Fundamentals of Convective Heat Transfer
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Module – 03
Convective Heat Transfer in External Flows - I
Lecture – 08
Pohlhausen Solution: Heat Transfer Parameters

Hello, everyone. So, in the last lecture, we have derived the Pohlhausen equation starting from the energy equation. Today, we will find the different Heat Transfer Parameters like heat flux, heat transfer coefficient, Nusselt number, from the solution of Pohlhausen equation.

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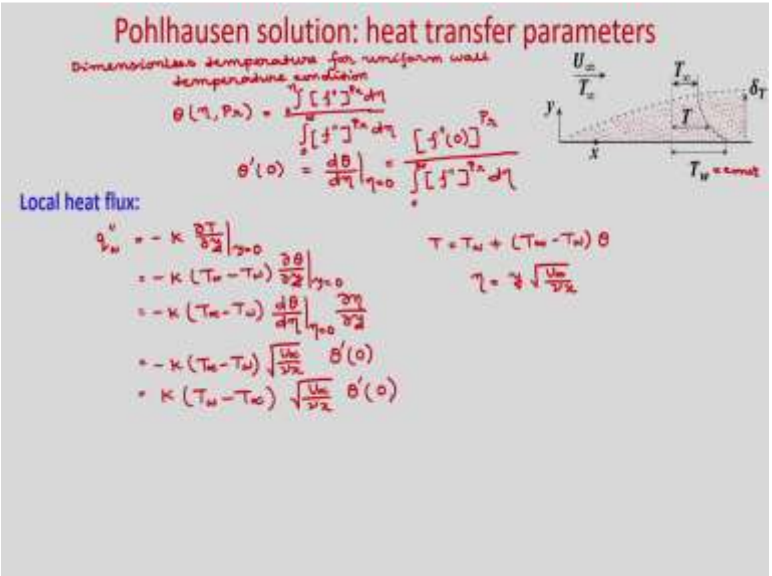

So, you can see that dimensionless temperature we have derived in last class, temperature for uniform wall temperature condition; that means, T_w is constant θ which is your dimensionless temperature which is function of η and Prandtl number,

$$\theta(\eta, \Pr) = \frac{\int_0^{\eta} [f'']^{\Pr} d\eta}{\int_0^{\infty} [f'']^{\Pr} d\eta}.$$

And the derivative of θ we have also derived and that η = 0, the expression is, $\theta'(0) = \frac{d\theta}{d\eta}|_{\eta=0} = \frac{[f''(0)]^{Pr}}{\int_0^\infty [f'']^{Pr} d\eta}$.

So, once you know the temperature distribution from the Pohlhausen solution then you will be able to calculate the local heat flux as well as local heat transfer coefficient and local Nusselt number and you need the value of $\frac{d\theta}{d\eta}|_{\eta=0}$; that means, the temperature gradient at the wall to find the heat flux. So, first let us find what is the local heat flux.

So, you know by definition, $q_w'' = -K\frac{\partial T}{\partial y}|_{y=0}$. So, at this wall the temperature gradient $\frac{\partial T}{\partial y}$ and that is coming from Fourier's law of heat conduction. And we have taken the dimensionless temperature such a way that $T = T_w + (T_\infty - T_w)\,\theta$. And the similarity, variable $\eta = y\sqrt{\frac{U_\infty}{\nu x}}$.

So, now, you can see that this we can write - K $\frac{\partial T}{\partial y}$. So, T_w is constant in this case. So, you can write $-K(T_\infty - T_w)\frac{\partial \theta}{\partial y}|_{y=0} = -K(T_\infty - T_w)\frac{\partial \theta}{\partial \eta}|_{\eta=0}\frac{\partial \eta}{\partial y}$

So, $\frac{\partial \eta}{\partial y}$ you can see it is $\sqrt{\frac{U_\infty}{\nu x}}$. So, you can write $-K(T_\infty - T_w)\sqrt{\frac{U_\infty}{\nu x}}\theta'(0)$. So, this you can write this minus you can take it inside. So, you can write, $K(T_w - T_\infty)\sqrt{\frac{U_\infty}{\nu x}}\theta'(0)$.

So, we have found the local heat flux. Now, we will calculate the local heat transfer coefficient. So, local heat transfer coefficient you know from the equation from Newton's law of cooling you will be able to calculate.

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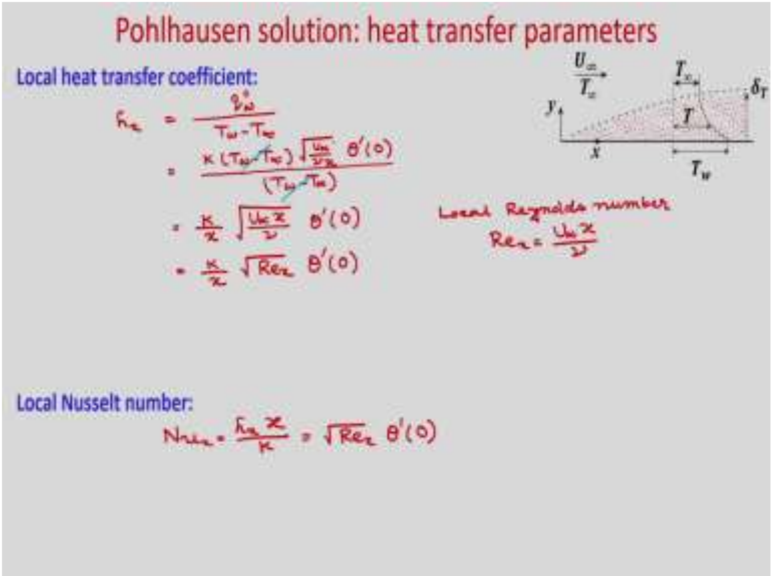

So, local heat transfer coefficient $h_x = \frac{q_w^{"}}{T_w - T_\infty}$ and this we are writing from the Newtons law of cooling. So, $q_w^{"}$, just we have calculated. So, this is your, $\frac{K(T_w - T_\infty)\sqrt{\frac{U_\infty}{\nu x}}\theta'(0)}{(T_w - T_\infty)}$.

So, this you can cancel so you will get we will just write $\frac{K}{x}$ and we will multiply in the numerator x and we will take inside this inside this root. So, it will be x^2. So, you will get $\sqrt{\frac{U_\infty}{\nu x}}\theta'(0)$. What is $\frac{U_\infty}{\nu x}$?

So, you know the Reynolds number definition right. So, this will be local Reynolds number at any location x. So, local Reynolds number R_{e_x}, you can write $\frac{U_\infty}{\nu x}$. So, this now you can write $\frac{K}{x}\sqrt{R_{e_x}}\theta'(0)$.

Now, once you know the local heat transfer coefficient you can calculate the local Nusselt number because you know the definition of local heat transfer as $\frac{hx}{K}$. So, local Nusselt number is definition by $\frac{h_x x}{K}$. So, you can see this x $\frac{x}{K}$ if you take this side. So, this will be left with $\sqrt{R_{e_x}}\theta'(0)$. So, once you can find the derivative $\theta'(0)$, then you will be able to find the value of local heat transfer coefficient and local Nusselt number.

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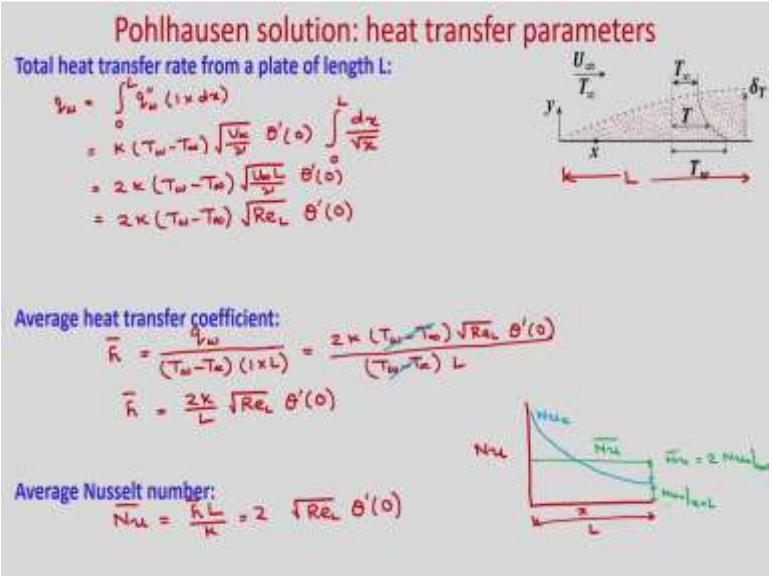

Now, let us calculate the total heat transfer from the plate of length L and we will calculate it for unit width. So, total heat transfer rate from a plate of length L you can calculate q_w. So, just local heat flux you have calculated into area. So, per unit width so, it will be, $q_w = \int_0^L q_w^{"}(1dx)$. L is the length of the plate.

So, if you put the expression of $q_w^{"}$. So, the constant you can take it outside the integral. So, it will be $K(T_w - T_\infty)\sqrt{\frac{U_\infty}{\nu}}\theta'(0)\int_0^L \frac{dx}{\sqrt{x}}$. So, if you integrate it you will get twice $K(T_w - T_\infty)$.

And if you put the limit L then you will get $\sqrt{L}$ in the numerator. So, that we will take $2K(T_w - T_\infty)\sqrt{\frac{U_\infty L}{\nu}}\theta'(0)$. So, from here you can see $2K(T_w - T_\infty)$ and what it is? It is Reynolds number right $\frac{U_\infty L}{\nu}$ so, based on the plate length. So, $\sqrt{R_{e_L}}\theta'(0)$. Once you know the total heat transfer rate you will be able to calculate the average heat transfer coefficient.

So, average heat transfer coefficient we will calculate $\bar{h} = \frac{q_w}{(T_w - T_\infty)(1.L)}$, right this is Newton's law of cooling. So, per unit to it so, 1xL. So, if you put the value so, you will get $\frac{2K(T_w - T_\infty)\sqrt{R_{e_L}}\theta'(0)}{(T_w - T_\infty)L}$.

So, this you cancel. So, you will get, $\bar{h} = \frac{2K}{L}\sqrt{R_{e_L}}\theta'(0)$. From here average Nusselt number you will calculate. So, average Nusselt number $\overline{N_u} = \frac{\bar{h}L}{K}$. So, from this expression you can see it will be $2\sqrt{R_{e_L}}\theta'(0)$.

So, all these expression we have written in terms of $\theta'(0)$. So, $\theta'(0)$ is still unknown because you need to find it from the temperature distribution. Here you notice the local heat transfer coefficient and average heat transfer coefficient. So, you can see that your average heat transfer coefficient is double of the local heat transfer coefficient at x = L. And similarly, average heat transfer coefficient is twice of the local Nusselt number at x= L.

So, if you see, let us say this is the Nusselt number distribution with length L. So, this is the plate length L. So, how it varies? So, let us say your local Nusselt number varies like this. So, if this is the local Nusselt number variation, then your average Nusselt number will be twice at x =L. So, this will be the average Nusselt number it is average Nusselt number and you can see it is value is double of this.

So, what is this value? So, this is your Nusselt number x at x = L. And this $\overline{N_u} = 2N_{u_x}|_{x=0}$. So, this is your double, so this is the same distance. So, it will be the same value, so obviously, it will be the $\overline{N_u} = 2N_{u_x}|_{x=0}$.

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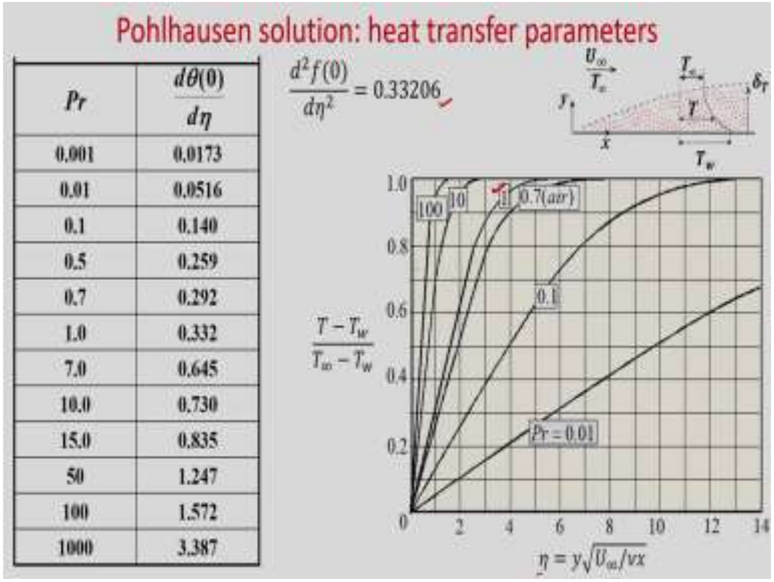

Pr	$\frac{d\theta(0)}{d\eta}$
0.001	0.0173
0.01	0.0516
0.1	0.140
0.5	0.259
0.7	0.292
1.0	0.332
7.0	0.645
10.0	0.730
15.0	0.835
50	1.247
100	1.572
1000	3.387

So, in this slide we are showing the temperature gradient at η = 0 for different Prandtl number. So, this has been evaluated numerically for a range of Prandtl numbers by Pohlhausen. That the numerical solutions and $\frac{d^2 f}{d\eta^2}$ = 0, already we have found from the Blasius solution.

And if you see the temperature distribution θ which is $\frac{T - T_w}{T_\infty - T_w}$ versus η. So obviously, at different x location all the temperature profile falls in the same curve, but it varies for different Prandtl numbers.

So, you can see Prandtl number = 1. So, this is the case where Prandtl number = 1. So, in this profile temperature profile will be same as the velocity profile, that we have already discussed. And for other Prandtl number you can see how it varies with η. So, at different Prandtl numbers.

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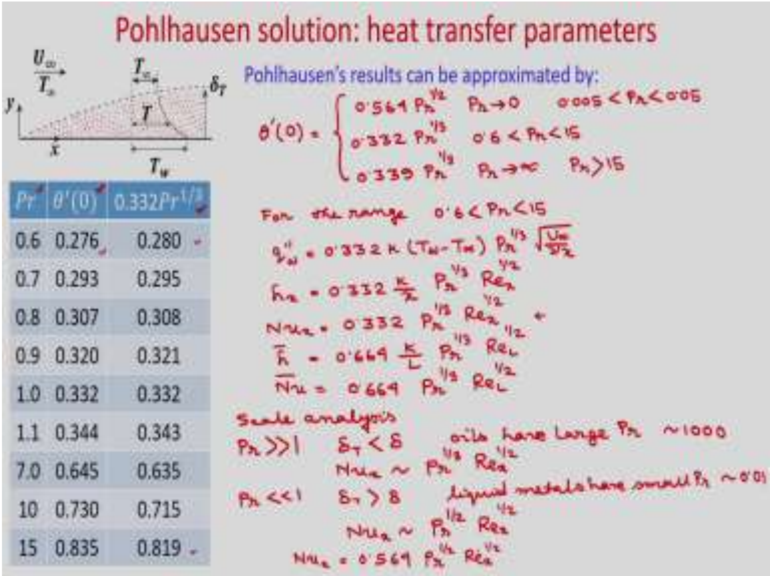

So, now, whatever results we have shown Pohlhausen's results now can be approximated by for different Prandtl number, this is the numerical value of temperature gradient at η = 0. So, this actually is approximated by Pohlhausen as $0.332\,\mathrm{Pr}^{1/3}$. So, you can see if you see 0.6. So, if you put 0.6 Prandtl number here so $0.6^{1/3}\,X\,0.332 = 0.280$.

And for different Prandtl number you can see that almost this is comparable. So, this is the numerical solution and this is approximate, $0.332\,\mathrm{Pr}^{1/3}$. So, 0.6 to 15 we have shown here and in this range so, $\theta'(0) = 0.332\,\mathrm{Pr}^{1/3}$.

So, you can approximate the first derivative of dimensionless temperature at η = 0 as $0.564\,\mathrm{Pr}^{1/2}$. For other range I am writing where Prandtl number →0. Generally, it will be valid in the range of 0.005 < Pr < 0.005. Then in this range 0.6 to 15 this is the approximation.

So, $0.332\,\mathrm{Pr}^{1/3}$ in the range of 0.6 and 15 and it will be $0.339\,\mathrm{Pr}^{1/3}$ for high Prandtl number. So, generally it is Prandtl number greater than 15. So, now, you got some approximate value of $\theta'(0)$ in terms of Prandtl number. So, from the Pohlhausen numerical solutions that is approximated in the power of Prandtl number.

So, for the range 0.6 and 15 you can write $q_w^{"}$ because we have already found the expression. So, if we put the value of $\theta''(0) = 0.332\,\text{Pr}^{1/3}$. So, you can write as, $q_w^{"} = 0.332K(T_w - T_\infty)\,\text{Pr}^{1/3}\sqrt{\frac{U_\infty}{\nu x}}$.

Local heat transfer coefficient you can write $h_x = 0.332\frac{K}{x}\text{Pr}^{1/3}\,\text{Re}_x^{1/2}$. So, local Nusselt number you can write $Nu_x = 0.332\,\text{Pr}^{1/3}\,\text{Re}_x^{1/2}$. And average heat transfer coefficient it will be twice of h_x at x = L. So, it will be $\bar{h} = 0.664\frac{K}{L}\text{Pr}^{1/3}\,\text{Re}_L^{1/2}$. And, average Nusselt number will be $\overline{Nu} = 0.664\,\text{Pr}^{1/3}\,\text{Re}_L^{1/2}$.

You can see that already we have seen this scale of this Nusselt number and the heat transfer coefficient using that scale analysis. So, if you recall using scale analysis we have written for high Prandtl number fluids, high Prandtl number of fluids; that means, your thermal boundary layer thickness will be less than hydro dynamic boundary layer thickness and, generally oils have large Prandtl number of the order of 1000.

So, here if you recall we have written $Nu_x \sim \text{Pr}^{1/3}\,\text{Re}_x^{1/2}$. So, you can see here same order with a value 0.332. Similarly, for low Prandtl number fluids $\delta_T > \delta$ generally, liquid metals have small Prandtl number of the order of 0.01.

So, if you recall, we have found $Nu_x \sim \text{Pr}^{1/3}\,\text{Re}_x^{1/2}$. So, you can see the power of Prandtl number for high Prandtl number fluids the power is 1/3 and low Prandtl number of fluids the power of Prandtl number is 1/2. So, that we have already found from the scale analysis.

And from this numerical solution you can see that for low Prandtl number fluids, Prandtl number →0 your $\theta'(0) = 0.564\,\text{Pr}^{1/2}$. So, if you write the local Nusselt number for low Prandtl number fluids then, $Nu_x = 0.564\,\text{Pr}^{1/2}\,\text{Re}_x^{1/2}$.

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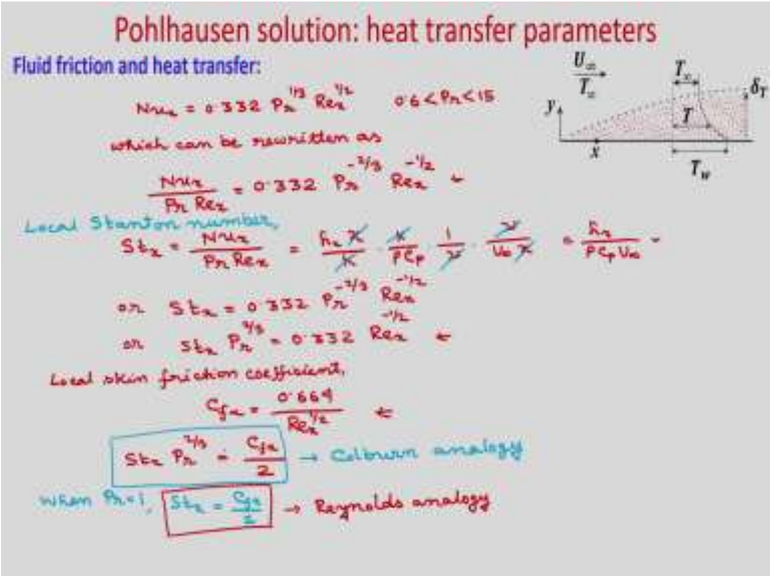

Now, we will discuss about the fluid friction and heat transfer relation. So, you can see that we have found the local Nusselt number $Nu_x = 0.332\Pr^{1/3}\mathrm{Re}_x^{1/2}$ in the range of Prandtl number 0.6 and 15. Now, this we can rewrite as, which can be rewritten as $\frac{Nu_x}{\Pr \mathrm{Re}_x}$. So, if you divide it and you will get in the right-hand side as $0.332\Pr^{-2/3}\mathrm{Re}_x^{-1/2}$.

So, you can see left hand side is the dimensionless group right. So, this is called local Stanton number. So, the left-hand side it is Stanton number. So, this is $\frac{Nu_x}{\Pr \mathrm{Re}_x}$. So, this is known as local Stanton number. So, from this relation you can see that $Nu_x = \frac{h_x x}{K}$.

Prandtl number is $\frac{\nu}{\alpha}$. So, you can write $\alpha = \frac{K}{\rho C_p}$ and $\frac{1}{\nu}$ and Reynolds number. So, $\frac{U_\infty x}{\nu}$; so from here you can rearrange it. So, you can write it as $\frac{h_x}{\rho C_p U_\infty}$. So, you can see the local Stanton number is given by this relation $\frac{h_x}{\rho C_p U_\infty}$.

So, now you can write $St_x = 0.332\Pr^{-2/3}\mathrm{Re}_x^{-1/2}$ or you can write $St_x \Pr^{2/3} = 0.332\mathrm{Re}_x^{-1/2}$.

From the Blasius solution, we have found the local skin friction coefficient. So, if you write down the expression for local skin friction coefficient which is known as friction coefficient because for flow over flat plate the friction is the dominant drag due to sheer stress.

So, you can write $C_{f_x} = \frac{0.664}{\text{Re}_x^{1/2}}$. So, you can see from this expression and this expression you can write $St_x \Pr^{2/3} = \frac{C_{f_x}}{2}$. So, this relation is known as Colburn analogy. So, this relation is known as Colburn analogy.

So, this is between the fluid friction and heat transfer for laminar flow on a flat plate you can write this expression. So, what is the advantage of using this analogy because, if you know the friction coefficient then you will be able to calculate the heat transfer right from this expression.

Now, when Prandtl number = 1; so this expression will become $St_x = \frac{C_{f_x}}{2}$ and this expression is known as Reynolds analogy. So, you can see you can calculate local heat transfer coefficient, when local friction coefficient is known on a flat plate under the conditions in which no heat transfer is involved.

So, now, let us consider the variable wall temperature what we started during the derivation and we have shown from the analysis that T_w varies as $T_w = T_\infty + Cx^\lambda$ C x to the power λ where C is the constant.

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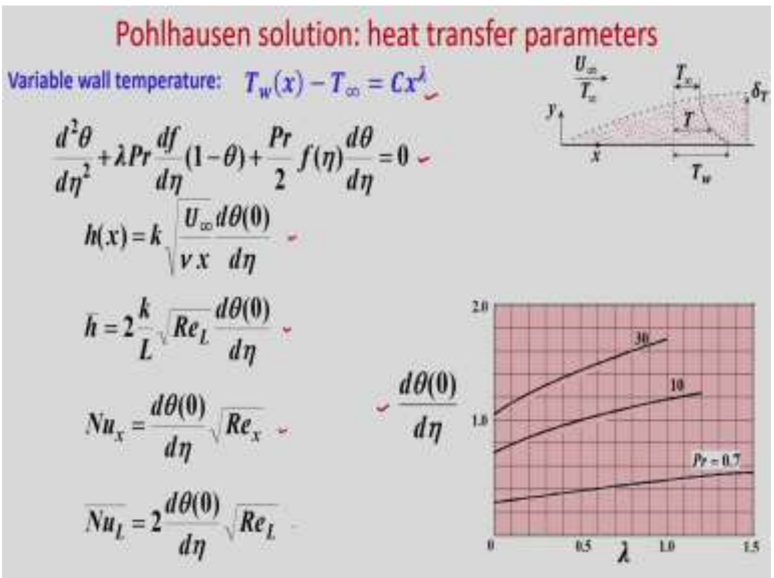

So, you can see variable wall temperature $T_w(x) - T_\infty = Cx^\lambda$, and this is the Pohlhausen equation right. So, this is the ordinary differential equation. So, for a special case we put λ = 0, where wall temperature become constant; so we drop this term.

So, from this equation if you numerically solve then if you can calculate the nondimensional temperature derivative at η = 0; that means, at the wall for different value of λ then for different Prandtl number this is the variation. So, you can see this is the dimensionless temperature gradient variation at the wall, for different value of λ, and at different Prandtl number. So, you know that λ = 0, these denotes for flow over flat plate with uniform wall temperature case.

So, by numerical techniques if you can solve then you can plot this and once you know this value then the same expression for local heat transfer coefficient, average heat transfer coefficient, local Nusselt number, and average Nusselt number you will be able to calculate.

Now, when you consider variable wall temperature, then can you find the value of λ for which the wall will be maintained that constant heat flux condition.

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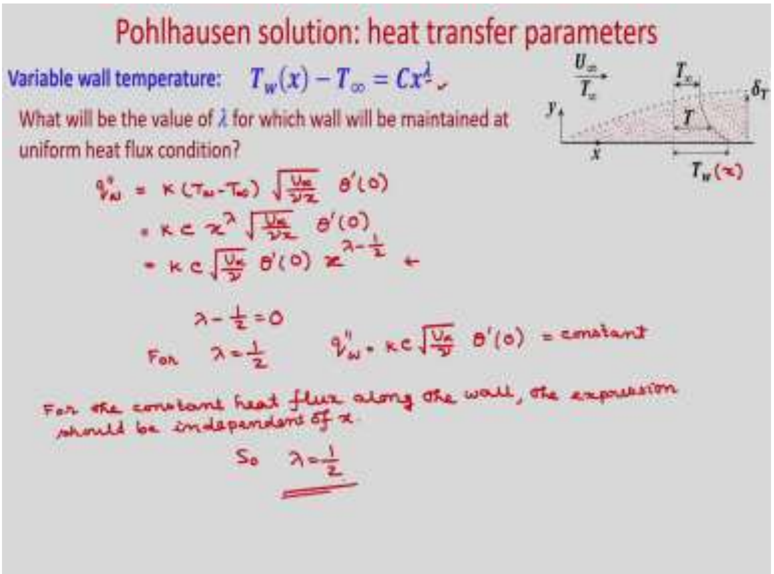

So, this is your variable wall temperature T_w is function of x. Now, what will be the value of λ, for which wall will be maintained at uniform heat flux condition? So, we have to find what is the value of λ? So, the heat flux local heat flux you can write, $q_w^{''} = K(T_w - T_\infty)\sqrt{\frac{U_\infty}{\nu x}}\theta'(0)$.

So, this expression just in previous slide we have derived now, you can see $T_w - T_\infty$, so you have T_w function of x so, $T_w - T_\infty$ you can put Cx^λ. So, you can put $KCx^\lambda\sqrt{\frac{U_\infty}{\nu x}}\theta'(0)$.

So, now, you can write it as $KC\sqrt{\frac{U_\infty}{\nu x}}\theta'(0)x^{\lambda-\frac{1}{2}}$ because here in the denominator you have $x^{1/2}$. So, here x^λ so, $x^{\lambda-\frac{1}{2}}$; so this is the expression for local heat flux right. So, now, to maintain at uniform wall heat flux condition it should be independent of x right, then only the wall will be maintained at constant heat flux condition.

So, you can see here K is the thermal conductivity that is constant, C is the integration constant, U_∞ free steam velocity is constant. This is your fluid kinematic viscosity that is also constant; $\theta'(0)$ which is your temperature gradient at η = 0 so, that having some numerical value. So, that is also constant, but it varies with $x^{\lambda-\frac{1}{2}}$.

So, these terms would be 1 right. Then only it will be independent of x. When it will be 1? When $\lambda - \frac{1}{2}$ will be 0; so that means, $\lambda - \frac{1}{2}$ will be 0 then it will be x^0; that means, 1. So, it will become independent of x. So, λ = 1/2.

So, for λ =1/2 what will be $q_w^{''}$. So, you can see this will be 1. So, it will be $KC\sqrt{\frac{U_\infty}{\nu x}}\theta'(0)$ which is constant. So, for the constant heat flux along the wall, the expression should be independent of x; so λ will be 1/2.

So, in today's class we have found the local heat transfer coefficient, average heat transfer coefficient, local Nusselt number, average Nusselt number, in terms of temperature gradient at η=0. Then later we have shown the numerical solution of Pohlhausen equation, and there we have tabulated the value of $\theta'(0)$ for different Prandtl number. Then Pohlhausen approximated this temperature gradient at the wall with

Prandtl number relation and we have shown that for the range of Prandtl number between 0.6 and 15.

The Nusselt number varies with $Pr^{1/3}$ and $Re^{1/2}$. The same we have shown earlier from the scale analysis as well. Later we have defined the Stanton number and from there we have shown the Colburn analogy as well as the Reynolds analogy. So, this can be used to find the heat transfer coefficient if you know the friction coefficient.

Then, for variable temperature boundary condition we have shown the numerical solution and later we have found the value of λ for which your flat plate wall will be maintained at uniform wall heat flux condition and the value of λ is 1/2.

Thank you.