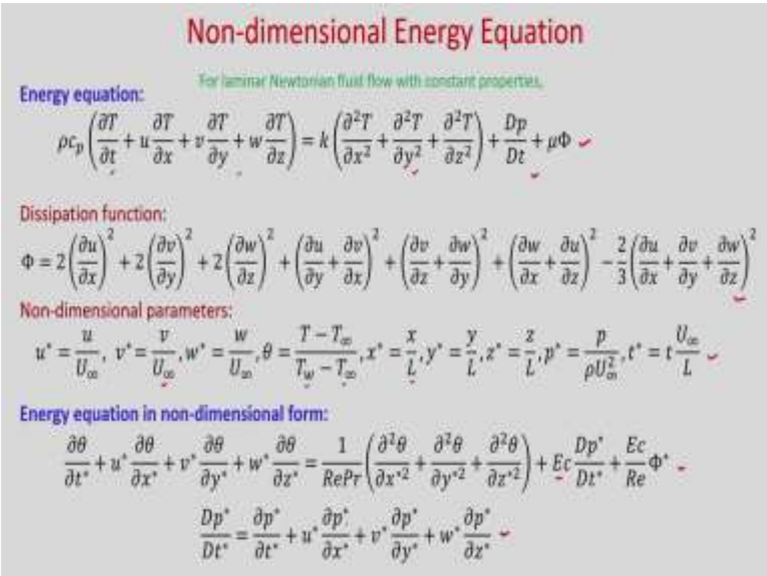
Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 03
Convective Heat Transfer in External Flows – I
Lecture – 06
Blasius solution: similarity method

Hello everyone. So, in today's class first we will derive the energy equation in non-dimensional form, then we will solve the governing equation for flow over flat plate using similarity method. This solution is known as Blasius solution. Blasius first solved this equation, using similarity method and derived the ordinary differential equation which you can solve using some numerical technique.

(Refer Slide Time: 01:02)



So, first let us see what we derived, the energy equation in dimensional form. So, if you see for laminar Newtonian fluid flow with constant properties, this is the energy equation we derived right. So, this is the temporal term, this is the convective term, this is the diffusion term and as it is constant properties so, the thermal conductivity we can take it outside and this is $\frac{Dp}{Dt}+\mu\phi$, where Φ is the dissipation function.

So, dissipation function in general is given in this form. If it is an incompressible flow; then obviously, $\nabla V=0$. So, last term will become 0. Using this non dimensional parameters now you please convert this energy equation in non-dimensional form. So, what we will use? We will use some reference velocity U_∞ and reference length L.

So, these are some characteristic length L and characteristic velocity U_∞. For external flows generally, we take the free stream velocity as reference velocity that is your U_∞ and for any geometry we find the characteristic length and that length we take as reference length L. So, you can see using this reference velocity we have non-dimensionalized the velocity u* as $\frac{u}{U_\infty}$.

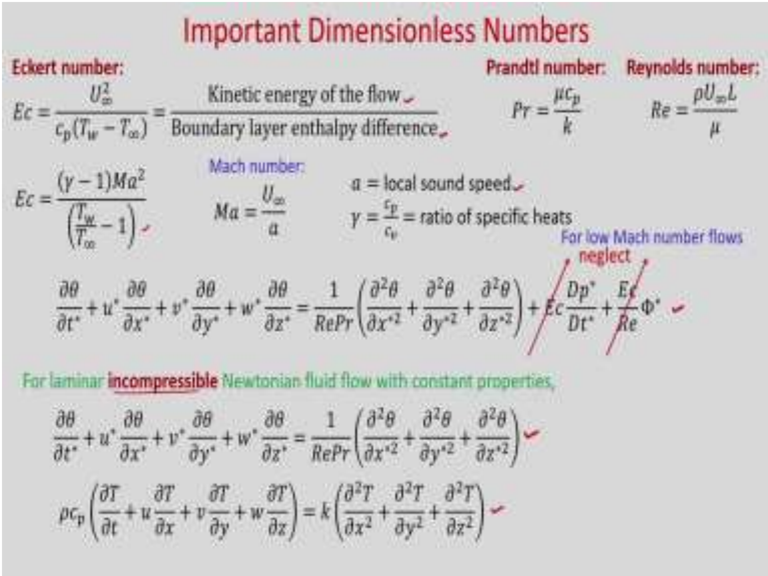
Similarly, $v^* = \frac{v}{U_\infty}$, $w^* = \frac{w}{U_\infty}$, the temperature non dimensional temperature we have defined as $\theta = \frac{T - T_\infty}{T_w - T_\infty}$, where T_W is the wall temperature and T_∞ is the free stream temperature.

Similarly, the coordinate, we have non-dimensionalized using the characteristic length L. So, $x^* = \frac{x}{L}$, $y^* = \frac{y}{L}$, $z^* = \frac{z}{L}$, and the pressure, pressure we have used $p^* = \frac{p}{\rho U_\infty^2}$ and the time $t^* = t\frac{U_\infty}{L}$. So, these are some non-dimensional parameters used to convert this dimensional form of the energy equation to non-dimensional form of the energy equation.

So, as homework you just do it and finally, you will get these equations. So, you can see we have written in non-dimensional form. So, this is the convective term. In diffusion term, these non-dimensional numbers comes one by Reynolds number into Prandtl number.

In this term, one new non dimensional number comes that is your Eckert number and with the dissipation function you will get $\frac{E_c}{R_e}$. And obviously, you know how the $\frac{Dp^*}{Dt^*}$ is defined. So, this is the material derivative. So, now let us define the non-dimensional numbers.

(Refer Slide Time: 04:06)



So, this conversion from dimensional form to non-dimensional form using these non-dimensional parameters, you can do as a homework. And now, you can see the definition of these non-dimensional numbers, which we encountered in non-dimensional form of the energy equation. So, first non-dimensional number is Eckert number. So, what is Eckert number? Eckert number $E_c = \frac{U_\infty^2}{c_p(T_w - T_\infty)}$.

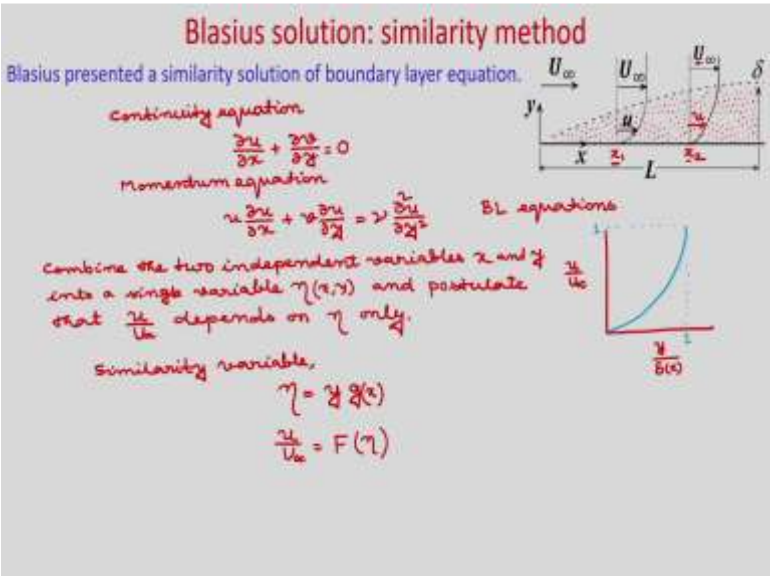
So, that means, it is the temperature difference. So, you can see that in the numerator you have kinetic energy of the flow and in denominator you have boundary layer enthalpy difference. So, you can see Eckert number is the ratio of kinetic energy of the flow to the boundary layer enthalpy difference. And Prandtl number you know, Prandtl number $P_r = \frac{\mu c_p}{k}$ and Reynolds number, $R_e = \frac{\rho U_\infty L}{\mu}$, where rho is the fluid density, U∞ is the reference velocity, L is the reference length and mu is the dynamic viscosity of the fluid.

So, these two non-dimensional numbers you are familiar with, but this is the new number Eckert number which we can write in this form. So, with some mathematical algebra, you can write Eckert number, $E_c = \frac{(\gamma -)Ma^2}{(\frac{T_w}{T_\infty} - 1)}$.

So, you know what is Mach number? Mach number is the ratio of fluid velocity to the local sound velocity. So, that means, Mach number you can defined, $Ma = \frac{U_\infty}{a}$, where a is the local sound speed and $\gamma = \frac{c_p}{c_v}$. So, you can see in this equation that, if Mach number is very low then Eckert number also will become very low. So, here you can see whatever energy equation we have written in non-dimensional form. So, if Mach number is very low then these last two terms you can neglect, because Eckert number will be very-very low.

So, you can see these two terms in the energy equation you can neglect. And, for laminar incompressible flow generally Mach number is very low and for that you can drop these two terms and you can write the energy equation in this form. So, it is in non-dimensional form and if you write in dimensional form, the energy equation for incompressible fluid flow, then this will be your energy equation.

(Refer Slide Time: 07:03)



Now, let us consider flow over flat plate. So, this is the simplest solution we can do for flow over flat plate, because your pressure gradient is 0. So, first let us write the governing equation. We have already derived the boundary layer equations for flow over flat plate. So, continuity equation; so, what is continuity equation? $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

So, while deriving the boundary layer equation for flow over flat plate what are the assumptions we took? We have assumed that it is a steady flow, two dimensional flow, and incompressible Newtonian fluid flow. So, for that you can write the continuity equation like this and momentum equation you can write $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$.

So, this is the boundary layer equations right, these are boundary layer equations. Now, you can see these equations are non-linear, because the momentum equation you can see these terms are non-linear. So, it is difficult to solve analytically. Blasius first time solved these equations using similarity method.

So, when can you use similarity method? First you see the velocity distribution for flow over flat plate. So, if you see this is the length of the plate L, x is the axial direction, y is measured from the flat plate, U_∞ is the free stream velocity and if you see at any location x_1, so this will be your velocity profile. So, velocity from 0, it will develop and at the edge of the boundary layer.

So, this is the edge of the boundary layer it will become free stream velocity U_∞. Similarly, if you see at location x_2; so, here also the velocity will vary; velocity will vary u from 0 to U_∞, because U_∞ is the free stream velocity. So, at the free stream velocity U_∞ is constant for flow over flat plate. So, you can see in these two locations that, velocity profile looks similar except it is with a scaling factor.

So, if you see at x_2 location, whatever velocity profile you can see if you scale it down, it will be similar to the velocity profile at x_1 location. So, we can see that as this velocity profile looks like similar with a scaling factor so we can use the similarity method to solve these governing equations. So, Blasius first observed that, if these two independent variables x, y; if you convert it into a single variable η, where η is function of x and y then, with the scaling factor all the velocity profile at different locations will fall in a same curve.

So, here if you see that if you plot the velocity u by U_∞; so, u by U_∞, so you can see that you will be 0 at y = 0 and at the edge of boundary layer you have U_∞, so it will become 1. And, if you plot this velocity distribution with $\frac{y}{\delta}$, δ is the boundary layer thickness.

So, at any location δ is the distance from the flat plate to the edge of bond layer, where the velocity becomes close to free stream velocity U_∞. And, at different location if you plot this u by U_∞ versus $\frac{y}{\delta(x)}$, because δ will vary with x.

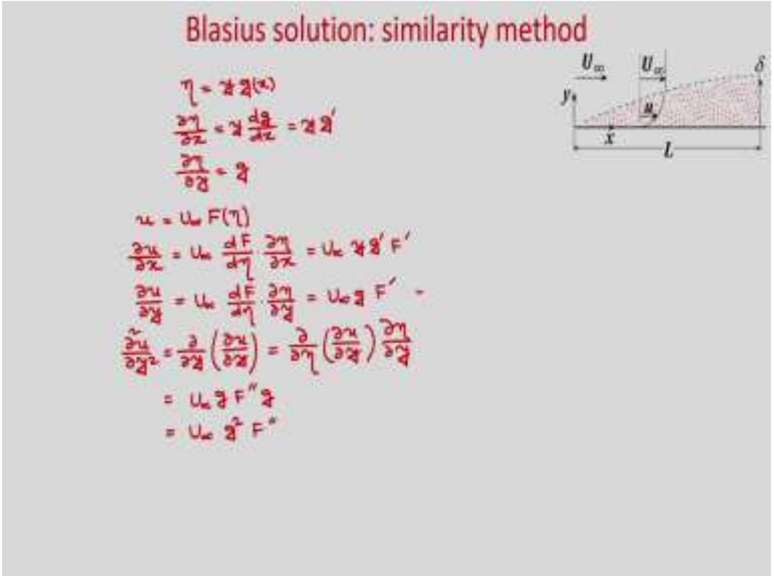
So, δ is function of x, then he observed that it will fall in the same curve. So, it will look like. So, you can see this is the velocity 1 and this will be also 1, because at y = δ, this will become 1. So, it will fall in the same curve at different location, along the axial direction at different x location if you plot the velocity in this curve $\frac{u}{U_\infty}$ versus $\frac{y}{\delta(x)}$, it will fall in the same curve.

So, you can see that velocity profiles are similar. So, that is why we can use the similarity approach to solve these governing equations. Now, you can see that we have defined $\frac{y}{\delta}$, δ is function of x; so obviously, we can see that we can define one variable η combining the two independent variables x and y such that, it will become y into g(x). So, combine the two independent variables; two independent variables x and y into a single variable η, which is function of x and y and postulate that $\frac{u}{U_\infty}$ depends on η only.

So, now this η will be function of x and y. We do not know; what is the function, but it will be function of y and x. So, we will write, because we can see from here that $\delta(x)$. So, it will be function of x, but that is unknown. So, let us define the similarity variable η; similarity variable $\eta = yg(x)$, we do not know what is the function; so, which is function of x.

So, now what we can do? So, now we can see that these independent variables x and y we have combined into a single variable η such that, $\frac{u}{U_\infty}$ is function of eta only; some function, we do not know the function, but function of η only. Now, we have the governing equations we know u with some function of F. So, we will just substitute all these in these equations and we will derive the Blasius equation.

(Refer Slide Time: 15:31)



So, we have seen that η=yg(x). So, we can write $\frac{\partial \eta}{\partial x} = y\frac{dg}{dx}$. So, you can write ordinary derivative; that means, yg'. So, g' represent the derivative of g with respect to x. Similarly, $\frac{\partial \eta}{\partial y} = g$. Now, we know the velocity profile $u = U_{\infty}F(\eta)$. And, we will see the derivative of u with respect to x and y.

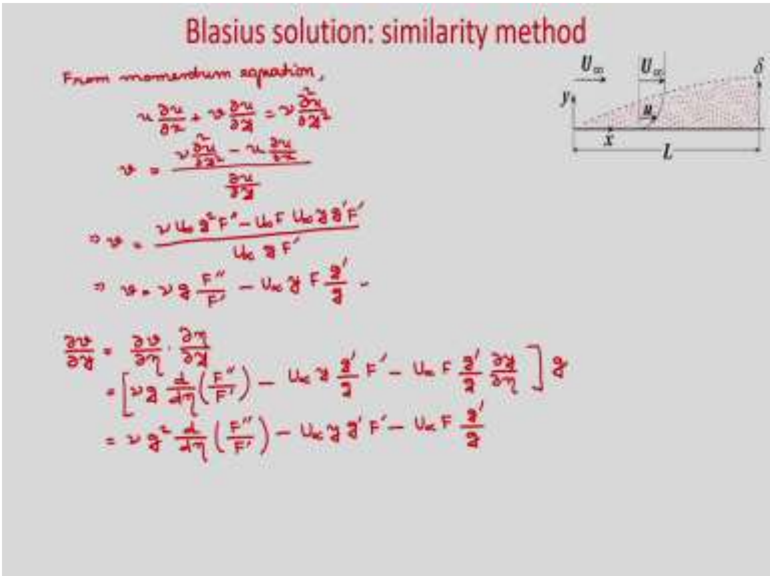
So, u = U_∞, where U_∞ is the free stream velocity into F. F is function of η only. So, it is function of η only and g is function of x only. So, first let us write $\frac{\partial u}{\partial x} = U_{\infty}\frac{dF}{d\eta}\frac{\partial \eta}{\partial x}$. So, $\frac{\partial \eta}{\partial x} = yg'$ and $\frac{dF}{d\eta} = F'$. So, you can write $U_{\infty}yg'F'$.

Similarly, you take the derivative of u with respect to y. So, $\frac{\partial u}{\partial y}$. So, it will be $U_{\infty}\frac{dF}{d\eta}\frac{\partial \eta}{\partial y}$. So, $\frac{\partial \eta}{\partial y} = g$. So, you can write $U_{\infty}gF'$. Now, take the derivative of $\frac{\partial u}{\partial y}$ with respect to y. So, we can write $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial u}{\partial y})$.

So, what you can write; $\frac{\partial u}{\partial y}$ we know right. So, we can write $\frac{\partial}{\partial \eta}(\frac{\partial u}{\partial y})\frac{\partial \eta}{\partial y}$. So, this you can write. So, now you see, this is the $\frac{\partial u}{\partial y}$. So, you write the derivative of it with respect to η. So, U_∞ is constant, g is function of x only; so, it will be F'' and $\frac{\partial \eta}{\partial y} = g$. So, you can write $U_\infty g F''$ and $\frac{\partial \eta}{\partial y} = g$.

So, you can write it as $U_\infty g^2 F''$. So, $\frac{\partial^2 u}{\partial y^2} = U_\infty g^2 F''$. Now, let us substitute this in the momentum equation.

(Refer Slide Time: 18:48)



So, from momentum equation, you can write $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$. So, here you can see, here we have already found $u\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial y^2}$, but v is unknown.

So, let us find v. So, $v = \frac{\nu\frac{\partial^2 u}{\partial y^2} - u\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$. So, you can write v is equal to; so, $\nu\frac{\partial^2 u}{\partial y^2}$.

So, we have already derived $U_\infty g^2 F'' - u$, u =U∞ F right and $\frac{\partial u}{\partial x}$ we have derived as $U_\infty y g' F'$ and divided by $\frac{\partial u}{\partial y}$.

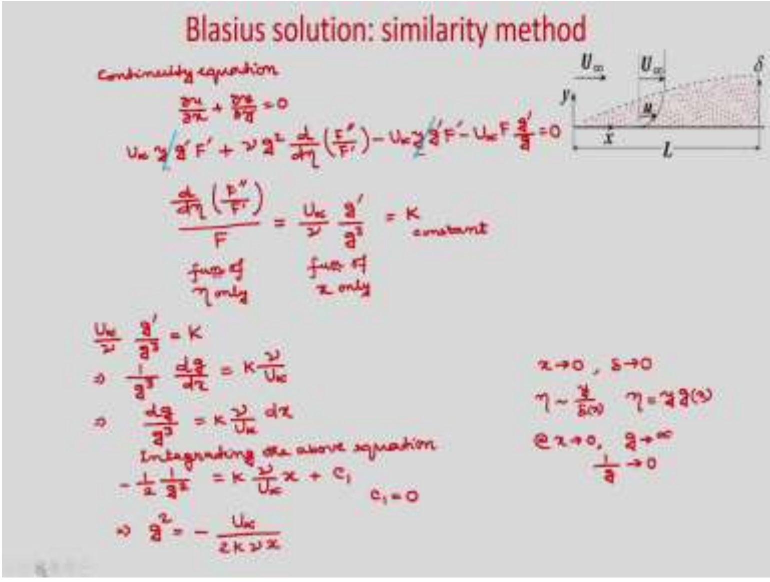
So, this is your $U_\infty g F'$. So, from here you can see that, you can write v is equal to. So, you see U∞ U∞ will cancel out, one g will get cancelled, so you will get $\nu g \frac{F''}{F'}$ and the second term one U∞ will get cancelled and $F' F'$ will get cancelled, so you will get $U_\infty y F \frac{g'}{g}$.

So, we have found the v. Now, if you see the continuity equation so, in the continuity equation we have $\frac{\partial v}{\partial y}$. So, let us find $\frac{\partial v}{\partial y}$ from here. So, $\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y}$. So, now you see $\frac{\partial v}{\partial \eta}$ from here you just find. So, νg. So, now we can write $\frac{d}{d\eta}(\frac{F''}{F'})$.

So, this is your first term and minus now, $\frac{\partial}{\partial \eta}$ of this so it will be so, you can see; so, there are two terms y and F. So, g is function of x only. So, you can write $U_\infty y \frac{g'}{g} F'$. So, we have taken the; we have taken the derivative of F minus. So, $U_\infty F \frac{g'}{g}\frac{\partial y}{\partial \eta}$ and we have $\frac{\partial \eta}{\partial y}$. So, what is $\frac{\partial \eta}{\partial y}$? $\frac{\partial \eta}{\partial y}$ is nothing but, g and here $\frac{\partial \eta}{\partial y}$ is there. In the denominator if you see it will be $\frac{\partial \eta}{\partial y}$ and that will be also g.

So, now you multiply g and write it. So, it will be $\nu g^2 \frac{d}{d\eta}(\frac{F''}{F'})$; so, these g g will get cancelled. So, it will be $U_\infty y g' F'$ minus you can see this, $\frac{\partial \eta}{\partial y}$ one g will be there and one g will get cancelled here. So, you can write $U_\infty F \frac{g'}{g}$. So, this is your $\frac{\partial v}{\partial y}$. Now, let us use the continuity equation. So, you can write the continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

(Refer Slide Time: 23:32)



So, this is the continuity equation. Now, we have already found $\frac{\partial u}{\partial x}$ and also we have found $\frac{\partial v}{\partial y}$. So, you can see $\frac{\partial u}{\partial x}$ whatever we have found $\frac{\partial u}{\partial x}$ is $U_\infty y g' F'$ and $\frac{\partial v}{\partial y}$ is this expression. So, let us put in the continuity equation.

So, if you put it here so, $U_\infty y g' F' + \nu g^2 \frac{d}{d\eta}(\frac{F''}{F'}) - U_\infty y g' F' - U_\infty F \frac{g'}{g} = 0$.

So, if you can see that this term and this term is same with a minus sign so, you can cancel it. And, now you notice, you see g is function of x only and F is function of η only. So, you rearrange such a way and separate the variables such that, that in the left hand side will be function of η only and right hand side will be function of x only. So, if you write and rearrange then, you can write $\frac{\frac{d}{d\eta}(\frac{F''}{F'})}{F}$.

So, this will be divided by F and if you take this in the right hand side, it will become positive and you can write as $\frac{U_\infty}{\nu}\frac{g'}{g^2}$; so, it you are dividing so it will be g^3. Now, you can see in this equation we have separated the variables left hand side is function of eta only and right hand side is function of x only and these are equal. So obviously, it will be equal to some constant right.

So, you can write that equal to some constant K. So, it will be some constant K, because it is function of η only; function of η only left hand side and this term is function of x only.

So obviously, because U_∞ and ν are constant for the flow over flat plate and g is function of x only. So, it will be equal to some constant K and that we need to find. Now, if you see; so, if you consider this term equal to K, then you can rearrange it as $\frac{U_\infty}{\nu}\frac{g'}{g^3} = K$. So, you can see you will be able to integrate it $\frac{1}{g^3}\frac{dg}{dx} = K\frac{\nu}{U_\infty}$. If you rearrange it again so you can write $\frac{dg}{g^3} = K\frac{\nu}{U_\infty}dx$.

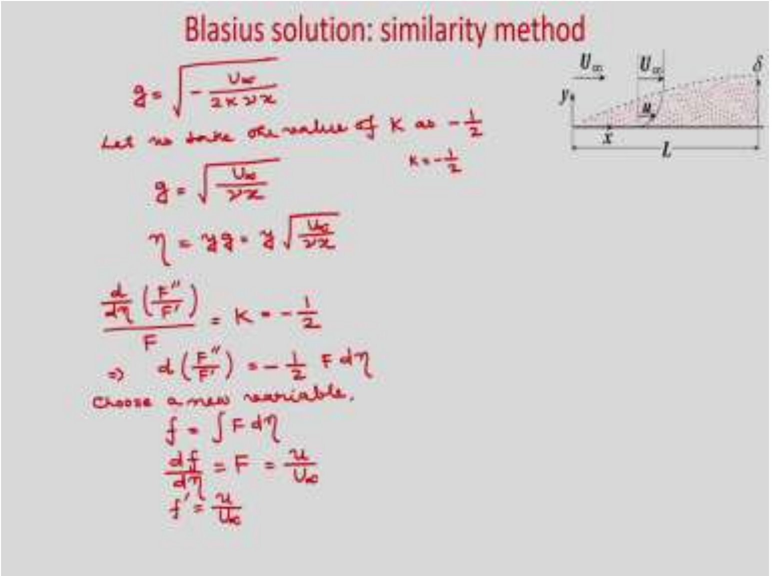
So, now you can see that, this equation now you will be able to integrate. So, integrating the above equation. So, you can see you can write $-\frac{1}{2}\frac{1}{g^2} = K\frac{\nu}{U_\infty}x + c_1$. So, this integration constant c_1 what will be the value? Now, you see physically, if you see this figure as x →0, your boundary layer thickness δ→0.

So, you can see x → 0, at the leading edge of the plate your boundary layer thickness will be tending to 0. It is not actually 0; although, we write that x = 0 ,δ = 0, but at the edge at the leading edge of the flat plate delta will not be exactly 0.

So, that is why we are writing x →0, δ →0 and δ is function of x and we have seen that, $\eta \sim \frac{y}{\delta(x)}$, because we have already seen that, the plot for the similar velocity so it will be $\frac{y}{\delta}$ and that we have written $\eta = yg(x)$, because δ is function of x and g is function of x. So, you can see as x →0, δ → 0; obviously, at x →0, g →∞; that means, $\frac{1}{g} \to 0$.

So, from here you can see that, x →0, $\frac{1}{g} \to 0$; so obviously, the integration constant c_1=0. So, now if you see from here you can write $g^2 = -\frac{U_\infty}{2K\nu x}$.

(Refer Slide Time: 30:16)



So, you can write $g = \sqrt{-\frac{U_\infty}{2K\nu x}}$. So, now you can see in this expression, K is constant right. So, for simple calculation we can take any value of K.

So, let us take the value of K as $-\frac{1}{2}$. So, that this minus will go away, because minus minus will become positive and here 2 is there and if you take half then, these 2 2 will get cancel and you will get a simple expression. So, let us take the value of K which is constant as $-\frac{1}{2}$. So, if you put K = $=-\frac{1}{2}$, then you will get $g = \sqrt{\frac{U_\infty}{\nu x}}$. In some book, you will find this $\eta = \frac{U_\infty}{2\nu x}$.

So, depending on how they have taken this constant it will vary, but in our calculation we are assuming the constant as $-\frac{1}{2}$, so that we will have the simple expression. So, g now we can write as $\sqrt{\frac{U_\infty}{\nu x}}$. So, now you can write η, $\eta = yg$. So, you can see that it will be $y\sqrt{\frac{U_\infty}{\nu x}}$. So, this is the similarity variable. Now, we got the expression of g which is a function of x as $\sqrt{\frac{U_\infty}{\nu x}}$. So, that we have found.

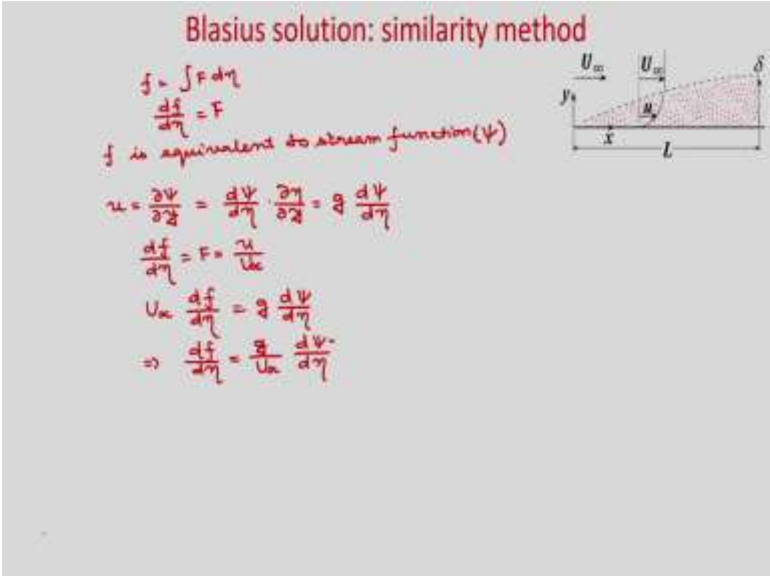
Now, let us take the other part. So, this part $\frac{\frac{d}{d\eta}(\frac{F''}{F'})}{F} = K$. And now, K we have already assumed $-\frac{1}{2}$. So, this will be equal to $-\frac{1}{2}$.

So, it will be $-\frac{1}{2}$. So, if you see it will be $d(\frac{F''}{F'}) = -\frac{1}{2}Fd\eta$. So, you can see in this equation still F is unknown; but F is function of η. So, now let us take that you can integrate it. So, if you integrate this equation, then you will get $\int Fd\eta$ and now, as F is unknown it will be difficult. So, let us assume or choose a new variable $f = \int Fd\eta$.

So, we are choosing $f = \int Fd\eta$ such that, $\frac{df}{d\eta} = F$. So, you can see that, we are choosing $f = \int Fd\eta$ as a new variable F such that, you will get $\frac{df}{d\eta} = F$. So, here you can see in this integration when we are writing it; so obviously, in here this it includes the constant so that, when we are integrating these so obviously, there will be integration constant and if you take the derivative. So, this constant will become 0; so, it will be $\frac{df}{d\eta} = F$.

So, now we can see, what is $\frac{df}{d\eta}$. So, we have written $F = \frac{u}{U_\infty}$. So, $\frac{df}{d\eta}$ which you can represent as $f' = \frac{u}{U_\infty}$. Now, what is the physical significance of f. So, we have chosen a new variable f. So, what is the physical significance of it? Let us see.

(Refer Slide Time: 35:17)



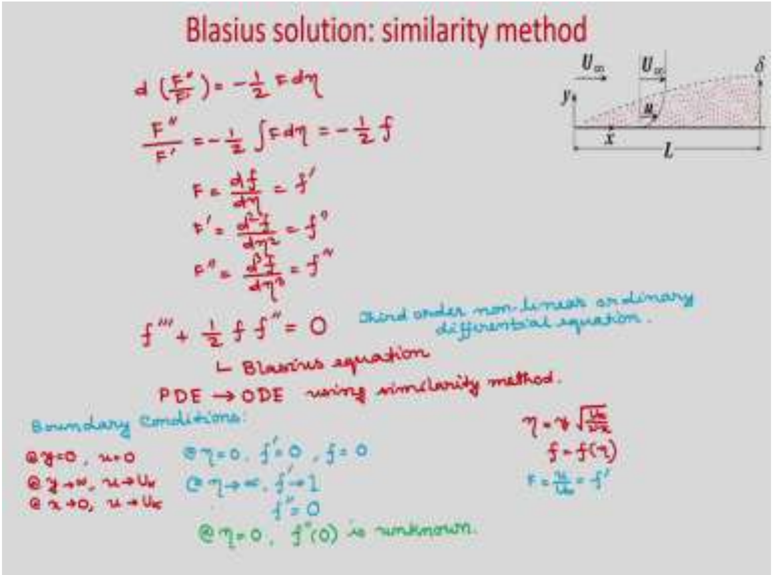
So, we have defined $f = \int F d\eta$ such that, $\frac{df}{d\eta} = F$. So, this F, whatever F we are defining this is having the physical significance. So, f is equivalent to stream function. So, whatever new variable we have defined f, it is having the physical significance because it is equivalent to stream function. So, let us see, so how do you define the velocity u? u you define if you have Ψ is the stream function then how you define u, $u = \frac{d\psi}{dy}$.

So, $u = \frac{d\psi}{dy}$. Now, you can write $\frac{d\psi}{dy}\frac{d\eta}{dy} = g\frac{d\psi}{d\eta}$. And, in earlier slide you can see we have seen that, $\frac{df}{d\eta} = F = \frac{u}{U_\infty}$.

So, you can see this u, you can write as $U_\infty \frac{df}{d\eta} = g\frac{d\psi}{d\eta}$ so that means, you can write $\frac{df}{d\eta} = \frac{g}{U_\infty}\frac{d\psi}{d\eta}$. So, you can see f is equivalent to stream function with a factor g by U_∞, but the physical significance of f is that, it is equivalent to stream function.

So, at the wall in this plate, you can assume the value of f as any constant, because it is a you can assume flat plate has a stream line and along a streamline your stream function will be constant and any value we can take. So, for simplification we will take the value of stream function or F as 0 on the flat plate. So, now, what we are getting from this expression. So, you can see now $f = \int F d\eta$.

(Refer Slide Time: 38:16)



So, you can write $d(\frac{F''}{F'}) = -\frac{1}{2} F d\eta$. So, if you integrate it. So, you will get $\frac{F''}{F'} = -\frac{1}{2}\int F d\eta$ such that, it will be $-\frac{1}{2} f$ and integration constant I have told that it is already included in the f. So, you can write $-\frac{1}{2} f$. And, from here you can see F we have defined $F = \frac{df}{d\eta}$.

So, $F' = \frac{d^2 f}{d\eta^2}$. So, it will be f'' and $F'' = \frac{d^3 f}{d\eta^3}$; that means, f'''. So, now if you put it here and rearrange what you are going to get? $f''' + \frac{1}{2} ff'' = 0$, which is known as Blasius equation.

So, now you can see we started with the partial differential equations, because continuity equation and momentum equation which are boundary layer equations. So, those equations were partial differential equations. Using the similarity method we converted this partial differential equation to ordinary differential equation, because this you can see, this is your ordinary differential equation, because they are ordinary derivative right $\frac{df}{d\eta}, \frac{d^2 f}{d\eta^2}$. So, this is ordinary derivative.

So, you have converted PDE to ODE using similarity method. Now, you see about this equation ordinary differential equation; so obviously, you can see it is a non-linear equation, because you have ff'' and it is a third order ordinary differential equation because you have f'''. So, you can see this is a third order non-linear ordinary differential equation. And, here f is function of η only that is why it is ordinary derivative.

Now, how to solve this equation. So, Blasius used a power series to solve this equation, but in this course we will not elaborate that, but you can use some numerical approach to solve this ordinary differential equation, because you know different numerical techniques right. So, you can use Rungekutta method to solve this equation with some initial and boundary conditions.

So, in this case; obviously, it is a boundary value problem, you can see from the boundary conditions which are known and which are not known. So, first let us write the boundary condition boundary conditions. So obviously, you see that at y = 0 at the wall, you have u = 0 and Ψ = constant at y →∞; that means, y →δ right.

So, you have u →U_∞, because it will approach free stream velocity. And, at x → 0, where at the leading edge of the flat plate here also your u →U ∞. Now, you convert the boundary condition in terms of f.

So, at y = 0. Now, you see what is the expression of η? So, you see η we have already written $\eta = y\sqrt{\frac{U_\infty}{\nu x}}$ and f is function of η only. Now, you can see here. So, at y = 0; obviously, η = 0. So, at η = 0.

So, u = 0; means you can see u, how we have defined $f = \frac{u}{U_\infty}$. And f is nothing but, f'.

So, from here you can see at η = 0 u = 0; that means, $f' = 0$ right. Now, if you see in terms of stream function, because we have already shown that, f is equivalent to stream function. So, you can consider the flat plate as a stream line and along a stream line stream function will be constant and that constant value we can choose 0 for convenience.

So, we can write f = 0. Now, if you see these two equations, now at $y \to \infty$ right, y_δ means ∞ also you can write means away from the wall. So, you can write at $y \to \infty$ or $x \to 0$ in both the cases $\eta \to \infty$ right, because $y \to \infty$ then, $\eta \to \infty$ and $x \to 0$ that also $\eta \to \infty$. So, what will be your value? So, you can see $u \to U_\infty$; so that means, f' will become 1.

So, $f' \to 1$. Now, you can see $f' \to 1$ in this case. And, what is the physical significance of f''. So, in the f'', so from here you can see. So, at $\eta \to \infty$ what will be the velocity gradient? So, at the edge of the boundary layer your and outside the boundary layer you have free stream velocity U_∞. So obviously, you can see the velocity gradient will be 0 and what will be the velocity gradient you can see f'' will represent the velocity gradient.

So, $f'' = 0$, ok. So, at the edge of boundary layer and outside the boundary layer you can see that, $f'' = 0$, because velocity gradient will become 0. So, now we can see from this boundary condition we have converted the boundary condition in terms of f and η.

So, you can see $\eta = 0$, your stream function is 0, we have just assumed 0 value, $f' = 0$ and at $\eta \to \infty$, $f' \to 1$ and velocity gradient. So, f'' will represent the velocity gradient.

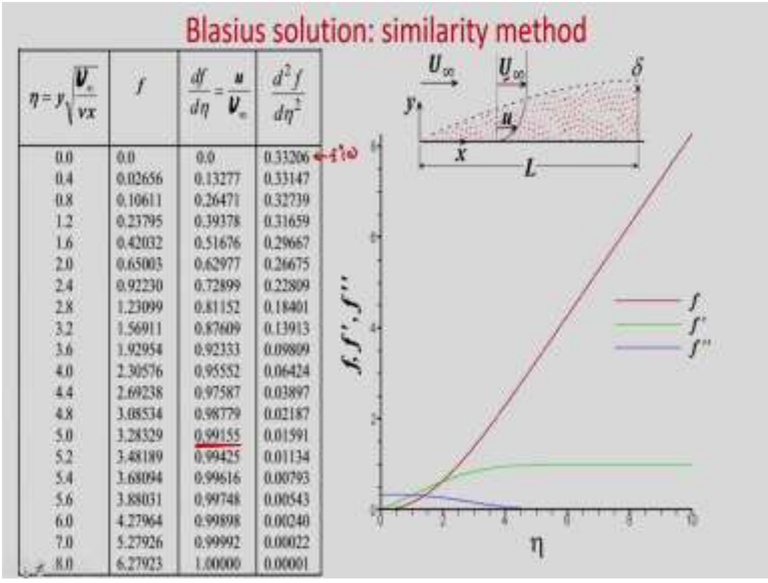
So, $f'' = 0$. Now, you can see that this ordinary differential equation you can solve using some numerical technique. So, you can refer some book, you can use Rungekutta method to solve this ordinary differential equation with this condition. So, you can see here, your the value of f'' at $\eta = 0$ is unknown. So, at $\eta = 0$, $f''(0)$ is unknown.

So, you solve such a way that, you assume the value of $f''(0)$ at $\eta = 0$, because to solve this equation you need it and satisfy that at $\eta \to \infty$, ∞ you cannot have. So, as far distance for η high value f'' will become 0, because velocity will become 0.

So, you can see f'' f values are known at $\eta = 0$, but at $\eta = 0$, f'' is unknown. So, you assume the value of $f''(0)$ and such a way that, you satisfy f'' at $\eta \to \infty$ means far away from the wall it will become 0.

So, it is known as shooting technique, because you have to assume and check whether it is satisfying that condition or not if it does not satisfy you change the value of $f''(0)$. So, in this way you solve these equations you solve these ordinary differential equation using Rungekutta method and shooting technique and satisfy f'' at $\eta \to \infty$ as 0, assuming the value of f'' at $\eta = 0$.

(Refer Slide Time: 48:23)



$\eta = y\sqrt{\frac{V_\infty}{\nu x}}$	f	$\frac{df}{d\eta} = \frac{u}{V_\infty}$	$\frac{d^2 f}{d\eta^2}$
0.0	0.0	0.0	0.33206 ← f''(0)
0.4	0.02656	0.13277	0.33147
0.8	0.10611	0.26471	0.32739
1.2	0.23795	0.39378	0.31659
1.6	0.42032	0.51676	0.29667
2.0	0.65003	0.62977	0.26675
2.4	0.92230	0.72899	0.22809
2.8	1.23099	0.81152	0.18401
3.2	1.56911	0.87609	0.13913
3.6	1.92954	0.92333	0.09809
4.0	2.30576	0.95552	0.06424
4.4	2.69238	0.97587	0.03897
4.8	3.08534	0.98779	0.02187
5.0	3.28329	0.99155	0.01591
5.2	3.48189	0.99425	0.01134
5.4	3.68094	0.99616	0.00793
5.6	3.88031	0.99748	0.00543
6.0	4.27964	0.99898	0.00240
7.0	5.27926	0.99992	0.00022
8.0	6.27923	1.00000	0.00001

If you solve this then, you will get these values. You can see that if you assume f'' value at $\eta = 0$ as 0.33206, then you can see at $\eta \to \infty$ at a higher value it is becoming 0.

So, this is the value η. So, this will be U_∞. So, you can see this is the η value. So, 0 is at wall and as η increases; that means, you are going away from wall. So, and you can see the value of f, f is equivalent to stream function; $\frac{df}{d\eta}$ is U_∞. So, this is your U_∞ and $\frac{d^2 f}{d\eta^2}$ it is the velocity gradient. So, that you are assuming this value such that, your at $\eta \to \infty$ it will become 0. So, if you plot it. So, you can see the plot here.

So, this is the plot of f. So, we have assumed f = 0. So, it is satisfying and away from wall means, η at higher value; it is increasing. f' is the representation of the velocity. So, now you can see f'. So, f' at $\eta = 0$, it is 0, but at $\eta \to \infty$; obviously, outside the boundary layer it will become constant value right, 1. So, you can see this is almost 1.

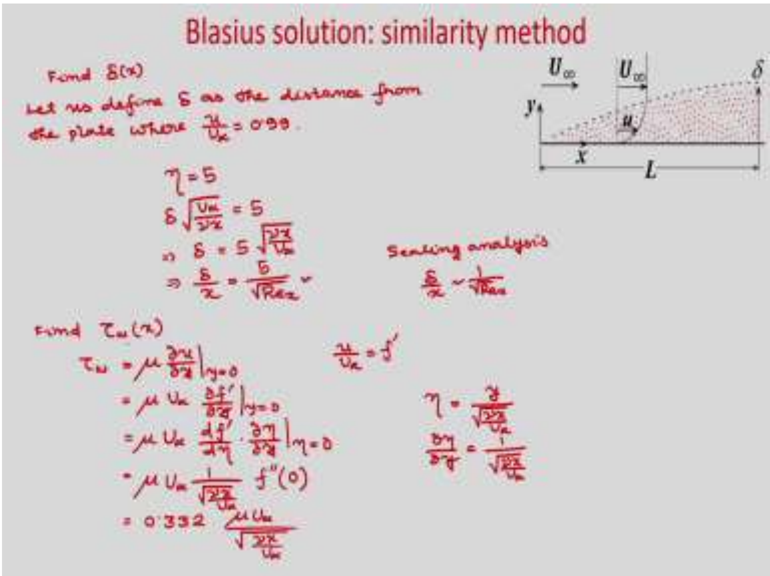
So, this is your u = U_∞ right; so that means, $\frac{u}{\infty} = 1$ and you can see around η = 5 it is becoming almost constant, because it is outside the boundary layer and it is maintaining the free stream velocity U_∞. And, f'' is the representation of velocity gradient.

So obviously, you can see velocity gradient will be higher at the wall, because wall shear stress will be there. And, as you go away from the wall your velocity gradient will be will decrease and outside the boundary layer; obviously, it will become 0 and you can see you have higher value at η = 0 as η increases your velocity gradient decreases and at the edge of the boundary layer almost, it is becoming 0.

So, you can see almost at η = 5, your velocity is becoming almost 99 % of U_∞ and that will define the hydrodynamic boundary layer thickness. So, δ will define where u becomes almost 99 % of U_∞. So, that you can see, it is becoming 1 and velocity gradient also is becoming 0.

So, you can tabulate it, you can solve this ordinary differential equation using Rungekutta method and plot this table as well as this figure, ok. Now, from the solution Blasius solution at η = 5 your velocity is becoming almost 99 % of U_∞. So, you can see from this table. So, at η = 5, you can see it is becoming almost 99 % of the U_∞.

(Refer Slide Time: 51:46)



So, let us find what is $\delta(x)$. So, let us define δ as the distance from the plate, where $\frac{u}{U_\infty} = 0.99$. So obviously, you can see. So, around η = 5, it is becoming 99 %. So, at y = δ; obviously, it is becoming this. So, η we know. So, at place of y you put $\delta \frac{U_\infty}{\nu x}$. So, this is the expression of η we are writing is equal to 5 and you can write $\delta = 5\sqrt{\frac{\nu x}{U_\infty}}$ and if you rearrange it so, you can see $\frac{\delta}{x} = \frac{5}{\sqrt{R_{e_x}}}$.

So, now you can see from the Blasius solution, we got the value of $\frac{\delta}{x} = \frac{5}{\sqrt{R_{e_x}}}$ and if you recall that when we use scaling analysis, we have shown that delta by x is order of 1 by root R e x. So, from scaling analysis $\frac{\delta}{x} \sim \frac{1}{\sqrt{R_{e_x}}}$. So, now you can see that $\frac{\delta}{x} \sim \frac{1}{\sqrt{R_{e_x}}}$ and you can see from the Blasius solution. In some book, you can see that this value 5 whatever we have taken, it can be taken as 5.2 or 4.9, because from this table you can see this is also 0.99.

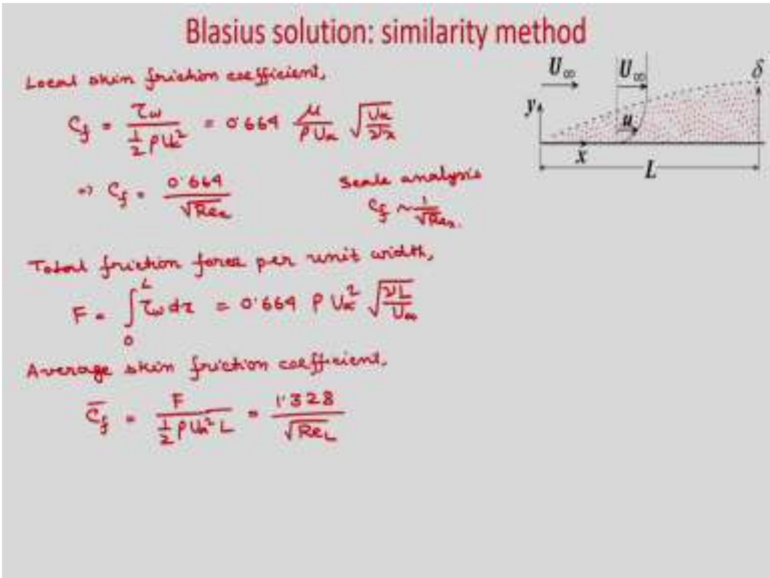
So, in different books you can see that, these value of η is taken as 5.2 or 4.9 kind of thing, but we have chosen just 5. So, $\frac{\delta}{x} = \frac{5}{\sqrt{R_{e_x}}}$. Now, let us find what is the wall shear stress? Right. So, find wall shear stress, which is function of x. So, now, $\tau_w = \mu \frac{\partial u}{\partial y}|_{y=0}$ right, at the wall y = 0 we are finding τ_w and $\frac{u}{U_\infty}$ we know that it is f'.

So, you can see now you can write μ $\tau_w = \mu U_\infty \frac{\partial f'}{\partial y}|_{y=0}$. Now, this you write in terms of η derivative right. So, you can write $\tau_w = \mu U_\infty \frac{df'}{dy}\frac{\partial \eta}{\partial y}|_{\eta=0}$; so, you can see $\eta = \frac{y}{\sqrt{\frac{\nu x}{U_\infty}}}$. So, $\frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{\frac{\nu x}{U_\infty}}}$.

So, if you put these values. So, you can see this will be $\tau_w = \mu U_\infty \frac{1}{\sqrt{\frac{\nu x}{U_\infty}}} f''(0)$. So, these value f'', what is the physical significance of f''? We have shown that it is a velocity gradient and velocity gradient now you know from the Blasius solution, what is the value at η = 0.

So, if you go back to this table, you can see this is the value of f'' at η = 0. So, if you put this value here and rearrange you will get 0.332; 0 6 I am not writing. So, it will be $\tau_w = 0.332 \frac{\mu U_\infty}{\sqrt{\frac{\nu x}{U_\infty}}}$. So, this is your wall shear stress.

(Refer Slide Time: 56:06)



Now, let us write local skin friction coefficient. So, this you can define as $C_f = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2}$.

So, if you put that you will get $0.664 \frac{\mu}{\rho U_\infty} \sqrt{\frac{U_\infty}{\nu x}}$ and if you rearrange it you will get $C_f = \frac{0.664}{\sqrt{R_{e_x}}}$.

So, if you remember that using scaling analysis also we have found $C_f \sim \frac{1}{\sqrt{R_{e_x}}}$. So, from scale analysis already we have shown that, $C_f \sim \frac{1}{\sqrt{R_{e_x}}}$. Now, let us find what is the total friction force per unit width? So, total friction force per unit width of the plate.

So, $F = \int_0^L \tau_w dx$, because this is the length of the plate. So, if you do it you will get $0.664\rho U_\infty^2 \sqrt{\frac{\nu L}{U_\infty}}$. Now, average skin friction coefficient now you can write; average skin friction coefficient $\overline{C_f} = \frac{F}{\frac{1}{2}\rho U_\infty^2 L}$.

So, that is L .1. So, $L = \frac{1.328}{\sqrt{R_{e_L}}}$. So, in today's class, we have seen that we have started with the boundary layer equation for flow over flat plate. We used similarity method, proposed by Blasius and we converted the partial differential equations to ordinary differential equation using similarity method.

So, similarity variable we have used as η, which is function of two independent variables x and y and we have used $\frac{u}{U_\infty}$ as capital F which is function of η only. And, we have converted these boundary equations to Blasius equation which is your third order non-linear ordinary differential equation.

Then, we have written down the boundary conditions and you know that, the value of F'', f'' which is the representation of velocity gradient is unknown at the wall. So, using some numerical technique like Rungekutta method and shooting technique you can assume the value of f'' at η = 0 and satisfy that f'' at η →∞ will become 0, because at the near to the edge of the boundary layer and outside the boundary layer your velocity gradient will become 0 for flow over flat plate.

Then, we have tabulated the values of f f'' with η and we have also shown the plot of f f'' and f'', where f is the representation of stream function f' is representation of the velocity and f'' is the representation of velocity gradient.

Then, we have found the hydrodynamic boundary layer thickness δ from the Blasius solution and we have shown that $\frac{\delta}{x} = \frac{5}{\sqrt{R_{e_x}}}$ and also we have found the wall shear stress and from wall shear stress we have found the local skin friction coefficient and average skin friction coefficient.

Thank you.