

Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 03
Convective Heat Transfer in External Flows - I
Lecture – 07
Pohlhausen solution: similarity method

Hello everyone. So, today we will consider thermal boundary layer over a flat plate and we will find the temperature distribution using Similarity Method and this solution is known as Pohlhausen solution.

We will consider two-dimensional, steady, laminar flow with constant properties. The free stream temperature T_∞ is constant and the wall temperature T_w in general will consider that it varies with x which is the axial direction, later we will consider a special case where we will assume T_w as constant.

In last class, we have already solved the velocity distribution using similarity method. As we are assuming that properties are constant so, velocity distribution is independent of temperature distribution. Here, we are considering low speed flow which is incompressible flow. So, we can neglect viscous dissipation effect.

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Pohlhausen solution: similarity method

The solution of boundary layer energy equation was obtained by Pohlhausen solution using similarity transformation.

Energy equation

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$

Let us introduce non-dimensional temperature,

$$\theta = \frac{T - T_w}{T_\infty - T_w}$$

$$T_w = T_w(x)$$

$$u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \alpha \frac{\partial^2 \theta}{\partial y^2}$$

BCs:

- @ $y=0$, $T=T_w$, $\theta=0$
- @ $y \rightarrow \infty$, $T=T_\infty$, $\theta=1$
- @ $x=0$, $T=T_\infty$, $\theta=1$

For $Pr=1$, $v=0$

$$u \frac{\partial \theta}{\partial x} = \alpha \frac{\partial^2 \theta}{\partial y^2}$$


$$u \frac{\partial \theta}{\partial x} = \alpha \frac{\partial^2 \theta}{\partial y^2}$$

For $Pr=1$, $\theta = u'$

$$u \frac{\partial u'}{\partial x} = \alpha \frac{\partial^2 u'}{\partial y^2}$$

BCs:

- @ $y=0$, $u'=0$
- @ $y \rightarrow \infty$, $u'=1$
- @ $x=0$, $u'=1$



First let us write down the energy equation and let us introduce non-dimensional temperature, $\theta = \frac{T - T_w}{T_\infty - T_w}$. So, energy equation for steady laminar flow we can

write $u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$. So, this is the thermal boundary layer equation right.

So, we are neglecting the viscous dissipation effect and now let us introduce non-dimensional temperature; non-dimensional temperature $\theta = \frac{T - T_w}{T_\infty - T_w}$. So, here T_∞ is the free stream velocity and that is constant, but T_w which is your wall temperature in general, we will consider T_w as function of x and as a special case, later we will consider that T_w is constant and you can see if $T_w > T_\infty$ so, temperature distribution will look like this because at the wall, we have maximum temperature and free stream temperature as $y \rightarrow \infty$.

So, with this now, if I write the energy equation, then you can write $u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \alpha \frac{\partial^2 \theta}{\partial y^2}$.

So, now, let us write the boundary conditions. You can see at the wall where $y = 0$, you have $T = T_w$ and $y \rightarrow \infty$ means away from the boundary, you have free stream temperature T_∞ .

You can see that as $x \rightarrow 0$ at the leading edge of the flat plate, we have free stream temperature T_∞ . So, at $y = 0$, you have $T = T_w$ so, θ will be 0 at or $y \rightarrow \infty$ so obviously, it is away from the wall you can see at the edge of the bond layer, you have T_∞ so, $T = T_\infty$ and you can write $\theta = 1$ and at $x \rightarrow 0$, you have $T = T_\infty$ so, θ will be 1.

Now, we have already solved the velocity distribution using similarity method. Can we have some similarity with the momentum equation and the energy equation when $\nu = \alpha$. So, when ν is when will be the $\nu = \alpha$? When Prandtl number, $Pr = 1$.

So, as a special case let us say for Prandtl number= 1 your ν you can write is equal to α and the momentum equation whatever we have so, you can write down

$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \alpha \frac{\partial^2 u}{\partial y^2}$, and this u you write in terms of non-dimensional velocity

$u^* = \frac{u}{U_\infty}$ then, you can see you can write $u \frac{\partial u^*}{\partial x} + v \frac{\partial u^*}{\partial y} = \alpha \frac{\partial^2 u^*}{\partial y^2}$.

And what are the boundary conditions for this equation at $y = 0$, $u^* = 0$ no slip boundary condition, at $y \rightarrow \infty$, you have $u^* = 1$ and at $x \rightarrow 0$, you have $u^* = 1$.

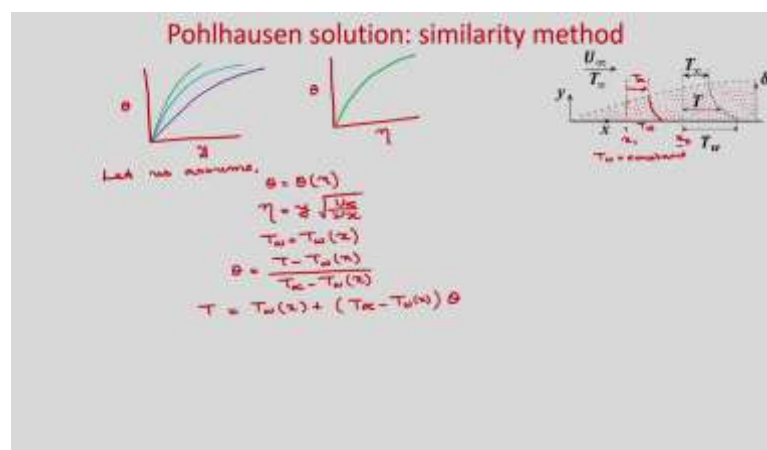
Now, you see these equations. This is the energy equation where θ is the non-dimensional temperature with these boundary conditions and this is the momentum equation where u^* is the non-dimensional velocity and these are the boundary conditions. So, if you compare these two for Prandtl number = 1 so, these are the same equation because if you replace $\theta = u^*$, then you can see the boundary conditions are same and $\nu = \alpha$. So, both equations are same.

So, you can write for Prandtl number = 1, θ can be replaced with u^* that means, you can write $\theta = u^*$; that means, $\frac{T - T_w}{T_\infty - T_w} = \frac{u}{U_\infty}$. So, you can see already we have solved the velocity distribution using similarity method and we could convert the partial differential equation to ordinary differential equation using similarity approach.

In this case, whatever we have shown now that for Prandtl number = 1 as a special case, the governing equations, energy equation as well as the momentum equation are same, and $\theta = u^*$. So obviously, we can have the similarity solution at least for Prandtl number = 1 for the temperature distribution.

So, we will use the similarity method to solve these energy equation and we will find the temperature distribution and if we can convert for other Prandtl number for Prandtl number $\neq 1$, this partial differential equation to ordinary differential equation; that means, similarity solution exist for Prandtl number $\neq 1$.

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So, you can see that the temperature distribution at any location say let us say x_2 and another location if you see here so, it will have free stream temperature T_∞ and you have let us say T_w so, this is your T_∞ , this is your T_w and this is location x_1 .

So, you can see that the temperature distribution if you see here and here for let us say T_w is equal to constant; you can see that these are similar profiles. So, if you can scale down the temperature distribution at location x_2 with a scaling factor, then you can get the same temperature distribution at x_1 . So that means, so, whatever temperature distribution say we are getting here θ versus y at different x location.

So, this will be you can see that at $y=0$, you have $\theta = 0$ and $y \rightarrow \infty$, it will be 1. So, you will have another profile. So, you can see these are the temperature profile at different x locations. Now, if you use this similarity variable, then you can see that if you plot θ versus η , then it will all the temperature profile will fall in same curve. So, it will fall in same curve.

So, let us start with whatever similarity variable η we have derived in earlier class and we will assume that θ is function of η only because here you can see it might fall in the same profile and using similarity method, if we can convert this partial differential equation to ordinary differential equation; that means, your similarity solution exists. If you see the energy equation, in the energy equation u and v are known from the solution of velocity distribution; that means, this is linear equation because velocity profiles are known, only you need to find the temperature distribution.

So, let us assume that θ is function of η only and η already we have found in last class

as $\eta = y \sqrt{\frac{U_\infty}{\nu x}}$. So, this is the similarity variable. So, we have written in terms of two independent variables x and y and properties ν is constant and U_∞ is the free stream velocity that is also constant.

Also, let us assume in general that T_w is a function of x . So, $\theta = \frac{T - T_w(x)}{T_\infty - T_w(x)}$ and now

$T = T_w(x) + (T_\infty - T_w(x))\theta$. So, let us write the energy equation the derivatives in terms of these with respect to the derivative with respect to η .

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Pohlhausen solution: similarity method

$$\eta = y \sqrt{\frac{U_\infty}{\nu x}}$$

$$\frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{\nu x}}$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{2} \frac{y}{x} \sqrt{\frac{U_\infty}{\nu x}} = -\frac{\eta}{2x}$$

$$T = T_w(x) + (T_\infty - T_w(x)) \theta$$

$$\frac{\partial T}{\partial x} = \frac{dT_w}{dx} - \frac{dT_w}{dx} \theta + (T_\infty - T_w) \frac{\partial \theta}{\partial x}$$

$$\frac{\partial T}{\partial x} = (1 - \theta) \frac{dT_w}{dx} + (T_\infty - T_w) \frac{\partial \theta}{\partial x}$$

$$\frac{\partial T}{\partial x} = (1 - \theta) T_w' + (T_\infty - T_w) \theta' \left(-\frac{\eta}{2x}\right)$$

$$\frac{\partial T}{\partial x} = (T_\infty - T_w) \frac{\partial \theta}{\partial \eta} \frac{\partial \eta}{\partial x} = (T_\infty - T_w) \theta' \sqrt{\frac{U_\infty}{\nu x}}$$

$$\frac{\partial T}{\partial \eta} = (T_\infty - T_w) \theta' \frac{U_\infty}{\nu x}$$

So, $\eta = y \sqrt{\frac{U_\infty}{\nu x}}$. So, $\frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{\nu x}}$ and $\frac{\partial \eta}{\partial x} = -\frac{1}{2} \frac{y}{x} \sqrt{\frac{U_\infty}{\nu x}} = -\frac{\eta}{2x}$.

Now, let us find the derivative of temperature. So, we have $\frac{\partial T}{\partial x}$. So, it will be. So,

$T = T_w(x) + (T_\infty - T_w(x))\theta$. So, $\frac{\partial T}{\partial x}$, you can see. So, this will be total derivative;

$$\frac{\partial T}{\partial x} = \frac{dT_w}{dx} - \frac{dT_w}{dx} \theta + (T_\infty - T_w) \frac{\partial \theta}{\partial x}.$$

So, now $\frac{\partial T}{\partial x}$ you can write as so, you can see this you

can $\frac{\partial T}{\partial x} = (1 - \theta) \frac{dT_w}{dx} + (T_\infty - T_w) \frac{d\theta}{d\eta} \frac{\partial \eta}{\partial x}$. So, now, $\frac{\partial T}{\partial x} = (1 - \theta) T_w' + (T_\infty - T_w) \theta' \left(-\frac{\eta}{2x}\right)$.

Now, similarly you find the derivative of T with respect to y. So, $\frac{\partial T}{\partial y}$ you can write now

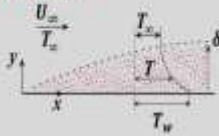
you see T_w is function of x. So, its derivative with respect to y will be 0 so, you will

get $\frac{\partial T}{\partial y} = (T_\infty - T_w) \frac{\partial \theta}{\partial \eta} = (T_\infty - T_w) \frac{\partial \theta}{\partial \eta} \frac{\partial \eta}{\partial y}$.

So, you can see, you can write $\frac{\partial T}{\partial y} = (T_\infty - T_w) \theta' \sqrt{\frac{U_\infty}{\nu x}}$ and $\frac{\partial^2 T}{\partial y^2} = (T_\infty - T_w) \theta'' \frac{U_\infty}{\nu x}$.

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Pohlhausen solution: similarity method



$$u = U_{\infty} f'$$

$$v = \nu g \frac{F''}{F'} - U_{\infty} y F' \frac{g'}{g}$$

$$\frac{F'''}{F'} = -\frac{1}{2} \quad \eta = yg \quad g = \sqrt{\frac{U_{\infty}}{2\nu x}}$$

$$\frac{g'}{g^2} = -\frac{1}{2} \frac{\nu}{U_{\infty}} \Rightarrow \frac{g'}{g^2} = -\frac{1}{2} \frac{\nu}{U_{\infty}} g$$

$$F = f' = \frac{u}{U_{\infty}}$$

$$v = \nu g \left(-\frac{1}{2}\right) - U_{\infty} y f' \frac{g'}{g^2}$$

$$= -\nu g \frac{1}{2} - U_{\infty} y f' \left(-\frac{1}{2} \frac{\nu}{U_{\infty}} g\right)$$

$$= \frac{1}{2} \nu g \left[\eta f' - f\right]$$

$$= \frac{1}{2} \nu \sqrt{\frac{U_{\infty}}{2\nu x}} \left[\eta f' - f\right]$$

$$= \frac{1}{2} \sqrt{\frac{U_{\infty} \nu}{2x}} \left[\eta f' - f\right]$$

Now, we have also velocities u and v . So, that let us find. So, u obviously, you know that $u = U_{\infty} f'$ and in last class, we have already derived the velocity v in terms of F first let us write that and then we will write in terms of F . So, v if you see in last class, we have written $v = \nu g \frac{F'''}{F'} - U_{\infty} y F' \frac{g'}{g}$.

And if after separation of variables if you see that we have written $\frac{F'''}{F'} = -\frac{f}{2}$. H already

you know so, that we have written $\eta = yg$, here y we can write $\frac{\eta}{g}$ and g we have written

$g = \sqrt{\frac{U_{\infty}}{\nu x}}$ and $\frac{g'}{g^3}$, if you see the equation where we have separated the variables that we

have written $\frac{g'}{g^3} = -\frac{1}{2} \frac{\nu}{U_{\infty}}$. So, from here we can write $\frac{g'}{g^2} = -\frac{1}{2} \frac{\nu}{U_{\infty}} g$ and F obviously,

you can see $F = f' = \frac{u}{U_{\infty}}$.

So, now, if you substitute all these here, you will get velocity v . So, it will

be $v = \nu g \left(-\frac{f}{2}\right) - U_{\infty} \eta f' \frac{g'}{g^2}$. So, $v = -\nu g \frac{f}{2} - U_{\infty} \eta f' \left(-\frac{1}{2} \frac{\nu}{U_{\infty}} g\right)$.

So, if you take half ν g outside and it will be $\nu = \frac{1}{2} \nu g [\eta f' - f]$. So, it will

be $\nu = \frac{1}{2} \nu \sqrt{\frac{U_\infty}{\nu x}} [\eta f' - f]$. So, this ν if you take inside the root, then you will

get $\nu = \frac{1}{2} \sqrt{\frac{U_\infty \nu}{x}} [\eta f' - f]$.

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Pohlhausen solution: similarity method

$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$
 $U_\infty f'(1-\theta)T_w' - U_\infty f'(T_\infty - T_w)\theta' - \frac{1}{2} \frac{U_\infty}{x} (T_\infty - T_w)f\theta' = \alpha (T_\infty - T_w) \frac{U_\infty}{\nu x} \theta''$
 Multiply both side by $\frac{x}{U_\infty (T_\infty - T_w)}$
 $\frac{x f'(1-\theta)T_w'}{T_\infty - T_w} - \frac{\eta}{2} f'\theta' + \frac{\eta}{2} f'\theta' - \frac{1}{2} f\theta' = \frac{1}{Pr} \theta''$
 $\frac{x f'(\theta-1)T_w'}{T_\infty - T_w} = \frac{1}{2} f\theta' + \frac{1}{Pr} \theta''$
 Divide both side by $f'(\theta-1)$
 $\frac{x T_w'}{T_\infty - T_w} = \frac{1}{2} \frac{f}{f'} \frac{\theta'}{\theta-1} + \frac{1}{Pr} \frac{1}{f'} \frac{\theta''}{\theta-1} = \lambda$
 func of x func of η

So, let us put all these values in the energy equation. So, energy equation

is $u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$. So, now, one by one let us put. So, you have u ,

$U_\infty f'(1-\theta)T_w'$ and another term will be there. So, you can see this term we have written

and this term you can write now. So, it will be $-U_\infty f'(T_\infty - T_w)\theta' \frac{\eta}{2x}$.

Now, we have $\nu \frac{\partial T}{\partial y}$ so, if you write it

$$\frac{1}{2} \frac{U_\infty}{x} \eta f'(T_\infty - T_w)\theta' - \frac{1}{2} \frac{U_\infty}{x} (T_\infty - T_w) f\theta' = \alpha (T_\infty - T_w) \frac{U_\infty}{\nu x} \theta''.$$

So, now, you rearrange this equation. So, if you rearrange you will get so, first multiply

both side by $\frac{x}{U_\infty (T_\infty - T_w)}$. So, you can see in the first term. So, U_∞ , U_∞ will get cancelled

so, you can write $\frac{x f' (1-\theta) T_w'}{(T_\infty - T_w)}$. The next term you can see U_∞ will get cancelled

$T_\infty - T_w$ also will get cancelled and this x, x. So, you will get finally, $\frac{\eta}{2} f' \theta'$.

Then, here also $\frac{U_\infty}{x} (T_\infty - T_w)$ is there so, you will get $\frac{\eta}{2} f' \theta'$. Then, the next term you can

see here you will get only $-\frac{1}{2} f \theta'$ and here you can see $\frac{\alpha}{\nu}$ that means, it is $\frac{1}{P_r}$ because

Prandtl number is $\frac{\nu}{\alpha}$ right, Prandtl number is $\frac{\nu}{\alpha}$ moving from diffusivity to thermal

diffusivity. So, you can write $\frac{1}{P_r}$ and θ'' .

So, here you can see this is $T_\infty - T_w$, it is $1-\theta$ we will write here $\frac{x f' (\theta-1) T_w'}{T_w - T_\infty}$ and you

have minus we taken right hand side. So, it will be $\frac{1}{2} f \theta' + \frac{1}{P_r} \theta''$.

Now, divide both sides; divide both side by $f' (\theta-1)$. So, if you rearrange it, you will

$$\text{get } \frac{x T_w'}{T_w - T_\infty} = \frac{1}{2} \frac{f}{f'} \frac{\theta'}{\theta-1} + \frac{1}{P_r} \frac{1}{f'} \frac{\theta''}{\theta-1}.$$

So, you see we have separated the variables. If you see the left-hand side, it is function of x only and right hand side it is function of η only because AP is function of θ and θ is also a function of η . So, we have separated the variables.

So, you can see this is T_w . So, T_w is a function of x only so, all these terms are function of x so, this is function of x and this right-hand side terms are function of η only. So, as left-hand side equal to right-hand side and left-hand side is function of x and right-hand side function of η so, it will be equal to some constant and that constant let us say that it is λ . So, this will be equal to λ .

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Pohlhausen solution: similarity method

$$\frac{x T_w'}{T_w - T_\infty} = \lambda$$

$$\Rightarrow \frac{dT_w}{T_w - T_\infty} = \lambda \frac{dx}{x}$$

Integrate

$$\Rightarrow \ln(T_w - T_\infty) = \lambda \ln x + \ln c$$

$$\Rightarrow T_w - T_\infty = C x^\lambda$$

$$\text{or } T_w = T_\infty + C x^\lambda$$

If $\lambda = 0$, $T_w = T_\infty + C = \text{const}$ uniform wall temperature.

$$\frac{1}{2} \frac{f}{f'} \frac{\theta'}{\theta - 1} + \frac{1}{Pr} \frac{1}{f'} \frac{\theta''}{\theta - 1} = \lambda$$

Multiply both side by $Pr f'(\theta - 1)$

$$\frac{1}{2} Pr f \theta' + \theta'' = \lambda Pr f'(\theta - 1)$$

$$\Rightarrow \theta'' + \frac{Pr}{2} f \theta' - \lambda Pr f'(\theta - 1) = 0$$

Second order linear ordinary differential equation.
— Pohlhausen equation

For $\lambda = 0$, $T_w = \text{const}$, $\theta'' + \frac{Pr}{2} f \theta' = 0$

We have separated the variable so; you can write it as $\frac{x T_w'}{T_w - T_\infty} = \lambda$. So, you can write dT_w

so, T_w' is $\frac{dT_w}{dx}$. So, we will write $\frac{dT_w}{T_w - T_\infty} = \lambda \frac{dx}{x}$.

So, now, integrate. So, you will get $\ln(T_w - T_\infty) = \lambda \ln x + \ln c$. So, you can write $T_w - T_\infty = C x^\lambda$ or $T_w = T_\infty + C x^\lambda$. We have assumed that wall temperature T_w is function of x and it varies in this way.

As a special case, you can see that if $\lambda = 0$; if $\lambda = 0$, then you will get $T_w = T_\infty + C$. So, T_w and T_∞ is constant, and c is constant so, this is equal to constant. So, you can see that it is a case of uniform wall temperature. So, this is a special case. But in general, we have derived and T_w varies as $T_\infty + C x^\lambda$.

So, now let us consider the other terms which is a function of η only. So, if you consider that term, then you will get $\frac{1}{2} \frac{f}{f'} \frac{\theta'}{\theta - 1} + \frac{1}{Pr} \frac{1}{f'} \frac{\theta''}{\theta - 1} = \lambda$. Now multiply both side

by $Pr f'(\theta - 1)$. So, what you will get? You will get $\frac{1}{2} Pr f \theta' + \theta'' = \lambda Pr f'(\theta - 1)$. So, you

will get $\theta'' + \frac{Pr}{2} f \theta' - \lambda Pr f'(\theta - 1) = 0$.

So, you see that this equation is ordinary differential equation. So, we started with the partial differential equation and we used similarity transformation and transferred this partial differential equation to ordinary differential equation for Prandtl number $\neq 1$ as well because it is a function of Prandtl number.

So that means, similarity solutions for this temperature distribution exist and we could get this second order linear ordinary differential equation. You can see this is second order and as f is known f'' and f' are known from the velocity distribution, so obviously, this is linear equation. So, you can see this is second order linear ordinary differential equation and this equation is known as Pohlhausen equation.

And you can see as a special case for uniform wall temperature you can put $\lambda = 0$. So, for $\lambda=0$ which is a special case T_w is constant for that you can write the Pohlhausen equation as $\theta'' + \frac{Pr}{2} f' \theta = 0$. So, you can see here, this is your linear equation because f is known from the velocity distribution and this is the second order linear ordinary differential equation.

So, this equation now you can solve using some numerical technique and find the temperature distribution and once you get the temperature distribution, you will be able to calculate the heat flux and from there you can calculate the heat transfer coefficient and Nusselt number.

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Pohlhausen solution: similarity method

For $\lambda = 0$

$$\theta'' + \frac{Pr}{2} f' \theta = 0 \quad T_w = \text{constant}$$

BCs @ $\eta = 0, \theta = 0$
 @ $\eta \rightarrow \infty, \theta = 1$

$$\frac{\theta''}{\theta'} = -\frac{1}{2} Pr f'$$

$$\frac{d}{d\eta} \left(\frac{\theta'}{\theta'} \right) = -\frac{1}{2} Pr f'$$

Integrating the above equation

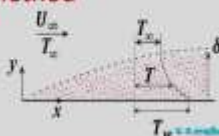
$$\int_0^\eta \frac{d(\theta')}{\theta'} = - \int_0^\eta \frac{Pr}{2} f' d\eta$$

$$\ln \left(\frac{\theta'}{\theta'(0)} \right) = - \int_0^\eta \frac{Pr}{2} f' d\eta$$

$$\frac{d\theta}{d\eta} = \theta'(0) e^{- \int_0^\eta \frac{Pr}{2} f' d\eta}$$

Integrate the above eqn

$$\int d\theta = \theta'(0) \int_0^\eta e^{- \int_0^\eta \frac{Pr}{2} f' d\eta} d\eta$$

$$\theta(\infty) - \theta(\eta) = \theta'(0) \int_\eta^\infty e^{- \int_0^\eta \frac{Pr}{2} f' d\eta} d\eta$$


Now, let us consider the special case when the flat plate is maintained at constant uniform temperature so; that means, T_w is constant and whatever Pohlhausen equation we got, now let us find the solution of this ordinary differential equation applying the boundary conditions.

So, for $\lambda = 0$, here now T_w is constant we can write $\theta'' + \frac{P_r}{2} f' \theta = 0$ where T_w is constant.

So, this is a special case for $\lambda = 0$.

Now, what are the boundary conditions? At $\eta = 0$ because $y = 0$ you will get $\eta = 0$ and it is your T_w so that means, $T_w - T_w = 0$ so; that means, θ will be 0 and $\eta \rightarrow \infty$ when $y \rightarrow \infty$ and $x \rightarrow 0$ these two boundary conditions are merged into 1 and for $\eta \rightarrow \infty$ you can write $\theta = 1$.

So, now, you can integrate this equation so, you can write $\frac{\theta''}{\theta'} = -\frac{1}{2} P_r f$ so, what is θ'' ?

You can write $\frac{d}{d\eta}$ right; $\frac{\frac{d}{d\eta}(\theta')}{\theta'} = -\frac{1}{2} P_r f$.

So, integrating the above equation so, what we will get? So, $\int_0^\eta \frac{d(\theta')}{\theta'} = -\int_0^\eta \frac{P_r}{2} f d\eta$. So, if

you integrate what you will get here? $\ln\left(\frac{\theta'}{\theta'(0)}\right) = -\int_0^\eta \frac{P_r}{2} f d\eta$.

So, unless you know the velocity distribution f so, f' and f , f is the stream function equivalent to a stream function, you will not be able to integrate so, you are keeping it in

terms of integral form. So, if you write this so, you can write $\frac{d\theta}{d\eta} = \theta'(0) e^{-\int_0^\eta \frac{P_r}{2} f d\eta}$.

So, integrate this above equation; integrate the above equation. So, we will

integrate $\int_\eta^\infty d\theta = \theta'(0) \int_\eta^\infty e^{-\int_0^\eta \frac{P_r}{2} f d\eta} d\eta$.

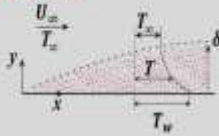
Here, we will put the limit from η to ∞ ; η to ∞ . So, if you put it, then what you will get?

θ at $\eta = \infty$ minus θ at $\eta = \theta'(0)$ and this is your η to ∞ , $\int_{\eta}^{\infty} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta$. So, now, you see θ at

$\eta \rightarrow \infty$ we have the boundary condition $\theta = 1$. So, you can put this is as 1. So, these value is 1.

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Pohlhausen solution: similarity method



$$\theta(\eta) = 1 - \theta'(0) \int_{\eta}^{\infty} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta$$

bc @ $\eta=0$, $\theta=0$

$$0 = 1 - \theta'(0) \int_0^{\infty} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta$$

$$\theta'(0) = \frac{1}{\int_0^{\infty} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta}$$

$$\theta(\eta) = 1 - \frac{\int_{\eta}^{\infty} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta}{\int_0^{\infty} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta}$$

$$\theta(\eta) = \frac{\int_0^{\eta} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta}{\int_0^{\infty} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta}$$

So, now, $\theta(\eta)$ we express in terms of other terms. So, you can

write $\theta(\eta) = 1 - \theta'(0) \int_{\eta}^{\infty} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta$.

So, now, we are left with another boundary condition at $\eta=0$, $\theta=0$. So, now, let us put that boundary condition. So, another boundary condition is there at $\eta=0$, $\theta=0$. So, if

you put that value so, $0 = 1 - \theta'(0) \int_0^{\infty} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta$.

Now, we can find the value of $\theta'(0)$. So, $\theta'(0) = \frac{1}{\int_0^{\infty} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta}$. So, once $\theta'(0)$ is known

so, now, we can find $\theta(\eta)$.

$$\text{So, } \theta(\eta) = 1 - \frac{\int_0^\eta e^{-\int_0^\eta \frac{P_r}{2} f d\eta} d\eta}{\int_0^\infty e^{-\int_0^\eta \frac{P_r}{2} f d\eta} d\eta}.$$

So, this is the temperature distribution. So, if you know the

function f , then you will be able to integrate and you will be able to find the temperature distribution using some numerical technique you can solve this equation.

$$\text{So, in this equation now you can see that if you write } \theta(\eta) = \frac{\int_0^\eta e^{-\int_0^\eta \frac{P_r}{2} f d\eta} d\eta - \int_\eta^\infty e^{-\int_0^\eta \frac{P_r}{2} f d\eta} d\eta}{\int_0^\infty e^{-\int_0^\eta \frac{P_r}{2} f d\eta} d\eta}.$$

So, you can see another $d\eta$ will be here another $d\eta$ will be here.

So, now you can see this integrant is same in both the integral, but the limits are different 0 to ∞ and η to ∞ . So, if you subtract that so obviously, you can get 0 to η . So,

$$\theta(\eta) \theta(\eta) = \frac{\int_0^\eta e^{-\int_0^\eta \frac{P_r}{2} f d\eta} d\eta}{\int_0^\infty e^{-\int_0^\eta \frac{P_r}{2} f d\eta} d\eta}.$$

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Pohlhausen solution: similarity method

Blasius equation

$$f''' + \frac{1}{2} f f'' = 0$$

$$\Rightarrow \frac{d}{d\eta} \left(\frac{f''}{f'} \right) = -\frac{f}{2}$$

Integrate

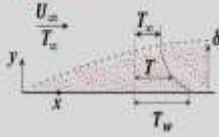
$$\int_0^\eta \frac{d(f'')}{f'} = - \int_0^\eta \frac{f}{2} d\eta$$

$$\ln \left(\frac{f''}{f'(0)} \right) = - \int_0^\eta \frac{f}{2} d\eta$$

Multiplying both side with P_n

$$\ln \left[\frac{[f'']^{P_n}}{[f'(0)]^{P_n}} \right] = - \int_0^\eta \frac{P_n}{2} f d\eta$$

$$\Rightarrow \frac{[f'']^{P_n}}{[f'(0)]^{P_n}} = e^{-\int_0^\eta \frac{P_n}{2} f d\eta}$$

$$f''(0) = 0.33206$$


that is your $f''' + \frac{f}{2} f'' = 0$. So, you can write $\frac{\frac{d}{d\eta}(f'')}{f''} = -\frac{f}{2}$. So, integrate this equation

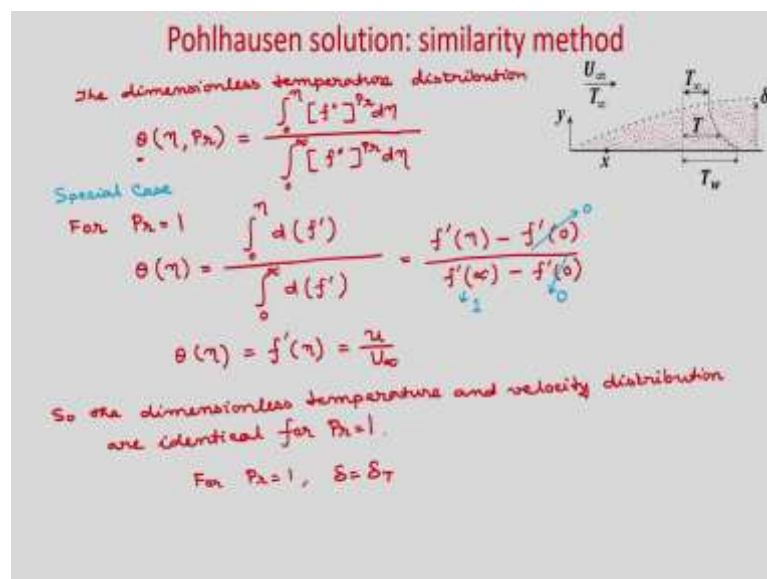
so, you will get $\int_0^\eta \frac{d(f'')}{f''} = -\int_0^\eta \frac{f}{2} d\eta$.

So, now, you can see that you will get $\ln\left(\frac{f''}{f''(0)}\right) = -\int_0^{\eta} \frac{f}{2} d\eta$. Now multiply both sides with Prandtl numbers; multiply both sides with Prandtl number. So, if you see here if you multiply Prandtl number so, you can write this as $\ln\left(\frac{[f'']^{Pr}}{[f''(0)]^{Pr}}\right) = -\int_0^{\eta} \frac{f}{2} d\eta$.

So, from here, you can write $\frac{[f^{(r)}]^{P_r}}{[f^{(0)}]^{P_r}} = e^{-\int_0^{\frac{P_r}{2}} f d\eta}$.

So, if you see in the last expression whatever you have $e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta}$ so, that expression we have written in terms of f'' . So, $f''(0)$ you know right because $f''(0)$ we have found from the velocity distribution so, this is the velocity gradient at $\eta = 0$ and that value is 0.33206.

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So, now, if you put this in this expression; in this expression so, what you will get? You will get so, the dimensionless temperature; dimensionless temperature distribution you

$$\text{will get } \theta(\eta, P_r) = \frac{\int_0^\eta [f']^{P_r} d\eta}{\int_0^\infty [f']^{P_r} d\eta}.$$

So, you can see here in this expression so obviously, $[f'(0)]^{P_r}$ is there and so, whatever

expression we have got here $e^{-\int_0^\eta \frac{P_r}{2} f' d\eta}$ so, that expression you put in the expression here in this expression and you just rearrange it finally, you can write the temperature distribution as the dimensionless temperature distribution.

So, after rearrangement you will get θ which is function of η and Prandtl number is equal

$$\text{to } \frac{\int_0^\eta [f']^{P_r} d\eta}{\int_0^\infty [f']^{P_r} d\eta}. \text{ So, after doing some rearrangement, you will get this as the final}$$

temperature distribution and you can see if you know the value of f' from the velocity distribution, you will be able to calculate the temperature distribution. So, this equation you can solve numerically.

So, now, let us see that as a special case whatever we started with that for Prandtl number =1 what is the temperature distribution is it same as the velocity distribution let us see. So, for Prandtl number =1 this is a special case. So, for Prandtl number= 1 you can see what you can write $\theta(\eta)$ for Prandtl number =1 is equal to 0 to η .

$$\text{So, Prandtl number =1. So, you can write } \theta(\eta) = \frac{\int_0^\eta d(f')}{\int_0^\infty d(f')} = \frac{f'(\eta) - f'(0)}{f'(\infty) - f'(0)}.$$

So, now what is f' prime? f' is the velocity $\frac{u}{U_\infty}$ and at $\eta = 0$ $f'(0)$. So, you can see this is your 0 and this is also 0 and what is $f'(\eta) \rightarrow \infty$? So, that is your 1 because $u \rightarrow U_\infty$. So,

this is your 1. So, you can see that from this expression that $\theta(\eta) = f'(\eta)$ and what is f' prime? It is nothing, but $\frac{u}{U_\infty}$.

So, the dimensionless temperature and velocity distribution are identical for Prandtl number = 1 that means for Prandtl number = 1, you have $\delta = \delta_T$. So, it will be the non-dimensional temperature distribution will be same as non-dimensional velocity distribution.

So, in today's class, we have started with the energy equation and we defined one non-dimensional temperature $\theta = \frac{T - T_w}{T_\infty - T_w}$. We have assumed in the beginning that T_w is function of x and we have derived the Pohlhausen equation in general which is your second order linear ordinary differential equation and we have shown that using similarity variable approach, we could convert the PDE to ODE for any Prandtl number.

For a special case, $\lambda = 0$, the wall temperature becomes constant and for that we have found the solution for temperature distribution and this temperature distribution we have expressed in terms of your velocity gradient f'' once you know the velocity distribution from the solution of Blasius equation you will be able to find the temperature distribution.

So, to you can solve this equation using some numerical technique and at last we have shown that for Prandtl number = 1, the temperature distribution and velocity distribution are identical and you can write that $\theta = \frac{u}{U_\infty}$

Thank you.