Fundamentals of Convective Heat Transfer
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Module – 11
Turbulent Flow and Heat Transfer
Lecture – 38
Integral solution for turbulent boundary layer flow over a flat plate

Hello everyone. So, in today's class, first we will use momentum integral equation which we derived for laminar flows and we will find the friction coefficient for turbulent flows, for flow over flat plate and then, we will find the heat transfer coefficient and Nusselt number for flow over flat plate.

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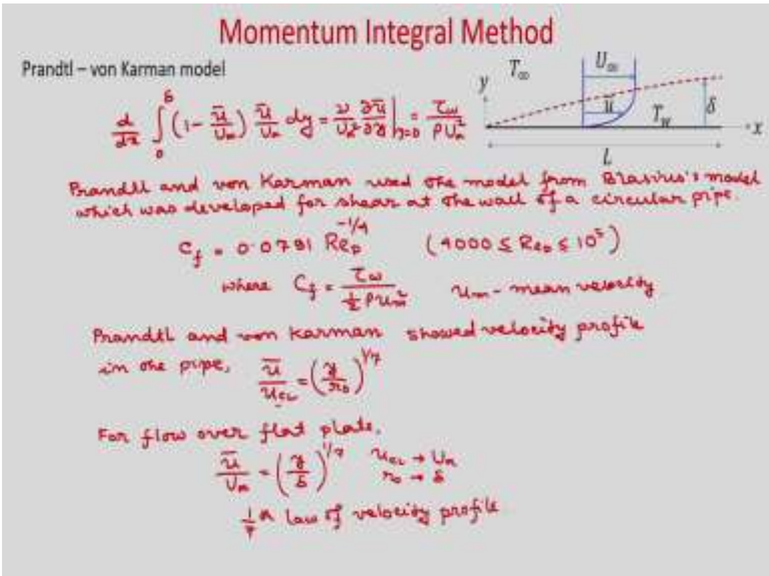

So, we can write the momentum integral equation which we derive for the laminar flows. So, this is $\frac{d}{dx}\int_0^\delta \left(1-\frac{\bar{u}}{U_\infty}\right)\frac{\bar{u}}{U_\infty}dy = \frac{\nu}{U_\infty^2}\left.\frac{\partial \bar{u}}{\partial y}\right|_{y=0} = \frac{\tau_w}{\rho U_\infty^2}$. So, this equation also can be used for time average velocities, in turbulent flows.

So, you just replace $u = \bar{u}$. So, if you replace $u = \bar{u}$ then these momentum integral equation we can use for this turbulent flows.

Now, you know that to use this integral equation, we need to find or we need to assume some velocity profile. In turbulent flows, it is very difficult to assume the velocity profile, so Prandtl and Von-Karman; what they did? They used very crude and simple method, but it gives very accurate result for flow over flat plate or for external flows.

So, they used the solution of Blasius for circular pipe case and that they used the velocity profile for the flow over flat plate. So, you can see Prandtl and Von-Karman used the model from Blasius model which was developed for the shear at the wall of a circular pipe.

So, Blasius proposed for circular pipe based on the dimensional analysis and experimental data the $C_f = 0.0781 \mathrm{Re}_D^{-1/4}$. And it is valid in the range $4000 \le \mathrm{Re}_D \le 10^5$.

So, this is for internal flows, pipe flow, where, $C_f = \dfrac{\tau_w}{\frac{1}{2}\rho u_m^2}$, where u_m is your mean velocity.

So, based on these empirical relation your Prandtl and Von-Karman developed the velocity profile inside a pipe as; so, Prandtl and Von-Karman showed the velocity profile in the pipe $\dfrac{\bar{u}}{u_{cL}} = \left(\dfrac{y}{r_0}\right)^{1/7}$. In this relation, y is measured from the pipe wall and u_{CL} is your centerline velocity, and r_0 is the radius of the pipe.

So, Prandtl and Von-Karman actually using this relation, they use the velocity profile for flow over flat plate just putting the r naught as δ, that is your boundary layer thickness. And u C L; u C L in this case ah there is no central line velocity, so u C L is substituted with the free stream velocity U_∞.

So, we will use for flow over flat plate. These velocity profile u by U ∞, so u_{cL} is actually replaced with U_∞ and $\dfrac{\bar{u}}{U_\infty} = \left(\dfrac{y}{\delta}\right)^{1/7}$, where r_0 is replaced with δ, boundary layer thickness to the power 1/ 7. So, you can see that it is a well-known one-seventh law of velocity profile; one-seventh law of velocity profile.

So, although they propose the velocity profile for the flow over flat plate like this, but it has some fundamental problem. So, if you calculate the shear stress at the wall, it will become almost ∞. So, to avoid this problem, they used the correlation for the C_f from the pipe flow relation.

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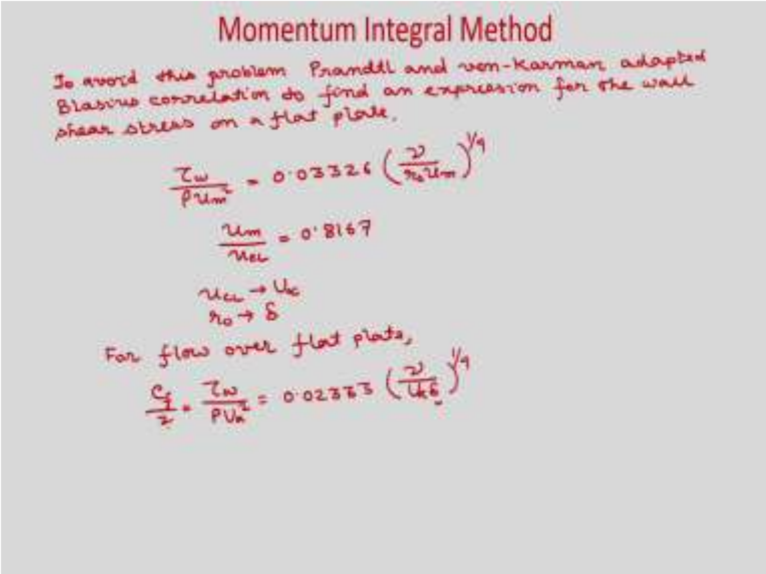

To avoid this problem Prandtl and Von-Karman adapted Blasius correlation to find an expression for the wall shear stress on a flat plate. So, they used $\frac{\tau_w}{\rho u_m^2} = 0.03326\left(\frac{\nu}{r_0 u_m}\right)^{1/4}$.

So, now you can substitute u_{CL} as U_∞ and r_0 as δ, but the mean velocity u_m. So, $\frac{u_m}{u_{cL}}$ it can be found for one-seventh law velocity profile inside a pipe as $\frac{u_m}{u_{cL}} = 0.8167$. So, these are valid for pipe flow. So, now you can use u_{CL} you can substitute with U_∞, and $r_0$you can substitute with δ.

So, now, if you substitute then you will get the shear stress relation for flow over flat plate. You can use now $\frac{C_f}{2} = \frac{\tau_w}{\rho U_\infty^2}$ after putting all these values you can rearrange and you will get as, $0.02333\left(\frac{\nu}{U_\infty \delta}\right)^{1/4}$. So, you see for flow over flat plate this $\frac{C_f}{2}$ is given in terms of the boundary layer thickness δ.

Now, you can use the momentum integral equation and we can substitute the velocity profile, one-seventh law velocity profit, and this shear stress relation, and we can find what is the boundary layer thickness.

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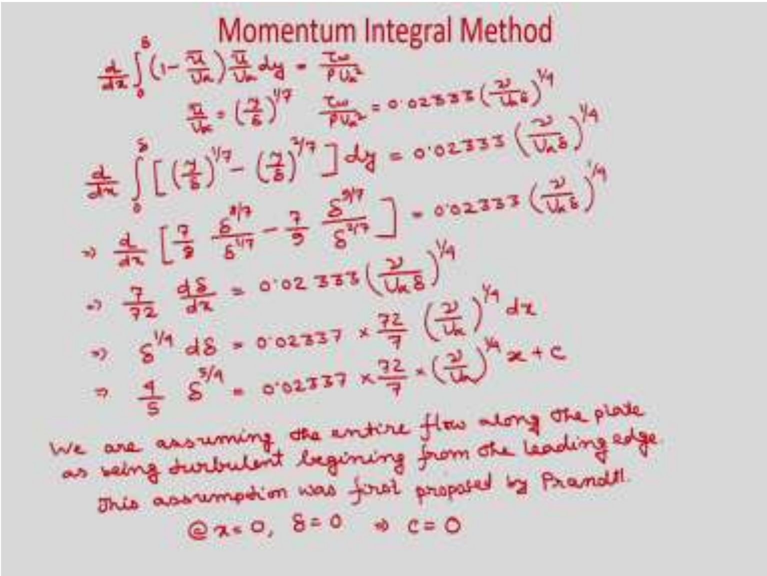

So, we have the momentum integral equations as $\frac{d}{dx}\int_0^\delta \left(1-\frac{\bar{u}}{U_\infty}\right)\frac{\bar{u}}{U_\infty}dy = \frac{\tau_w}{\rho U_\infty^2}$. So, now we have, $\frac{\bar{u}}{U_\infty}$, Prandtl and Von-Karman proposed as $\left(\frac{y}{\delta}\right)^{1/7}$ and you have, $\frac{\tau_w}{\rho U_\infty^2} = 0.02333\left(\frac{\nu}{U_\infty \delta}\right)^{1/4}$.

Now, you substitute these two in the momentum integral equation and find the value of δ. And once you know the value of δ you will be able to find, the skin friction coefficient because in the skin friction coefficient you have the unknown parameter δ.

So, if you substitute it you will get $\frac{d}{dx}\int_0^{\delta}\left[\left(\frac{y}{\delta}\right)^{1/7}-\left(\frac{y}{\delta}\right)^{2/7}\right]dy = 0.02333\left(\frac{\nu}{U_\infty \delta}\right)^{1/4}$. So, if you integrate it what you will get? $\frac{d}{dx}\left[\frac{7}{8}\frac{\delta^{8/7}}{\delta^{1/7}}-\frac{7}{9}\frac{\delta^{9/7}}{\delta^{2/7}}\right] = 0.02333\left(\frac{\nu}{U_\infty \delta}\right)^{1/4}$. So, this if you do the algebra you will get $\frac{7}{72}\frac{d\delta}{dx} = 0.02333\left(\frac{\nu}{U_\infty \delta}\right)^{1/4}$.

So, this δ you take in the left hand side, so you will get, $\delta^{1/4}d\delta = 0.02337\times\frac{72}{7}\left(\frac{\nu}{U_\infty}\right)^{1/4}dx$.

So, now if you integrate it you see ν, U_∞ are constant. So, you can integrate this. So, you will get $\frac{4}{5}\delta^{5/4} = 0.02337\times\frac{72}{7}\left(\frac{\nu}{U_\infty}\right)^{1/4}x + C$. So, we will assume here that you have the turbulent flow from the leading edge of the flat plate.

So, first Prandtl proposed this assumption, so that we can use the condition at x = 0; that means, at the leading edge of the flat plate you have boundary layer thickness δ = 0. So, these are the assumptions you have to take.

So, if you take these assumptions, then you can use the boundary condition at x = 0, δ = 0. So, we are assuming that you have a turbulent flow over this flat plate starting from the leading edge.

So, we are assuming the entire flow along the plate as being turbulent beginning from the leading edge. So, this assumption was first proposed by Prandtl. So, if you assume this then you can put at x = 0, δ = 0.

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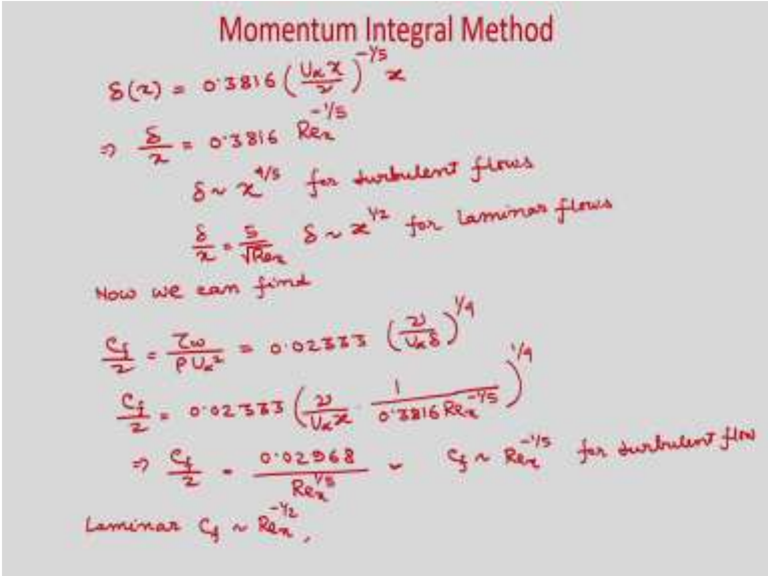

So, that means, your constant C = 0. So, if we put C = 0 and if you rearrange it you will get $\delta(x) = 0.3816\left(\frac{U_\infty x}{\nu}\right)^{-1/5} x$. So, you can write $\frac{\delta}{x} = 0.3816\,\mathrm{Re}_x^{-1/5}$.

So, here you can see that your boundary layer thickness varies $\delta \sim x^{4/5}$. So, your, in this case you can see that your boundary layer thickness $\delta \sim x^{4/5}$ for turbulent flows.

For laminar flows, do you remember what was the δ? So, it was $\frac{5}{\sqrt{\mathrm{Re}_x}}$, so that means, $\frac{\delta}{x} = \frac{5}{\sqrt{\mathrm{Re}_x}}$. So, that means, $\delta \sim x^{1/2}$, for laminar flows.

And now, if you find the C_f; $\frac{C_f}{2} = \frac{\tau_w}{\rho U_\infty^2} = 0.02333\left(\frac{\nu}{U_\infty \delta}\right)^{1/4}$. So, you substitute here this δ value. So, if you substitute it you will get $\frac{C_f}{2} = 0.02333\left(\frac{\nu}{U_\infty x}\frac{1}{0.3816\,\mathrm{Re}_x^{-1/5}}\right)^{1/4}$.

So, this you can write as, $\frac{C_f}{2} = \frac{0.02968}{\mathrm{Re}_x^{1/5}}$.

And for laminar flow you know $C_f \sim \mathrm{Re}_x^{-1/2}$, and in turbulent flows you can see $C_f \sim \mathrm{Re}_x^{-1/5}$, for turbulent flows.

So, now whatever expression we have derived for this $\frac{C_f}{2}$ that will use to find the heat transfer coefficient.

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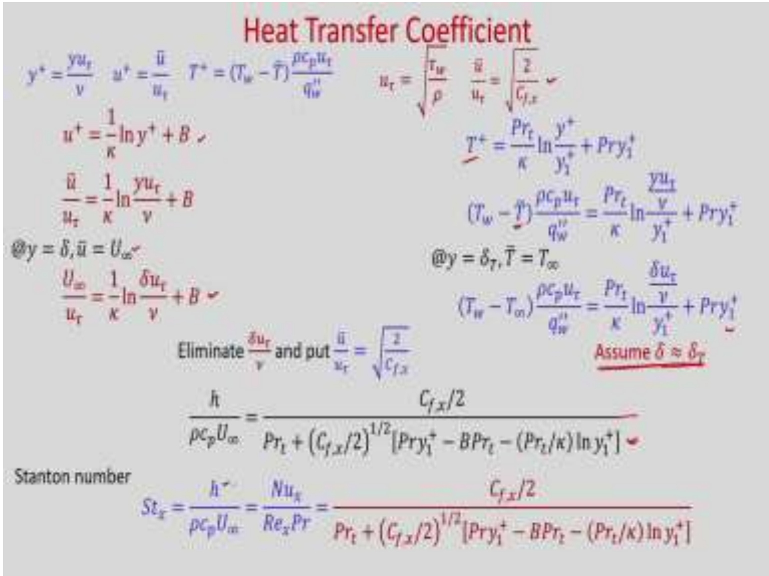

So, you can see we have already derived these for fully turbulent boundary layer $u^+ = \frac{1}{\kappa}\ln y^+ + B$. And, $T^+ = \frac{\Pr_t}{\kappa}\ln\frac{y^+}{y_1^+} + \Pr y_1^+$. Now, $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$. So, now, if you define, $\frac{\bar{u}}{u_\tau} = \sqrt{\frac{2}{C_{f,x}}}$.

So, here you can see $u^+ = \frac{\bar{u}}{u_\tau}$. So, $y^+ = \frac{yu_\tau}{\nu}$, plus the constant B. So, at y = δ, at the edge of the boundary layer you have free stream velocity U_∞.

So, $\frac{\bar{u}}{u_\tau}$ now this $\bar{u} = U_\infty$. So, $\frac{U_\infty}{u_\tau} = \frac{1}{\kappa}\ln\frac{\delta u_\tau}{\nu} + B$. So, this is we have derived.

Now, if you see here, here we will assume that $\delta \approx \delta_T$, so that our derivation will be simplified. So, with that we know that at the edge of the boundary layer we have $\overline{T} = T_\infty$.

So, here if you put $\overline{T} = T_\infty$,so this you can write, $\left(T_w - T_\infty\right)\dfrac{\rho C_p u_\tau}{q_w^{"}} = \dfrac{\Pr_t}{\kappa}\ln\dfrac{\delta u_\tau}{\nu} + \Pr y_1^+$ at y = δ.

So, now these two relations we have. Now, what you do you see in both the equations you have $\dfrac{\delta u_\tau}{\nu}$. Now, eliminate $\dfrac{\delta u_\tau}{\nu}$ and put $\dfrac{\overline{u}}{u_\tau} = \sqrt{\dfrac{2}{C_{f,x}}}$. So, if you do that then you will get, $\dfrac{h}{\rho c_p U_\infty} = \dfrac{C_{f,x}/2}{\Pr_t + \left(C_{f,x}/2\right)^{1/2}\left[\Pr y_1^+ - B\Pr_t - \left(\Pr_t/\kappa\right)\ln y_1^+\right]}$.

So, now, we will define Stanton number. Already we have discussed about this. So, $St_x = \dfrac{Nu_x}{\mathrm{Re}_x \Pr}$. So, $St_x = \dfrac{h}{\rho c_p U_\infty}$. So, this if you put, then now we will be able to find what is the heat transfer coefficient and from there we will be able to find what is the Nusselt number.

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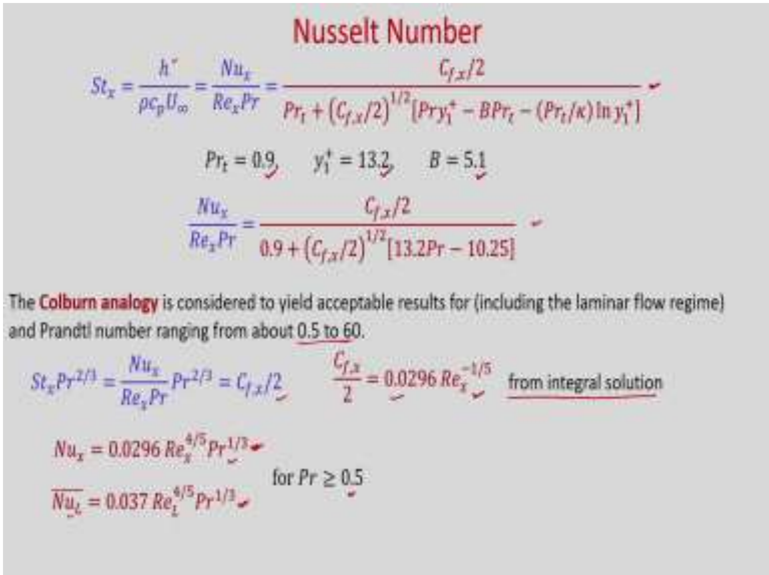

So, now we have just expressed Stanton number with this. Now, we will assume turbulent Pr= $_t$ 0.9, y_1^+ = 13.2 and the B = 5.1. So, these are from empirical values, from the experiments these values are found.

Now, if you put all these here then you will get, $\frac{Nu_x}{\text{Re}_x \text{Pr}} = \frac{C_{f,x}/2}{0.9 + \left(C_{f,x}/2\right)^{1/2}\left[13.2\,\text{Pr} - 10.25\right]}$. So, you can see from these expression now you will be able to find what is the Nusselt number or also heat transfer coefficient.

Now, this we have derived using the two layers model because we have used for fully turbulent flows what is the u^+ and T^+ expression and from there we have derived the expression for Nusselt number here.

You can also use Colburn analogy. So, that already we have discussed. So, the Colburn analogy is considered to yield acceptable results for including the laminar flow regime and Prandtl number ranging from about 0.5 to 60.

So, this ah Colburn analogy if you use, so you know that, $St_x \text{Pr}^{2/3} = C_{f,x}/2$. And, $St_x = \frac{Nu_x}{\text{Re}_x \text{Pr}}$.

And from the integral solution, just in this class we have derived, $\frac{C_{f,x}}{2} = 0.0296\,\text{Re}_x^{-1/5}$.

So, you can see in the Colburn analogy we will use this integra $\frac{C_{f,x}}{2}$ l solution of $\frac{C_{f,x}}{2}$, and if you put it here and if you find what is the Nusselt number you will get Nu_x in terms of Reynolds number and Prandtl number. So, you can see $Nu_x = 0.0296\,\text{Re}_x^{4/5}\,\text{Pr}^{1/3}$.

So, this is from Colburn analogy, just using the integral solution from, integral solution of $\frac{C_{f,x}}{2}$ you can find the Nusselt number. This is your local Nusselt number and it is valid for Pr $\geq$ 0.5.

And if you find the average Nusselt number just integrating from 0 to L, you will get average $\overline{Nu_L} = 0.037\,\mathrm{Re}_x^{4/5}\,\mathrm{Pr}^{1/3}$. So, this actually gives reasonable good results. This is simple simplified expression, but you can also use these expression as well.

So, now, if you have laminar region in the beginning then you have turbulent region. And you know that critical Re_{x_c} = 10^5, so in this region if you find the what is the average Nusselt number, considering both laminar and turbulent regime that now we will find.

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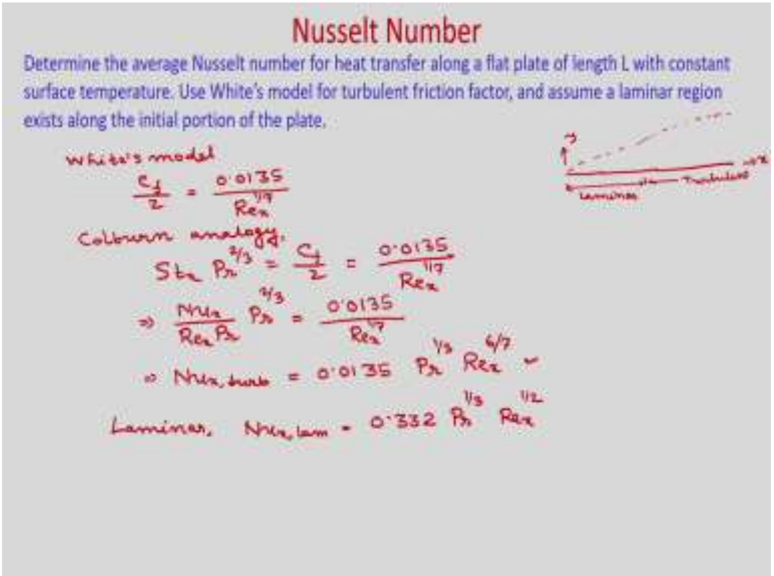

So, determine the average Nusselt number for heat transfer along a flat plate of length L with constant surface temperature. Use White's model for turbulent friction factor, and assume a laminar region exist along the initial portion of the plate.

So, if you consider a flat plate. So, we are considering that initial region you have laminar flow, and then you have turbulent. So, if you see the, so this is your boundary layer thickness and this is your y, this is your x.

Now, we will use White's model. So, if you see what is White's model, White's model we have found, $\frac{C_f}{2} = \frac{0.0135}{\mathrm{Re}_x^{1/7}}$. And from Colburn analogy, Colburn analogy and you can find the Nusselt number in turbulent flow regime. So, $St_x\,\mathrm{Pr}^{2/3} = \frac{C_f}{2} = \frac{0.0135}{\mathrm{Re}_x^{1/7}}$.

So, $St_x = \dfrac{Nu_x}{\text{Re}_x \Pr}$ and thus, $\dfrac{Nu_x}{\text{Re}_x \Pr}\Pr^{2/3} = \dfrac{0.0135}{\text{Re}_x^{1/7}}$. So, Nusselt number you can find. For turbulent flows as, $Nu_{x,turb} = 0.0135 \Pr^{1/3} \text{Re}_x^{6/7}$.

So, using White's model, we found the Nusselt number in the turbulent regime as this. And you know for laminar region, we have already derived this Nusselt number, $Nu_{x,\text{lam}} = 0.332 \Pr^{1/3} \text{Re}_x^{1/2}$. So, this we will use and we will find the average heat transfer coefficient for both laminar and turbulent regime.

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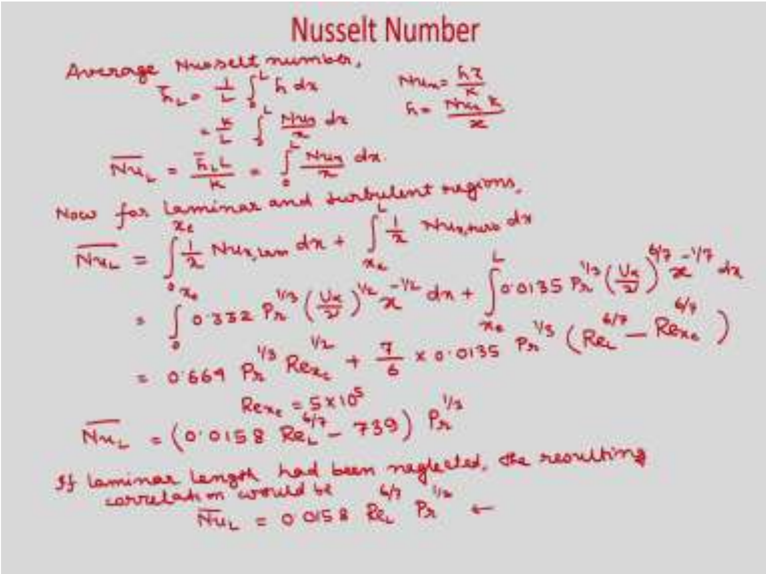

So, how to calculate average Nusselt number? Average Nusselt number we calculate as, $\bar{h} = \dfrac{1}{L}\int_0^L h dx$. And now, $Nu_x = \dfrac{hx}{K}$.

So, you can find $h = \dfrac{Nu_x K}{x}$. You substitute it here. So, you will get $\bar{h} = \dfrac{K}{L}\int_0^L \dfrac{Nu_x}{x}dx$.

Now, if you find the Nusselt number, average Nusselt number, if you find the average, $\overline{Nu_L} = \dfrac{\overline{h_L} L}{K}$. So, you can simply see this will be, $\int_0^L \dfrac{Nu_x}{x}dx$.

Now, considering the laminar and turbulent region, for laminar and turbulent region, so, $\overline{Nu}_L = \int_0^{x_c} \frac{1}{x} Nu_{x,lam} dx + \int_{x_c}^{L} \frac{1}{x} Nu_{x,turb} dx$. So, we have already found the expression that you substitute it here.

So, you will get, $\overline{Nu}_L = \int_0^{x_c} 0.332 \Pr^{1/3} \left(\frac{U_\infty}{\nu} \right)^{1/2} x^{-1/2} dx + \int_{x_c}^{L} 0.0135 \Pr^{1/3} \left(\frac{U_\infty}{\nu} \right)^{6/7} x^{-1/7} dx$.

So, if you perform the integration you will get, $\overline{Nu}_L = 0.664 \Pr^{1/3} \mathrm{Re}_{x_c}^{1/2} + \frac{7}{6} \times 0.0135 \Pr^{1/3} \left(\mathrm{Re}_L^{6/7} - \mathrm{Re}_{x_c}^{6/7} \right)$.

If you consider, $\mathrm{Re}_{x_c} = 5 \times 10^5$ and substitute it here and you will get, $\overline{Nu}_L = \left(0.0158 \mathrm{Re}_L^{6/7} - 739 \right) \Pr^{1/3}$.

And if you neglect the laminar length, if laminar length had been neglected the resulting correlation would be, $\overline{Nu}_L = 0.0158 \mathrm{Re}_L^{6/7} \Pr^{1/3}$. So, it is for fully turbulent flow considering the turbulent flow from the beginning of the flat plate.

So, in today's class we started with the momentum integral equation which we derived for laminar flows and we substituted u as time average velocity $\bar{u}$. From there with the correlation of pipe flow, we could find the velocity distribution $\frac{\bar{u}}{U_\infty} = \left(\frac{y}{\delta} \right)^{1/7}$. So, this is known as one-seventh velocity profile.

And also, we saw that the problem is finding the tau w, because if you use these velocity profile as well y = 0 you will get infinite shear stress. So, to avoid that again from Blasius relation of C_f, we use the turbulent friction factor for the flow over flat plate.

Now, that we expressed in terms of the unknown parameter boundary layer thickness δ, now we substituted this velocity profile as well as the turbulent friction factor in the momentum integral equation and we found the value of turbulent boundary layer thickness δ. And once you know the δ, so you could find the value of $C_{f,\,x}$.

Then, using the relation for fully turbulent layer this u^+ and T^+ values, we found using the Colburn analogy the expression for Nusselt number. And also, for using this integral solution whatever we found the value of $C_{f, x}$ that we used and use the Colburn analogy and we found the simplified expression for the Nusselt number, local Nusselt number as well as the average Nusselt number.

Thank you.