^{0L} \end{bmatrix} + \begin{bmatrix} D_{11}^{21} & D_{12}^{22} & o & o \\ D_{12}^{21} & D_{22}^{22} & o & o \\ o & o & D_{66}^{21} & o \\ o & o & o & D_{66}^{12} \end{bmatrix} \begin{bmatrix} \hat{\varepsilon}_1 \\ \hat{\varepsilon}_2 \\ \hat{\omega}_1 \\ \hat{\omega}_2 \end{bmatrix} + \begin{bmatrix} B_{11}^{211} & B_{12}^{222} & o & o \\ B_{12}^{111} & B_{22}^{122} & o & o \\ o & o & B_{66}^{212} & o \\ o & o & o & B_{66}^{122} \end{bmatrix} \begin{bmatrix} \varepsilon_1^{NL} \\ \varepsilon_2^{NL} \\ \omega_1^{NL} \\ \omega_2^{NL} \end{bmatrix}$$

These are the in-plane stresses and in terms of strains. Ultimately, for the case of a solution, we will write it explicitly and solve it. But first, we have to define it like this. It is also important to write like this from a programming point of view, we write a nonzero component of a matrix, and then later on we have to multiply with this.

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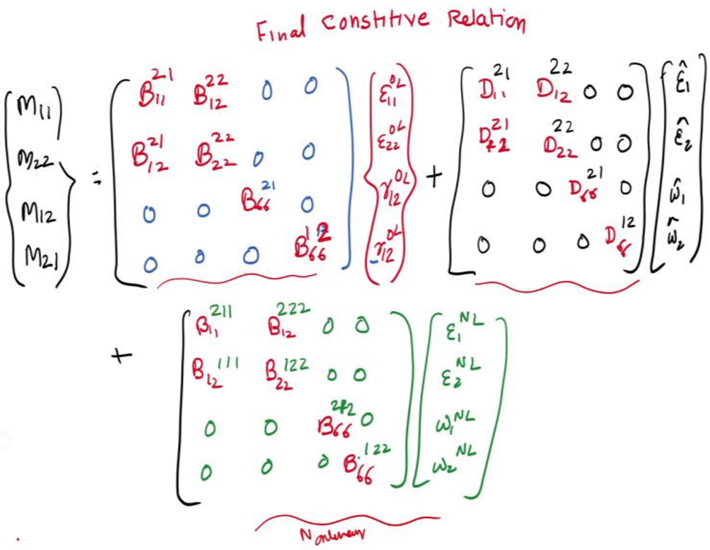

The same way, with the moment M_{11}, M_{22}, M_{12}, and M_{21}, we have to find it and arrange that in B_{11}^{21}, D_{11}^{21}, and B_{11}^{211} matrix. B_{11}^{21} is for bending, D_{11}^{21} is called stretching, and B_{11}^{211} is the non-linear part.

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$$\begin{Bmatrix} Q_2 \\ Q_1 \end{Bmatrix} = K_s \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon^{0}_{23} \\ \varepsilon^{0}_{13} \end{Bmatrix}$$

Shear correction factor

$$\left.\begin{matrix} \tau_{23} \\ \tau_{13} \end{matrix}\right\} \text{For accurate shear stress}$$

Now, we have the shear forces Q_1 and Q_2 can be written as A_{44} and A_{55} and the coefficient will be ε^{0}_{23} and ε^{0}_{13}. $\begin{bmatrix} Q_2 \\ Q_1 \end{bmatrix} = \begin{bmatrix} A_{44} & o \\ o & A_{55} \end{bmatrix} \begin{bmatrix} \varepsilon^{0}_{23} \\ \varepsilon^{0}_{13} \end{bmatrix}$

Here point to be noted that for the case of first-order shear deformation theory, it is multiplied or divided with some K_S depending upon the situation.

Here we are taking a general case, in the first-order theory the shear forces when you evaluate, do not come so accurately.

We have to multiply with a factor known as the shear correction factor. Based on the geometry and loading the shear correction factor is evaluated and it is multiplied with that so that we can get accurate shear stresses along with the thickness.

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$$A_{11}^{21} = \int_{-h/2}^{h/2} Q_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1} d\varsigma$$

For thin shells $\frac{\varsigma}{R_1}$ & $\frac{\varsigma}{R_2} = 0$

$(1+x)^{-1} = 1 - x + x^2 - x^3 - \text{to } \infty$

For shallow shells $= \frac{\varsigma}{R_1} = 0,\ \frac{\varsigma}{R_2} = 0$

For thick shells.

$$A_{11}^{21} = \int_{-h/2}^{h/2} Q_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1-\frac{\varsigma}{R_1}+\left(\frac{\varsigma}{R_1}\right)^2-\left(\frac{\varsigma}{R_1}\right)^3+\left(\frac{\varsigma}{R_1}\right)^4+\cdots\right) d\varsigma$$

$$A_{11}^{21} = \int_{-h/2}^{h/2} Q_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1-\frac{\varsigma}{R_1}+\left(\frac{\varsigma}{R_1}\right)^2\right) d\varsigma$$

Already, I have discussed that for a case of thin shell $\frac{\varsigma}{R_2}$ and $\frac{\varsigma}{R_1}$ are neglected and for the case of shallow shell these are neglected, but for the case of thick shell, these are considered. Now, how do you solve this $\left(1+\frac{\varsigma}{R_1}\right)^{-1}$? Ultimately, you are writing a program you need one number.

This term can be written like $(1+x)^{-1}$, the binomial expansion of that will be:

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + ...$$

If we expand that term and it will go up to infinity.

Based on the inclusion of these terms will give you an entirely different shell theory, whether we say that we have to take only the first term, the logic behind that is if the thickness is small and R_1 is large then, this can be neglected.

But, for the thick shell, generally, in the case of flugge's shell theory, up to the second-order terms are considered in this binomial expansion.

$$A_{11}^{21} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}+\left(\frac{\varsigma}{R_1}\right)^2-\left(\frac{\varsigma}{R_1}\right)^3+\left(\frac{\varsigma}{R_1}\right)^4+\ldots\ldots\right)d\varsigma$$

In most of the thin shell theories, only the first term $1+\frac{\varsigma}{R_1}$ is taken in the case of inverse and some researchers have considered $1-\frac{\varsigma}{R_1}$ also and some other researchers have considered up to this. Depending upon that compared to 1 the contribution of these terms is less, but sometimes they may give more accuracy at the stresses required zone.

If we talking about composite plates in that case that researcher has considered the term up to:

$$A_{11}^{21} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}+\left(\frac{\varsigma}{R_1}\right)^2\right)d\varsigma$$

Because at the interfaces the transverse stresses are there bending stresses are there, we need to consider these terms. If you take up to the second-order, then it is considered as flugge's shell theory.

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$$A_{11}^{21} = \int_{-h/2}^{h/2} Q_{11}\left(1 - \frac{\varsigma}{R_1} + \left(\frac{\varsigma}{R_1}\right)^2 + \frac{\varsigma}{R_2} - \frac{\varsigma}{R_1}\cdot\frac{\varsigma}{R_2} + \frac{\varsigma}{R_2}\cdot\left(\frac{\varsigma}{R_1}\right)^2\right] d\varsigma$$

$$= \int_{-h/2}^{h/2} Q_{11}\left[1 - \frac{\varsigma}{R_1} + \frac{\varsigma}{R_2} - \frac{\varsigma^2}{R_1 R_2} + \frac{\varsigma^2}{R_1^2} + \frac{\varsigma^3}{R_2 R_1^2}\right] d\varsigma$$

For composite plates.

$$A_{11}^{21} = \sum_{k=1}^{N} Q_{11}^{k}\Bigg[\left[\varsigma_{k+1} - \varsigma_k\right] - \frac{1}{2}\left[\frac{z_{k+1}^2}{R_1} - \frac{z_k^2}{R_1}\right] + \frac{1}{2}\left[\frac{z_{k+1}^2}{R_2} - \frac{z_{k+1}^2}{R_2}\right]$$

$$- \frac{1}{3}\left[\frac{z_{k+1}^3}{R_1 R_2} - \frac{z_k^3}{R_1 R_2}\right] + \frac{1}{3}\left[\frac{z_{k+1}^3}{R_1^2} - \frac{z_k^3}{R_1^2}\right]$$

$$+ \frac{1}{4}\left[\frac{z_{k+1}^4}{R_2 R_1^2} - \frac{z_{k+1}^4}{R_2 R_1^2}\right]\Bigg]$$

And if you multiply, ultimately, you get the terms like this

$$A_{11}^{21}=\int_{\frac{-h}{2}}^{\frac{h}{2}}Q_{11}\left(1-\frac{\varsigma}{R_1}+\frac{\varsigma}{R_2}-\frac{\varsigma^2}{R_1R_2}+\frac{\varsigma^2}{R_1^2}+\frac{\varsigma^3}{R_2R_1^2}\right)d\varsigma$$

You have to evaluate it through the thickness.

If it is an isotropic shell or a single layer shell, though it may be orthotropic material, if it is a single layer shell, then you have to integrate it. But, now here we are talking about a composite shell that may have an 'N' number of layers then how do you evaluate it?

In the case of the N layer, layer-wise you have to do addition that if the first term is Q_{11}, at each layer, the stiffness will be different. Let us say K goes from 1- N,

$\sum_{K=1}^{N}Q_{11}^{K}$, K-th layer will be 1, 2, 3, and so on.

Let us say, K = 1, then the integration will give you ς times and if you put the limits $\frac{-h}{2}\ to\ \frac{h}{2}$, the coordinates of the top layer minus the coordinates of the bottom.

If there is one layer let us say K-th layer, coordinate of upper layer minus lower coordinates gives you the thickness. Ultimately, it is the thickness of that layer, $\left(\varsigma_{K+1}-\varsigma_K\right)$ is the thickness of that layer.

Now, the second term $\frac{\varsigma}{R_1}$, if you integrate it will become $-\frac{1}{2}\left(\frac{\varsigma_{K+1}^2}{R_1}-\frac{\varsigma_K^2}{R_1}\right)$ and the third term will be $\frac{1}{2}\left(\frac{\varsigma_{K+1}^2}{R_2}-\frac{\varsigma_K^2}{R_2}\right)$ this, when we have a ς^2 then it will be a ς^3, therefore fourth term will be $-\frac{1}{3}\left(\frac{\varsigma_{K+1}^3}{R_1R_2}-\frac{\varsigma_K^3}{R_1R_2}\right)+\frac{1}{3}\left(\frac{\varsigma_{K+1}^3}{R_1^2}-\frac{\varsigma_K^3}{R_1^2}\right)$ and the fifth term will be $\frac{1}{4}\left(\frac{\varsigma_{K+1}^4}{R_2R_1^2}-\frac{\varsigma_K^4}{R_2R_1^2}\right)$.

In this way, we can evaluate the effective coefficient of A_{11}^{21} for a case of a multi-layered composite plate.

$$A_{11}^{21} = \sum_{K=1}^{N} Q_{11}^{K} \left[\begin{array}{l} \left(\varsigma_{K+1} - \varsigma_{K}\right) - \frac{1}{2}\left(\frac{\varsigma_{K+1}^{2}}{R_1} - \frac{\varsigma_{K}^{2}}{R_1}\right) + \frac{1}{2}\left(\frac{\varsigma_{K+1}^{2}}{R_2} - \frac{\varsigma_{K}^{2}}{R_2}\right) - \frac{1}{3}\left(\frac{\varsigma_{K+1}^{3}}{R_1 R_2} - \frac{\varsigma_{K}^{3}}{R_1 R_2}\right) \\ + \frac{1}{3}\left(\frac{\varsigma_{K+1}^{3}}{R_1^{2}} - \frac{\varsigma_{K}^{3}}{R_1^{2}}\right) + \frac{1}{4}\left(\frac{\varsigma_{K+1}^{4}}{R_2 R_1^{2}} - \frac{\varsigma_{K}^{4}}{R_2 R_1^{2}}\right) \end{array} \right]$$

Whenever we are going to develop a solution for a multi-layered plate based on FSDT theory or a classical shell theory or any higher-order shell theory, then we evaluate effective coefficients and adding through like this.

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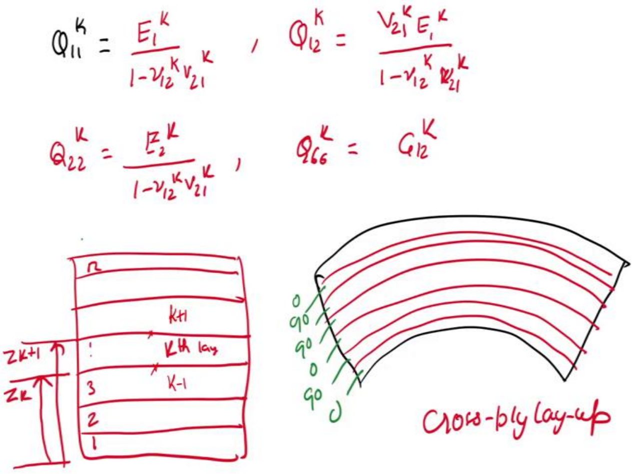

What is this Q_{11}^{K}? Q_{11}^{K} can be found in terms of stiffness in that direction:

$$Q_{11}^{K} = \frac{E_1^{K}}{1-\mu_{12}^{K}\mu_{21}^{K}}$$

These relations are given in books of composites, but for reference, I am presenting here:

$$Q_{12}^{K} = \frac{\mu_{21}^{K} E_1^{K}}{1-\mu_{12}^{K}\mu_{21}^{K}}, \quad Q_{22}^{K} = \frac{E_2^{K}}{1-\mu_{12}^{K}\mu_{21}^{K}}, \text{ and } Q_{66}^{K} = G_{12}^{K}.$$

A composite shell may have many layers and for a cross-ply, it may be 0˚, 90˚, 0˚, 90˚, and so on. What is the concept of Z_K and Z_{K+1}? Let us say we are giving numbers 1, 2, 3, 4, 5, up to n, and in between K-th comes.

For a K-th layer, this will be the bottom coordinate and this will be the top coordinates. Above that, there will be K +1-th layer, and here will be K - 1-th layer. This will be Z_K and this will be Z_{K+1}.

These are the standard procedure one can use. For a composite shell concept, these are the same thing Z_K layer instead of thickness, we decide the thickness of a shell is $\frac{-h}{2}\ to\ \frac{h}{2}$.

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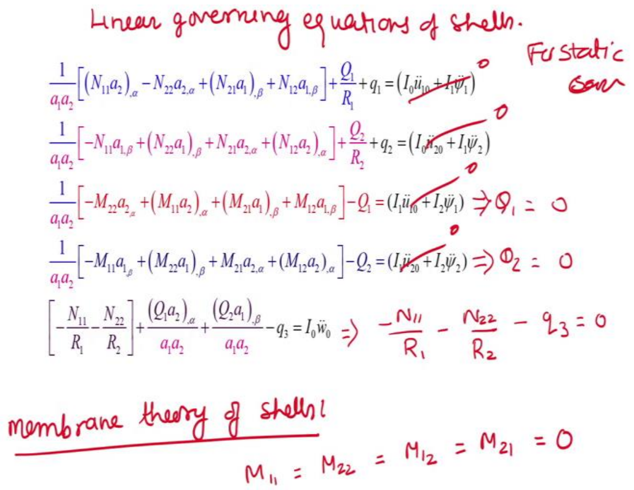

We have defined the constitutive relations, now, we will solve for a static case and first, we will try for the linear governing equations, and here all static terms are going to be 0.

$$\frac{1}{a_1a_2}\left[\left(N_{11}a_2\right)_{,\alpha}-N_{22}a_{2,\alpha}+\left(N_{21}a_1\right)_{,\beta}+N_{12}a_{1,\beta}\right]+\frac{Q_1}{R_1}+q_1=0 \quad equation\,(1)$$

$$\frac{1}{a_1a_2}\left[-N_{11}a_{1,\beta}+\left(N_{22}a_1\right)_{,\beta}+N_{21}a_{2,\alpha}+\left(N_{12}a_2\right)_{,\alpha}\right]+\frac{Q_2}{R_2}+q_2=0 \quad equation\,(2)$$

$$\frac{1}{a_1a_2}\left[-M_{22}a_{2,\alpha}+\left(M_{11}a_2\right)_{,\alpha}+\left(M_{21}a_1\right)_{,\beta}+M_{12}a_{1,\beta}\right]-Q_1=0 \quad equation\,(3)$$

So, why we

$$\frac{1}{a_1a_2}\left[-M_{11}a_{1,\beta}+\left(M_{22}a_1\right)_{,\beta}+M_{21}a_{2,\alpha}+\left(M_{12}a_1\right)_{,\alpha}\right]-Q_2=0 \quad equation\,(4)$$

$$\left(-\frac{N_{11}}{R_1}-\frac{N_{22}}{R_2}\right)+\frac{\left(Q_1a_2\right)_{,\alpha}}{a_1a_2}+\frac{\left(Q_2a_1\right)_{,\beta}}{a_1a_2}-q_3=0 \quad equation\,(5)$$

In the shell theories even after developing the basic governing equations, we can further simplify which means the solution of the first one is the membrane theory of shell and the second is the moment theory of shell or a flexural theory of shells. Why do we need separate treatments?

The reason behind that is we can solve a problem altogether, but that will be more complex. If we say that if loading in the boundary condition is such that it comes under the assumptions of membrane theory of shells, then we will apply the concept of membrane theory of shells. The idea behind that shell is very thin and it is subjected to only in-plane stretching, there is no bending moment.

Moments M_{11}, M_{22} and M_{12}, M_{21} are 0. If we assume this condition because we say that it cannot take a bending moment only the stretching in a plane can take place. Then, the equation (3) and equation (4), if you put all 0, then Q_1 and Q_2 will be 0.

The equation (3) and equation (4), become identically 0 and from that Q_1 and Q_2 gives 0. If you use this relation in equation (5) that will give you:

$$\left(-\frac{N_{11}}{R_1}-\frac{N_{22}}{R_2}\right)-q_3=0$$

And it will become the third equation.

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$$\frac{1}{a_1a_2}\left[\left(N_{11}a_2\right)_{,\alpha}-N_{22}a_{2,\alpha}+\left(N_{21}a_1\right)_{,\beta}+N_{12}a_{1,\beta}\right]+q_1=0$$

$$\frac{1}{a_1a_2}\left[-N_{11}a_{1,\beta}+\left(N_{22}a_1\right)_{,\beta}+N_{21}a_{2,\alpha}+\left(N_{12}a_2\right)_{,\alpha}\right]+q_2=0$$

$$\left[-\frac{N_{11}}{R_1}-\frac{N_{22}}{R_2}\right]-q_3=0$$

membrane state of stren →

That can not support bending and twisting momentum. The corresponding theory of thin shells.

Now, if you substitute in the first and second equation that will give you these three equations

$$\frac{1}{a_1a_2}\left[\left(N_{11}a_2\right)_{,\alpha}-N_{22}a_{2,\alpha}+\left(N_{21}a_1\right)_{,\beta}+N_{12}a_{1,\beta}\right]+q_1=0 \quad equation\,(1)$$

$$\frac{1}{a_1a_2}-N_{11}a_{1,\beta}+\left(N_{22}a_1\right)_{,\beta}+N_{21}a_{2,\alpha}+\left(N_{12}a_2\right)_{,\alpha}+q_2=0 \quad equation\,(2)$$

$$\left(-\frac{N_{11}}{R_1}-\frac{N_{22}}{R_2}\right)-q_3=0 \quad equation\,(3)$$

Depending upon a situation or requirement we can apply.

Now, you see that the number of governing equations is reduced, plus the number of variables also have been reduced. We have variable N_{11}, N_{22}, N_{12}, and N_{21}. Again, depending upon the cases, we can say that our variables will be also three, later on, some terms will going to be 0 and that equations will be 3 in numbers.

The membrane state of stress that cannot support the bending and twisting momentum, then the corresponding theory is of the thin shell in which we assume that there are no bending moments.

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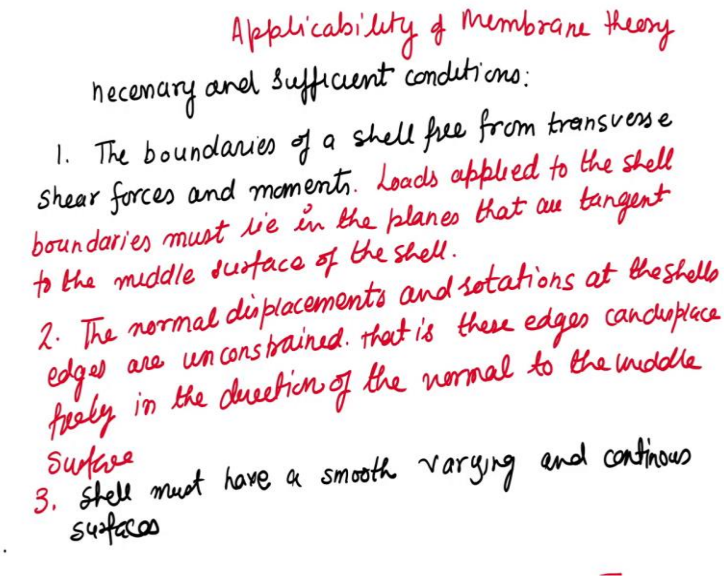

Now, what are the necessary and sufficient boundary conditions? When we mean concept, it is when we are going to apply the membrane theory of shells.

Based on your careful examination, suppose, some work come to your research work, you are going to solve a problem let us say in the case of a fluid-filled structure, for example, you have a very big cylinder in which the oil is filled and it is buried inside the soil.

It is carrying a vehicle like LPG storage tanks and through a trailer, it is moving. You want to know the stress is in the shell or let us say some aerospace is moving and their body of shell you want to study.

When do you analyze those shells under the membrane theory of shells? The very first condition: the boundaries of a shell-free from transverse shear forces and moments which means the boundary should be free from any moment and shear force.

Generally, the simply supported one is the most ideal situation. In the simply supported one, we may have no moments, but we may have the shear forces at the boundary. The free boundaries: no moments, no shear forces and the loads applied to the shell boundaries must lie in the plane that is tangent to the middle surface.

Load should be applied in such a way they lie in the plane of the middle surface so that they cause a pure stretching effect in the shell. They do not cause a bending effect. The next condition is the normal displacement and rotations at the shell edges are unconstrained which means they are allowed to move, i.e., their edges can displace freely in the direction of normal to the middle surface.

Boundary condition should be applied in such a way that the **w** which is the transverse deflection, can have the transverse deflection, which may allow moving in that direction. The shell must have a smooth varying and continuous surface. If there is a breakup in curvature, if there are abrupt changes in the dimensions or the curvature then their moments may arise.

We cannot apply those concepts.

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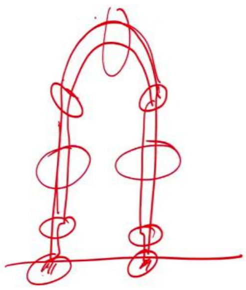
4. The components of the surface and edge loads must be also smooth and continuous functions of the coordinates.



And, then the component of the surfaces and edge load must be also smooth and continuous functions of coordinates. If we talk about a structure, inside that there may be some disruptions.

These are the points where curvature changes and then there are some certain changes in the thickness or some boundary conditions, clamped, it may have stresses. There we cannot apply the solution. We cannot say that the membrane theory of stresses will apply to those conditions.

It should be continuous like this. In this zone, the membrane theory of shells can be applied.

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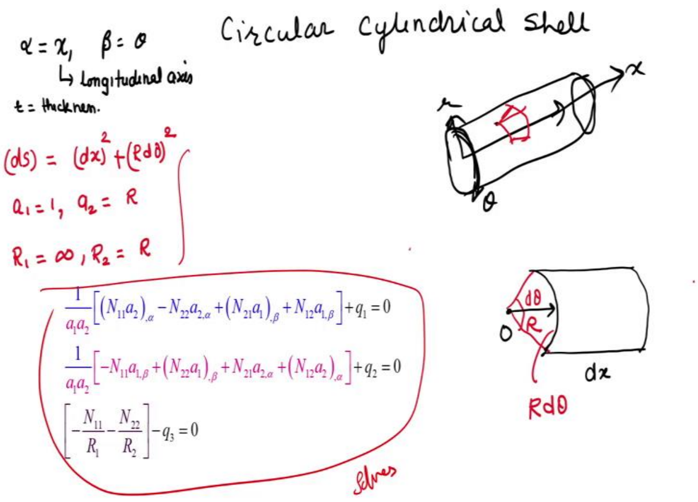

Now, we are going to perform the membrane theory of stresses in the next lecture. I will explain one or two examples of the circular cylindrical shell.

First, we will convert using this concept that the longitudinal axis α = x, $\beta = \theta$, a_1 = 1, a_2 = R, $R_1 = \infty$, and R_2 = R. We will first convert these three equations into a cylindrical equation, and depending upon the loading and other boundary condition we are going to solve these equations.

Thank you very much.