Fundamentals of Convective Heat Transfer

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Module - 02
Preliminary Concepts
Lecture - 03
Derivation of energy equation

Hello everyone. So, you know that there are three conservation laws; conservation of mass, conservation of momentum and conservation of energy, all these three conservation laws must be satisfied at a point in a moving fluid. So, today we will derive the equation for conservation of energy which is known as energy, equation starting from the Reynolds Transport Theorem.

Already you have learnt Reynolds transport theorem in fluid mechanics course. So, we will use Reynolds transport theorem and we will conserve the energy and we will derive the energy equation.

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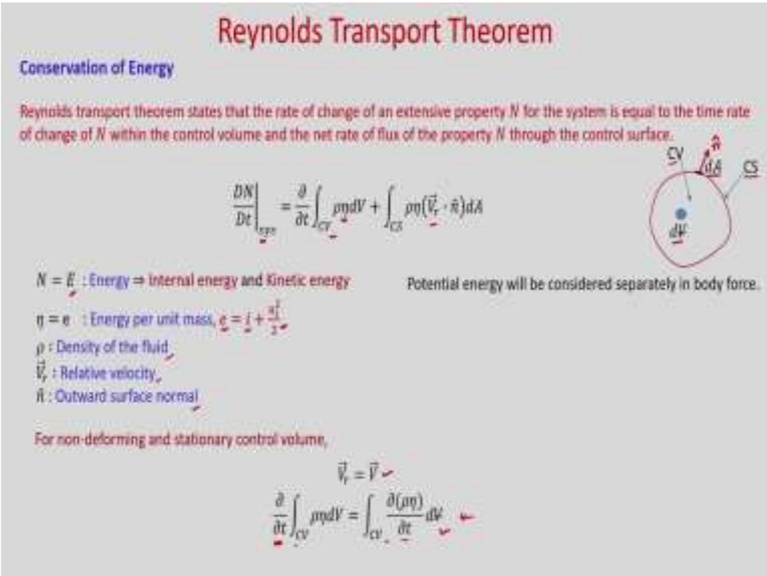

So, what is Reynolds transport theorem? Reynolds transport theorem states that the rate of change of an extensive property N for the system is equal to the time rate of change of

N within the control volume and the net rate of flux of the property N through the control surface. So, you can see we can write $\frac{DN}{Dt}\Big|_{sys} = \frac{\partial}{\partial t}\int_{CV} \rho\eta dV + \int_{CV} \rho\eta(\overrightarrow{V_r}.\hat{n})dA$

So, you consider any arbitrary control volume, where this is the control surface and if you consider one elemental volume that we are denoting with dV and elemental surface is dA. So, you can see that this N is your extensive property, here we will consider this N= E which is your energy.

In this case we will consider total energy as summation of internal energy and kinetic energy, we will not consider the potential energy here we will consider this potential energy in the source term separately and all other energy we have neglected.

And this η= e which is your energy per unit mass. So, you can write energy per unit mass =internal energy + kinetic energy . So, now, you can see in this case N = E which is your total energy summation of internal energy plus kinetic energy and η is your energy per unit mass which is $e = i + \frac{u_i^2}{2}$ and; obviously, rho is the density of the fluid and V_r is relative velocity and n is outward surface normal. So, n if you consider here so, always it is outward normal.

So, now, we will assume stationary and non-deforming control volume, in that scenario you can write the relative velocity as V. So, $\overrightarrow{V_r} = \overrightarrow{V}$ and as your volume is not changing. So, this time derivative you can take inside this integral of this control volume as you are assuming non - deforming and stationary control volume.

So, $\frac{\partial}{\partial t}\int_{CV} \rho\eta dV = \int_{CV} \frac{\partial(\rho\eta)}{\partial t} dV$. So, in this particular case it will be just $\frac{\partial(\rho e)}{\partial t} dV$. So, this is the term we have we can write like this.

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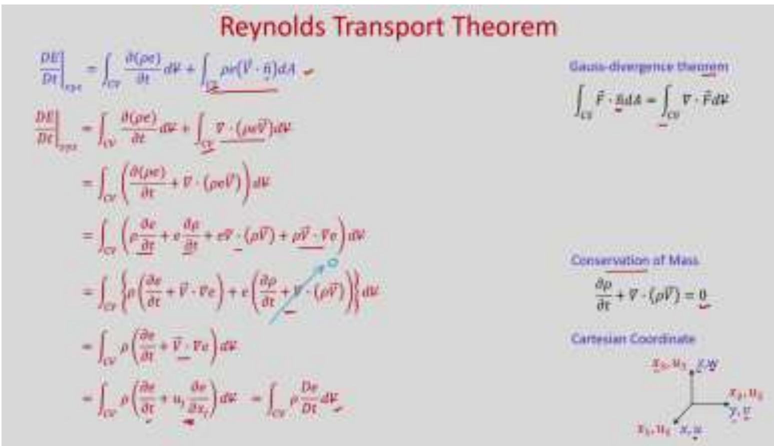

So, now if you put the total energy and energy per unit mass in the Reynolds transport equation then you will get this equation. So, $\frac{DE}{Dt}\Big|_{sys} = \int_{CV} \frac{\partial(\rho e)}{\partial t} dV + \int_{CS} \rho e(\vec{V}.\hat{n})dA$, where n is the outward surface normal.

Now, what we will do? We will change this surface integral to volume integral using Gauss divergence theorem. So, what is Gauss divergence theorem? You can see if you have a $\int_{CS} \vec{F}.\hat{n}\, dA = \int_{CV} \nabla \vec{F} dV$. So, n is your surface normal you need surface normal.

So, you can write divergence form when you write in the integral inside the integral. So, you can see now if you put it here. So, these term if you use Gauss divergence theorem you can write $\nabla \rho e\vec{V} dV$. So, using Gauss divergence theorem this surface integral we have converted to volume integral.

So, now we will do some numerical algebra. So, you can see just you simplify it. So, you can see you can write, $\frac{\partial(\rho e)}{\partial t} + \nabla \rho e\vec{V}$. So, this derivative you just write as $\rho\frac{\partial e}{\partial t} + e\frac{\partial \rho}{\partial t}$ and this divergence you write $e\nabla(\rho\vec{V}) + \rho\vec{V}.\nabla e$. So, after writing this if you rearrange then this term and this term you write together. So, what you can write, $\rho(\frac{\partial e}{\partial t} + \vec{V}.\nabla e)$ and this term now you write together. So, $e(\frac{\partial \rho}{\partial t} + \nabla(\rho\vec{V}))$.

So, now, what it is you can see if you use this Reynolds transport equation and conserve the mass then you will get the continuity equation and this is your $\frac{\partial \rho}{\partial t} + \nabla(\rho\vec{V}) = 0$. So, this is the continuity equation in general, it is applicable for compressible flow as well as incompressible flow. So, if you write this continuity equation. So, it is 0. So, this term will become 0.

So, you can write this as $\rho(\frac{\partial e}{\partial t} + \vec{V}.\nabla e)dV$ and in tensorial form if you write. So, $\vec{V}.\nabla e$ you can write $u_j\frac{\partial e}{\partial x_j}$. So, this term in tensorial form we have written like this or this term together you can write as $\int_{CV} \frac{De}{Dt} dV$. So, $\frac{De}{Dt}$ is material derivative.

So, which contains the temporal term as well as the convection term and in our study we are now considering only the Cartesian coordinate and you can see this is your x direction where velocity is u and y direction velocity v and z direction velocity w and when we will write the in tensorial form. So, you can write that u_1 x_1 is equivalent to u x, similarly y is equivalent to x_2 and v is equivalent to u_2 and similarly u_3 is equivalent to w and x_3 is equivalent to z.

So, now right hand side we have written in this form where left hand side still we need to determine. So, whatever the energy acting on the system that we need to consider.

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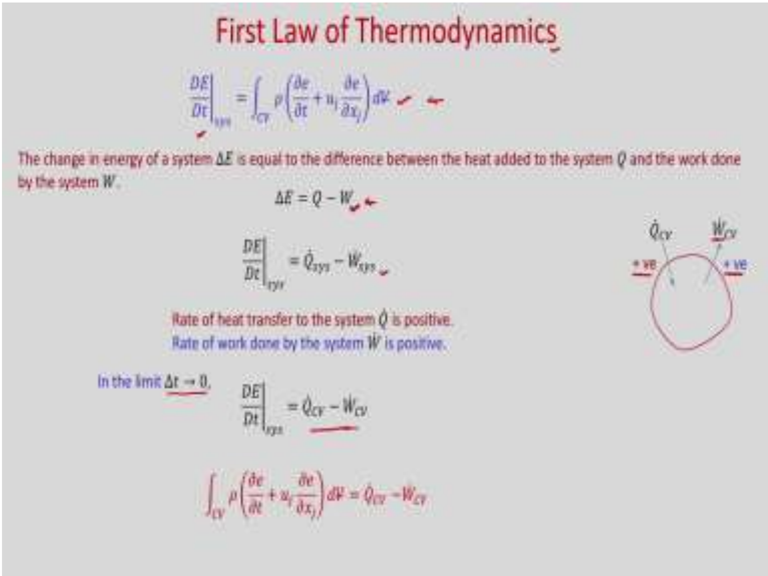

Now we will use first law of thermodynamics, the change in energy of a system ΔE is equal to the difference between the heat added to the system Q and the work done by the system W. So, from the first law of thermodynamics you can write that $\Delta E = Q - W$. So, that you have already studied in thermodynamics. So, now, if you write in a rate of change sense then you can write this equation as $\frac{DE}{Dt}\Big|_{sys} = \dot{Q}_{sys} - \dot{W}_{sys}$. So, now let us discuss about the sign convention. So, in heat transfer now we will consider that rate of heat transfer to the system is positive and rate of work done by the system is positive. So, you can see this is the control volume, in the control volume if your rate of heat transfer is to the system then we will consider as positive and rate of work done is by the system then we will consider as positive.

So, with this sense just will proceed the derivation and in the limit Δt →0, this system and volume will coincide. So, whatever we have written that $\frac{DE}{Dt}|_{sys} = \dot{Q}_{sys} - \dot{W}_{sys} = \dot{Q}_{CV} - \dot{W}_{CV}$. So, now you can see finally, if you put this $\frac{DE}{Dt}|_{sys}$ this expression in this equation then you can write $\int_{CV} \rho(\frac{\partial e}{\partial t} + u_j \frac{\partial e}{\partial x_j}) dV = \dot{Q}_{CV} - \dot{W}_{CV}$. Now we need to derive the expression for this rate of heat transfer and rate of work done.

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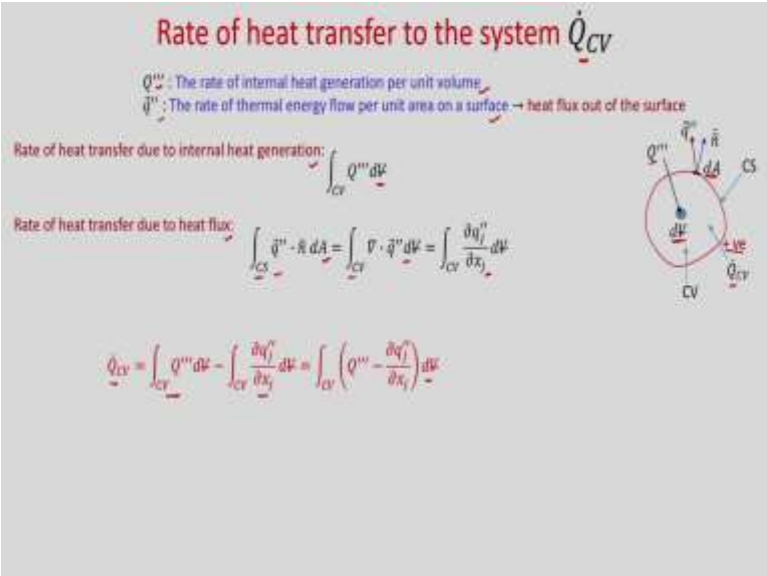

So, first let us consider rate of heat transfer to the system and we will derive the expression for $\dot{Q}_{CV}$. So, you can see there are two types of heating; one is volumetric heating and another is surface heating. So, you can see that if there is a volumetric heat generation then it is a volumetric phenomena and that you consider the rate of internal heat generation per unit volume.

So, if you consider this as a control volume and this is the control surface, in the control volume if you consider one elemental volume dV then this rate of internal heat generation is taking place in this elemental volume dV.

So, if $Q'''dV$ then you will get the rate of heat transfer inside this elemental volume and if you integrate over the whole volume then you will get the total rate of heat transfer due to heat generation. So, you can see. So, rate of heat transfer due to internal heat generation we can write $\int_{CV} Q''' dV$.

Now, there will be surface heating and surface heating will take place due to heat conduction. So, you can see if you consider one elemental area dA on the surface. So, dA and this is your normal unit normal n and your heat conduction is taking place in outward direction. So, it is heat flux $\vec{q''}$.

So, this q'' is the rate of thermal energy flow per unit area on a surface and it is heat flux out of the surface. Now you see this heat flux is the surface phenomena and it is going out of the surface, but we have considered that heat transfer to the system is positive so, but here your heat flux is going out of the surface. So, there will be a negative sign.

So, if you see rate of heat transfer due to heat flux. So, this q'' is acting on this surface. So, in the normal direction if you take then $\vec{q''}.\hat{n} dA$. So, it is acting on this surface dA, now if we integrate over the whole surface then you will get $\int_{CS} \vec{q''}.\hat{n} dA$.

Now you use Gauss divergence theorem to convert this surface integral to volume integral. So, you can write this $\int_{CV} \nabla.\vec{q''} dV$ and this if you write in tensorial form then you can write this ∇Q you can write $\frac{\partial q_j''}{\partial x_j}$, but this is negative, because Q dot cv we have taken if it is heat transfer is taking place to the system then it is positive, but it is going out.

So, when we will calculate $\dot{Q}_{CV}$ so, we will write this rate of heat transfer due to internal heat generation and minus because it is going out of the surface. So, this is the minus rate of heat transfer due to heat flux. So, together you can write $\int_{CV} (Q''' - \frac{\partial q_j''}{\partial x_j}) dV$.

So, now, let us consider rate of work done. So, rate of work done is $\dot{W}_{CV}$ and there are two types of forces acting on this fluid element. So, what are the forces, one is your body force which is your volumetric phenomena and you have surface force ok. So, that is your surface phenomena.

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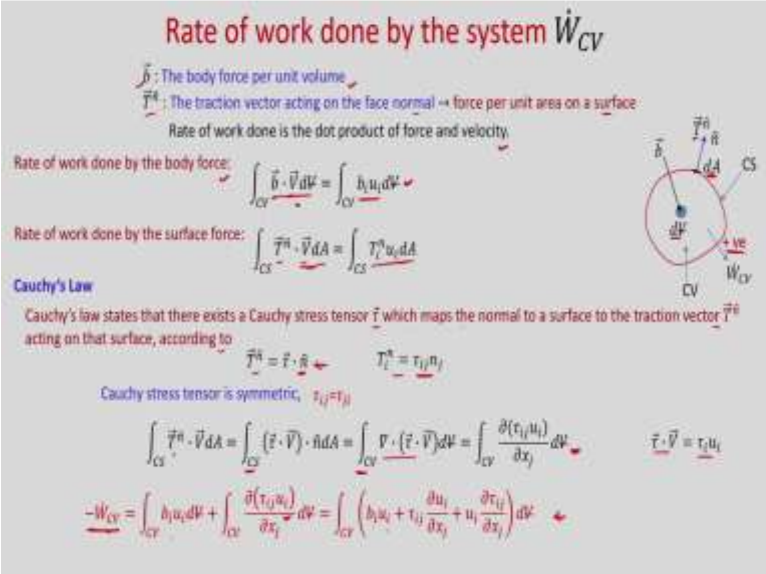

So, you can see the body force per unit volume we are writing as V and the surface force what is acting on this elemental surface dA that we will denote with the traction vector. So, this is denoted as $\vec{T}^{\hat{n}}$ the traction vector acting on the phase normal and it is force per unit area on a surface.

So, these are the forces acting on the volume as well as in the surface we need to calculate the work done right. So, work done is the dot product of this force and the velocity. So, rate of work done is the dot product of force and velocity. So, you can see rate of work done by the body force you can write.

So, V is the body force and you take the dot product of the velocity and if you consider this elemental volume dV. So, the rate of work done by the body force in this elemental volume is $\vec{b}.\vec{V}dV$ and if you integrate over the control volume then you can get the total rate of work done by the body force.

And this you can write in tensorial form as b_i u_i and rate of work done by the surface force. So, this is your normal direction to the elemental surface dA and this is the traction vector acting on this normal. So, you can see that rate of work done by the surface force you can write $\int_{CS} \vec{T}^{\hat{n}}.\vec{V} dA$ and this in tensorial form also you can write $\int_{CS} T_i^n u_i dA$.

So, now this traction vector we will relate with the stress tensor and we will use Cauchy's law. So, Cauchy's law states that there exists a Cauchy stress tensor tau which maps the normal to the surface to the $\vec{T}^{\hat{n}}$ acting on the surface according to this $\vec{T}^{\hat{n}} = \vec{\tau}.\hat{n}$,

n is the unit normal outward of the surface and tensorial form you can write $T_i^n = \tau_{ij} n_j$ and Cauchy stress tensor is symmetric. So, you can write $\tau_{ij} = \tau_{ji}$ and now this integral now you consider. So, this $\vec{T}^{\hat{n}}$ you substitute this expression.

Then you can write $(\vec{\tau}.\overrightarrow{V}).\hat{n} dA$, now this surface integral you convert to the volume integral using Gauss divergence theorem and you write $\nabla.(\vec{\tau}.\vec{V})dV$ and in tensorial form we can write $\nabla.(\vec{\tau}.\vec{V})$ as $\frac{\partial(\tau_{ij} u_i)}{\partial x_j}$ because this $\vec{\tau}.\vec{V}$ you can write $\tau_i u_i$ and this a there is a divergence. So, that if you write then you can write $\frac{\partial(\tau_{ij} u_i)}{\partial x_j}$. So, this is the volume integral we have converted you have a rate of work done by the body force and you have rate of work done by the surface force, if you add together then you will get the work done by the force.

So, you can see in thermodynamics we have calculated this work done by the body force and work done by the surface force, but when we considered the sign convention in today's class that work done by the system is positive, but here whatever we have derived these are work done by the force so; that means, there is a change in sign convention.

So, work done by the system we have considered as positive, but here we have calculated work done by the force so; obviously, work done by the system if we consider then there will be a negative sign. So, we have written $-\dot{W}_{CV}$ because this is the sign convention we have taken, because $\dot{W}_{CV}$ work done by the system is positive, but whatever we have considered those are rate of work done by the force.

So, now, if you see. So, if there is a negative sign. So, actually you should write $\dot{W}_{CV}$ is equal to negative of this, but as simplification we are writing $-\dot{W}_{CV} = \int_{CV} b_i u_i dV + \int_{CV} \frac{\partial(\tau_{ij} u_i)}{\partial x_j} dV = \int_{CV} (b_i u_i + \tau_{ij} \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial \tau_{ij}}{\partial x_j}) dV$. So, this is your $\dot{W}_{CV}$.

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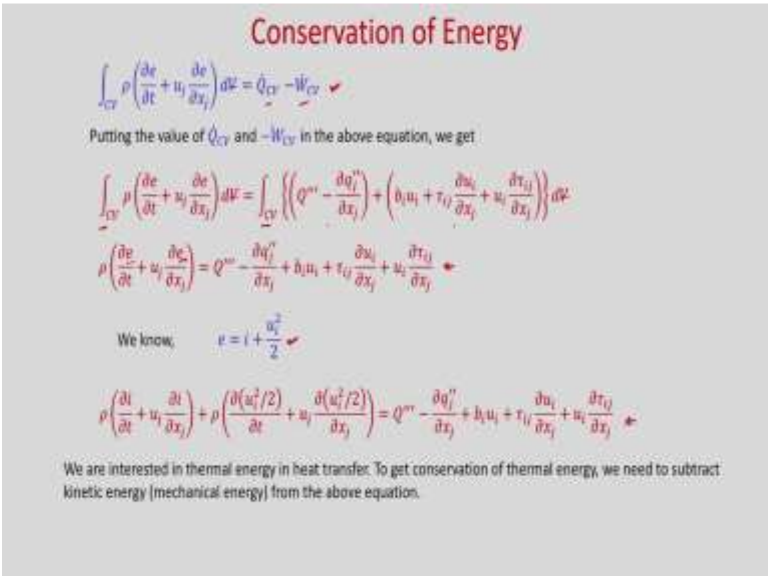

So, you can see we had this expression, now you substitute this $\dot{Q}_{CV}$ expression in $-\dot{W}_{CV}$ expression here then you can get. So, left hand side will be as it is in the right hand side this is the $\dot{Q}_{CV}$ and this is the $-\dot{W}_{CV}$.

So, we can see both side you have volume integral. So, this is also volume integral this is also volume integral. So, for any arbitrary control volume so, you can write as

$\rho(\frac{\partial e}{\partial t}+u_j\frac{\partial e}{\partial x_j})=Q''' -\frac{\partial q_j''}{\partial x_j}+b_i u_i+\tau_{ij}\frac{\partial u_i}{\partial x_j}+u_i\frac{\partial \tau_{ij}}{\partial x_j}$. Let us substitute the total energy at summation of internal energy plus kinetic energy ok. So, that is $e=i+\frac{u_i^2}{2}$.

So, if you put it in this expression. So, what you will get, you can see $\rho(\frac{\partial i}{\partial t}+u_j\frac{\partial i}{\partial x_j})+\rho(\frac{\partial(u_i^2/2)}{\partial t}+u_j\frac{\partial(u_i^2/2)}{\partial x_j})$ and write tensor terms will be as it is. Now you see in heat transfer we are interested in your internal energy, but we are having in this expression internal energy plus mechanical energy; that means, your kinetic energy.

So, now we have to subtract somehow this kinetic energy. So, that we can get the equation for conservation of thermal energy; that means, only internal energy will be present. So, to do that, now we will consider the Navier's equation.

So, from the Navier's equation if you multiply with the velocity u $_i$ then you can write the equation for kinetic energy and once we get the equation for kinetic energy and if you subtract that equation from this equation then you will get the equation for conservation of thermal energy.

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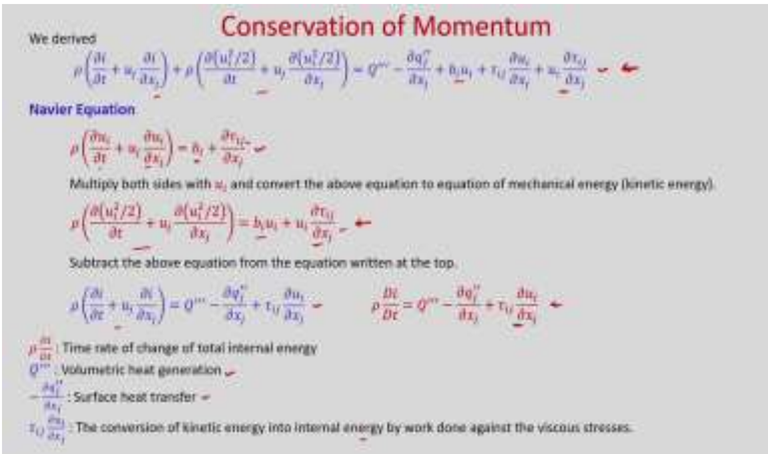

So, this is the equation we have derived now Navier's equation, you can see in general we can write like this $\rho(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j})=b_i+\frac{\partial \tau_{ij}}{\partial x_j}$, where b_i is the body force term and τ_{ij} is your stress tensor and this is temporal term and this is your convective term.

So, we have considered Navier equation. So, it is valid in general for compressible and incompressible fluid Newtonian and non - Newtonian fluid flow. So, multiply both sides with u_i and convert the above equation to equation of mechanical energy. So, if you multiply u_i here and if you take in the inside these derivative then you can write $\frac{\partial (u_i^2/2)}{\partial t}$ and here also u_i if you take inside these derivative then you can write $u_j \frac{\partial (u_i^2/2)}{\partial x_j}$ and here $b_i u_i + u_i \frac{\partial \tau_{ij}}{\partial x_j}$.

So, now, subtract these equation from this equation. So, what you will get now you are subtracting the kinetic energy from this equation so that you can get the equation for internal or thermal energy. So, you can see if you subtract.

So, this term and this term will get cancel b_i u_i will get cancel. So, you can see you can write $\rho(\frac{\partial i}{\partial t} + u_j \frac{\partial i}{\partial x_j}) = Q''' - \frac{\partial q_j''}{\partial x_j} + \tau_{ij} \frac{\partial u_i}{\partial x_j}$. So, this term also will get cancel with this. So, we will have only $+\tau_{ij} \frac{\partial u_i}{\partial x_j}$. So, this term if you write in terms of material derivative then

$\nabla.\vec{V}$

you can write $p \frac{\partial u_k}{\partial x_k}$ this is the left hand side and right hand side will be same as this

equation.

So, now you can see the left hand side you have time rate of change of total internal energy right and in the right hand side the first term Q'''. So, this is your volumetric heat generation, the second term it is due to surface heat transfer and the third term you can see the conversion of kinetic energy into internal energy by work done against the viscous stresses.

Thus so, in the fluid flow will occur and there will be friction between the two fluid layers and that friction will be converted to thermal energy. And due to stresses that you see there will be a generation of internal energy and that is the conversion of kinetic energy into internal energy. Now you can see this is the term we need to expand $\tau_{ij}\frac{\partial u_i}{\partial x_j}$.

So, now, first let us see the constitutive equation.

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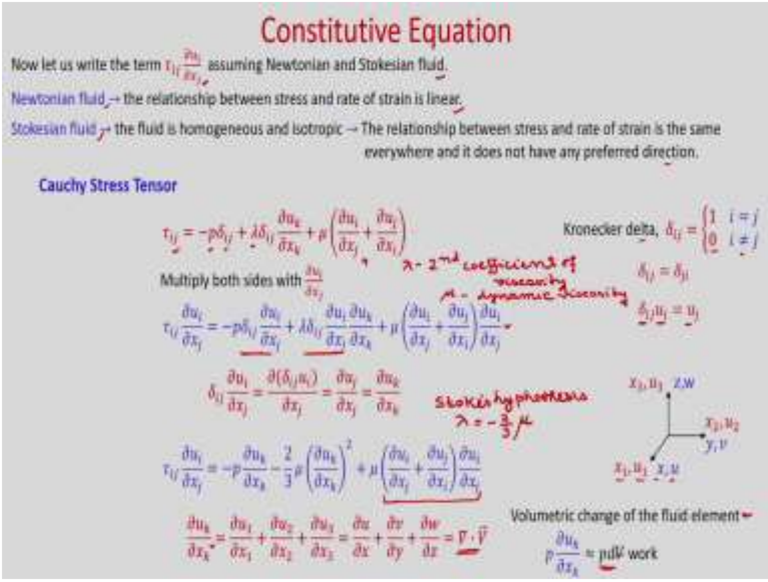

The constitutive equation gives the expression for the Cauchy's stress tensor. So, you will get the expression for τ_{ij}. So, before that now let us assume that we have Newtonian fluid and Stokesian fluid. So, now, we are assuming that we have the fluid flow for Newtonian and Stokesian fluid. What is Newtonian fluid? The relationship between the stress and the rate of strain is linear and Stokesian fluid the fluid is homogenous and isotropic.

What is homogenous? The relationship between stress and rate of strain is the same everywhere and isotropic means it does not have any preferred direction. So, assuming this Newtonian fluid and Stokesian fluid we can write τ_{ij} as $-p\delta_{ij}$, where p is your thermodynamic pressure, δ_{ij} is your Kronecker delta where δ_{ij} = 1, when i = j and δ_{ij} is =0 when i $\neq$ j.

And $+\lambda\delta_{ij}\frac{\partial u_k}{\partial x_k}$, where λ is your second coefficient of viscosity $+\mu(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})$. So, you can see we have written this τ_{ij} in terms of thermodynamic pressure and the velocity gradient and where μ is your dynamic viscosity.

So, now we need to determine the term $\tau_{ij}\frac{\partial u_i}{\partial x_j}$ right. So, this is the term we want to derive. So $\tau_{ij}\frac{\partial u_i}{\partial x_j}$. So, you multiply $\frac{\partial u_i}{\partial x_j}$ with these terms. So, what you will get, $-p\delta_{ij}\frac{\partial u_i}{\partial x_j}+\lambda\delta_{ij}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_k}$ + with this term you multiply $\frac{\partial u_i}{\partial x_j}$. This you can see that if you operate these δ_{ij} on u_i then you will get u_j.

So, δ_{ij} u_i you will get u_j. So, here now $\delta_{ij}\frac{\partial u_i}{\partial x_j}$. So, this is the term this is the same term we want to write $\delta_{ij}\frac{\partial u_i}{\partial x_j}=\frac{\partial(\delta_{ij}u_i)}{\partial x_j}$. So, I have taken this δ_{ij} here. So, you can see δ_{ij} u_i if you operate this δ_{ij} on u_i it will get u_j. So, $\frac{\partial u_j}{\partial x_j}$ and it is equivalent to write $\frac{\partial u_k}{\partial x_j}$ because both are same j. So, this j we have replaced with k. So, $\frac{\partial u_k}{\partial x_k}$.

So, you can see this term will become $-p\frac{\partial u_k}{\partial x_k}$ and this term will become $(\frac{\partial u_k}{\partial x_k})^2$. So, it will be $\lambda(\frac{\partial u_k}{\partial x_k})^2$,now we will use Stoke's hypothesis. So, in Stoke's hypothesis we can write the second coefficient of viscosity in terms of the dynamic viscosity as $-\frac{2}{3}\mu$. So, if you put it here.

So, it will be $-\frac{2}{3}\mu(\frac{\partial u_k}{\partial x_k})^2$. So, now, if you expand $\frac{\partial u_k}{\partial x_k}$ what it is, $\frac{\partial u_k}{\partial x_k}$. So, now, k u vary 1 2 3. So, it will be $\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}$. So, now, u_1 is equivalent to u. So, you can write $\frac{\partial u}{\partial x}$, x_1 is equivalent to x similarly $+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$ at which is nothing, but $\nabla.\vec{V}$.

So, what is $\nabla.\vec{V}$? $\nabla.\vec{V}$ is the volumetric change of the fluid element right, for incompressible fluid $\nabla.\vec{V} = 0$ right, but in general we are writing. So, there will be a volumetric change in the fluid element and that is represented by $\nabla.\vec{V}$ and these p $\nabla.\vec{V}$ is nothing, but the pdV work in thermodynamics.

So, you can see whatever we have written $p\frac{\partial u_k}{\partial x_k}$ which is your p $\nabla.\vec{V}$ and that is nothing, but your pdV work in thermodynamics. So, now, let us expand this term.

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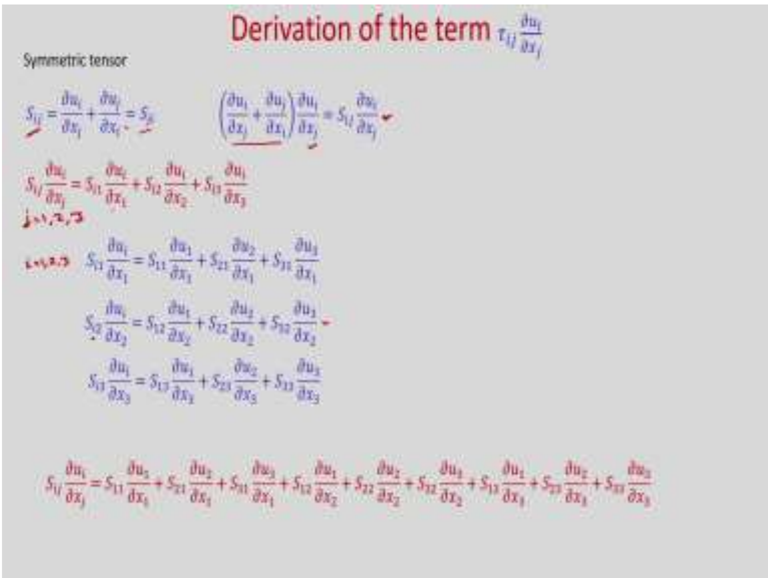

So, we will write this term $\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$ as symmetric tensor S_{ij} and it is equal to S_{ji} because it will remain same. So, it is a symmetric tensor S_{ij} is equal to S_{ji}. So, this term we can write as $S_{ij}\frac{\partial u_i}{\partial x_j}$ because this term we have written as S_{ij}. So, $S_{ij}\frac{\partial u_i}{\partial x_j}$.

So, now you can see $S_{ij}\frac{\partial u_i}{\partial x_j}$, first you vary j =1, 2 and 3. So, if you vary 1, 2, 3 so what will be there? So, it will be $S_{i1}\frac{\partial u_i}{\partial x_1} + S_{i2}\frac{\partial u_i}{\partial x_2} + S_{i3}\frac{\partial u_i}{\partial x_3}$, because j we have varied 1, 2 and 3. Now you vary i = 1, 2 and 3 so, in each term. So, if you take the these term.

So, i you vary 1 2 3. So, it will be $S_{11}\frac{\partial u_1}{\partial x_1}+S_{21}\frac{\partial u_2}{\partial x_1}+S_{31}\frac{\partial u_3}{\partial x_1}$.

Similarly here also you vary i=1,2,3. So, if you put i=2 $S_{12}\frac{\partial u_1}{\partial x_2}+S_{22}\frac{\partial u_2}{\partial x_2}+S_{32}\frac{\partial u_3}{\partial x_2}$ and for i=3, $S_{13}\frac{\partial u_1}{\partial x_3}+S_{23}\frac{\partial u_2}{\partial x_3}+S_{33}\frac{\partial u_3}{\partial x_3}$.

So, now, you see this term you can write as summation of these three. So, there will be nine components. So, we have written just these nine components in the expression of $S_{ij}\frac{\partial u_i}{\partial x_j}$. So, now, we have to find what is these S_{11}, S_{21} all these terms and S_{ij} already we have represented as this one.

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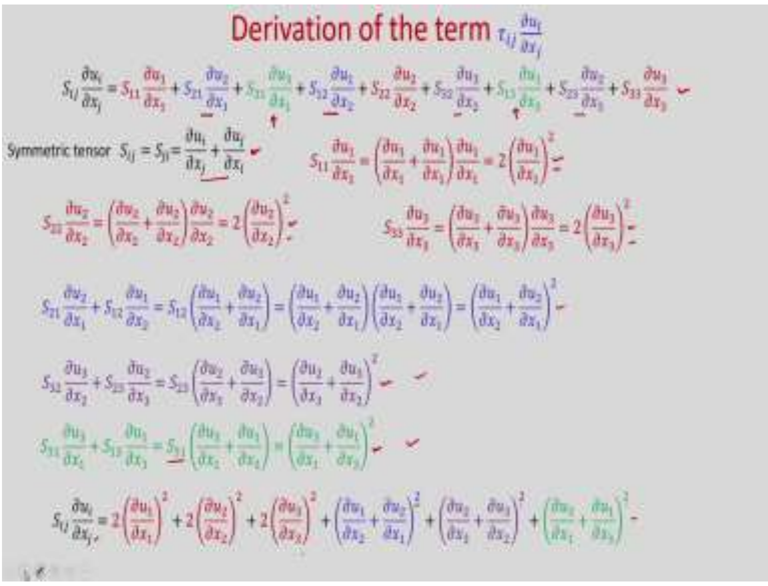

So, you can see. So, this is the term we have derived S_{ij} is the expression this one. So, now, you see the red colored terms. So, $S_{11}\frac{\partial u_1}{\partial x_1}$. Now S_{11} we have to find from this expression. So, if you put i and j both 1; that means, $\frac{\partial u_1}{\partial x_1}+\frac{\partial u_1}{\partial x_1}$ because i and j are 1 and you have $\frac{\partial u_1}{\partial x_1}$. So, this will become $2\frac{\partial u_1}{\partial x_1}$. So, it will be $2(\frac{\partial u_1}{\partial x_1})^2$.

Similarly the other two terms in the red colored you can write $S_{22}\frac{\partial u_2}{\partial x_2}$ as $2(\frac{\partial u_2}{\partial x_2})^2$ and $S_{33}\frac{\partial u_3}{\partial x_3}$ as $2(\frac{\partial u_3}{\partial x_3})^2$.

Now rest other terms let us find. So, you see this term and this term you consider, because you know that it is a symmetric tensor S_{ij} is equal to S_{ji}. So, $S_{21}=S_{12}$. So, these two terms we are considering together and we are writing. So, S_{12} you can take it outside and you can write $\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}$ and S_{12} now i=1, j=2.

So, you put it here i=1, j =2. So, $\frac{\partial u_1}{\partial x_2}$ and j =2. So, $\frac{\partial u_2}{\partial x_1}$. So, now, you can see these two terms are same. So, it will be $(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1})^2$. Similarly you consider rest of the term. So, you can see $S_{32}\frac{\partial u_3}{\partial x_2}$ and $S_{23}\frac{\partial u_2}{\partial x_3}$ you are considering these two terms together because we know S_{32} is equal to S_{23}.

So, if you consider this you can write $(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2})^2$ and similarly $S_{31}\frac{\partial u_3}{\partial x_1}$ and $S_{13}\frac{\partial u_1}{\partial x_3}$. So, S_{13} is equal to S_{31}. So, we can write $S_{31}(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3})$ and this S_{31} you can find here i=3 and j =1. So, you can see $(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3})^2$.

So, now, all these terms we have retained in terms of velocity gradients right. So, now, substitute all these terms in $S_{ij}\frac{\partial u_i}{\partial x_j}$. So, it will be you see this is one term, second, third. So, we have written here then we have these terms. So, we have written here. So, now, you look into this expression.

So, what is this? This is the generation of internal energy due to friction right, because you are converting this kinetic energy to thermal energy right. And all these gradients are having square, you see, $(\frac{\partial u_2}{\partial x_2})^2$, $(\frac{\partial u_3}{\partial x_3})^2$ and all these are having square.

So, whether velocity gradient is positive or negative always this term $S_{ij}\frac{\partial u_i}{\partial x_j}$ will be positive term, you will get always heating inside the fluid element. So, because due to the friction between two fluid elements it will generate heat and it will be always positive; that means, it will be heat, heating will be there. So, you can see from this expression.

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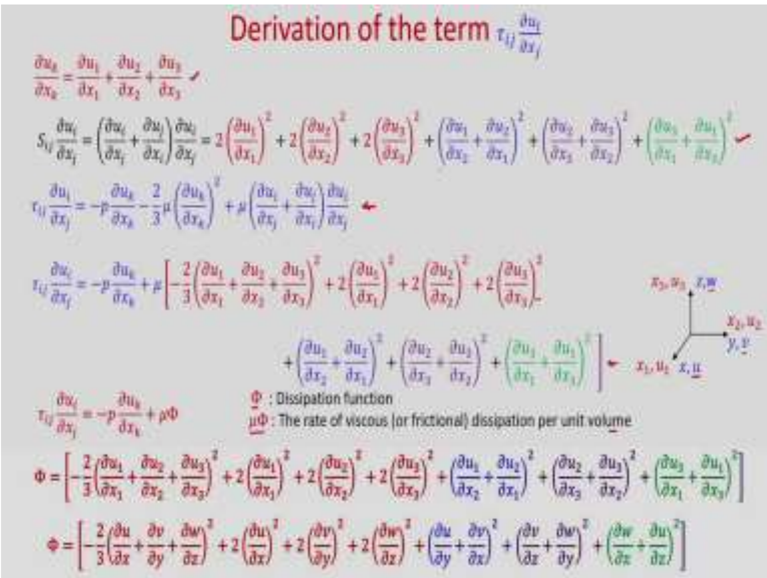

You see $\frac{\partial u_k}{\partial x_k}$ already we have written in this expression $S_{ij}\frac{\partial u_i}{\partial x_j}$ we have written in this expression and we have $\tau_{ij}\frac{\partial u_i}{\partial x_j}$ these expression now you put all these values right. So, $\frac{\partial u_k}{\partial x_k}$ this expression and this expression you write from here and you can write $\tau_{ij}\frac{\partial u_i}{\partial x_j}$ as this term + μ I am taking outside $-\frac{2}{3}$ then $\frac{\partial u_k}{\partial x_k}$ is this one. So, it is whole square plus this terms plus these three terms.

So, you can see we have written $\tau_{ij}\frac{\partial u_i}{\partial x_j}$ in terms of velocity gradients. So, now, we can see this you can write μ into all these terms you can write as Φ. So, this Φ is known as dissipation function and μΦ is the rate of viscous dissipation per unit volume. So, you can see this expression you can write as $\tau_{ij}\frac{\partial u_i}{\partial x_j} = -p\frac{\partial u_k}{\partial x_k} + \mu\phi$.

So, Φ is the dissipation function and μΦ is the rate of viscous dissipation. So, you can see now Φ will be this term. So, this we have written and in terms of u v w and x y z if you write then you will get this expression. And if you consider incompressible fluid flow then you can see this will be your 0 because this is your continuity equation $\nabla.\vec{V}$.

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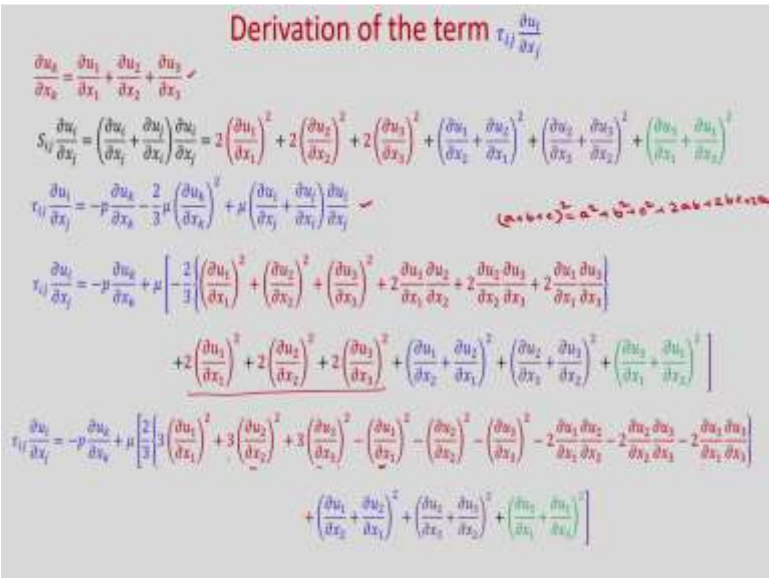

In other way also you can express this dissipation function. So, you can see now this is the expression we have, you take μ outside and $-\frac{2}{3}$ we have taken common. Now if you express these $(\frac{\partial u_k}{\partial x_k})^2$; that means, this is square.

So, we can write $(\frac{\partial u_1}{\partial x_1})^2 + (\frac{\partial u_2}{\partial x_2})^2 + (\frac{\partial u_3}{\partial x_3})^2 + 2ab + 2bc + 2ca$ right $(a+b+c)^2$. So, this we have written. So, it is $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.

So, this expression we have written and we have this term. So, we have written here and now we are taking $\frac{2}{3}$ outside. So, this terms we are writing here. So, as $\frac{2}{3}$ we have taken outside. So, it will be $3(\frac{\partial u_1}{\partial x_1})^2 + 3(\frac{\partial u_2}{\partial x_2})^2 + 3(\frac{\partial u_3}{\partial x_3})^2$ and $\frac{2}{3}$ is there and negative of these term you have to write. So, negative of this term we have written like this.

So, now, let us rearrange this. So, we can see you have $3(\frac{\partial u_1}{\partial x_1})^2$ and here 1 if you subtract then you will get $2(\frac{\partial u_1}{\partial x_1})^2$. Similarly for this term we will get 2 factor 2 as a coefficient.

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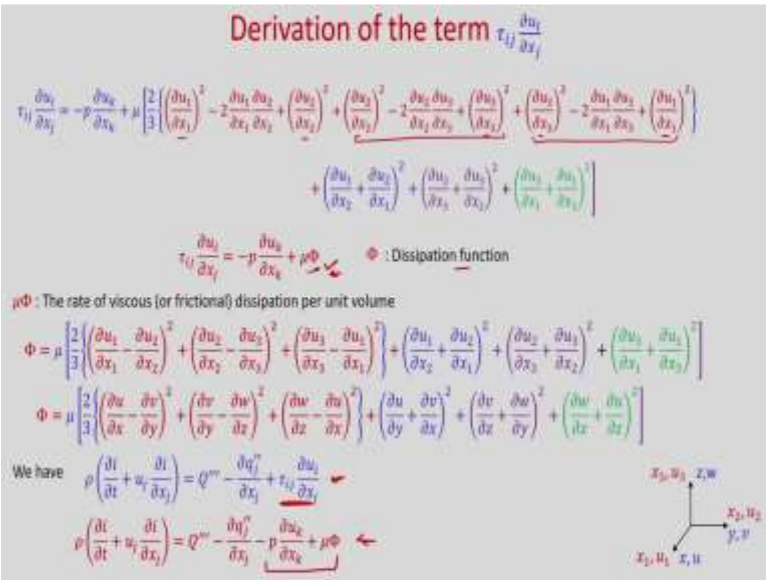

So, now you rearrange it like this. So, you have one you have written here, another you have written here, here you have written, one here, another so here one, here another. So, these you have written then $-2\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}$ is there. So, now, what is this? It is a $(-b)^2$ right. So, if you write in this form. So, it will become. So, $(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2})^2$.

Similarly, these term together you can write $(\frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3})^2$ and these three terms together you can write $+3(\frac{\partial u_3}{\partial x_3} - \frac{\partial u_1}{\partial x_1})^2$ and all these 3 terms will be there. So, all these terms we have written in terms of dissipation function and this $\tau_{ij}\frac{\partial u_i}{\partial x_j} = -p\frac{\partial u_k}{\partial x_k} + \mu\phi$, where phi is the dissipation function and $\mu\Phi$ is the rate of viscous dissipation per unit volume.

So, Φ now this full expression you can write in terms of velocity u v w and coordinate x y z if you write then it will be

$$[\frac{2}{3}\{(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y})^2+(\frac{\partial v}{\partial y}-\frac{\partial w}{\partial z})^2+(\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x})^2\}+(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})^2+(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y})^2+(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})^2].$$

So, this is your dissipation function. So, now, we had this one. So, now, we have derived this term as this. So, if you substitute in this equation. So, you can get this equation. So, this is the two terms whatever we have derived and phi is the dissipation function, it is a big expression in terms of velocity gradient.

So, we have considered Cartesian coordinates. So, this is the expression for dissipation function in Cartesian coordinate, but if you consider cylindrical coordinate or spherical coordinate then this expression for this dissipation function will be different.

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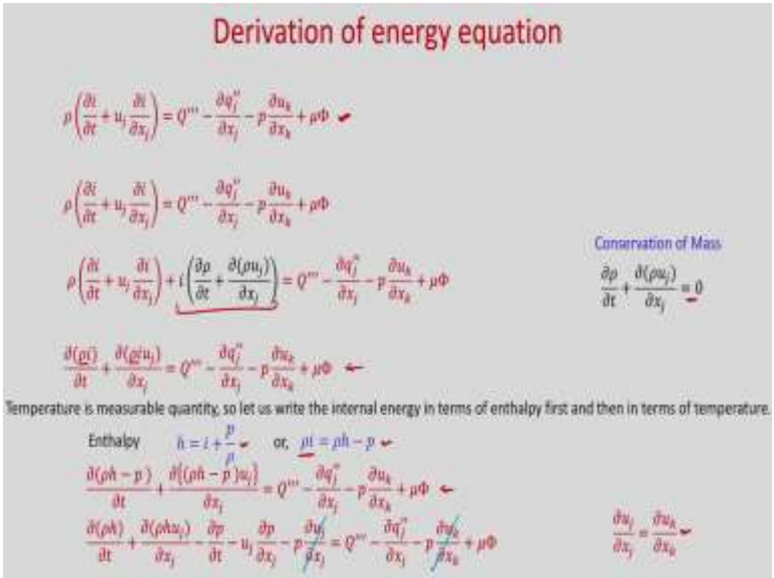

So, we have already derived this now what will do, we will just add this term $i(\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u_j)}{\partial x_j})$. So, if you add this term it is actually conservation of mass $\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u_j)}{\partial x_j}=0$ in general, it is applicable for both compressible and incompressible flow.

So, if you add this term then together if you write. So, you can write in a conservative form, because this $\rho \frac{\partial i}{\partial t} + i \frac{\partial \rho}{\partial t}$ you can write as $\frac{\partial(\rho i)}{\partial t}$ and $\rho u_j \frac{\partial i}{\partial x_j} + i \frac{\partial(\rho u_j)}{\partial x_j}$ you can write as $\frac{\partial(\rho i u_j)}{\partial x_j}$.

So, now, this is the equation we have derived in terms of internal energy, but now we need to write in terms of some measurable quantity. So, that is temperature. So, first we will write this equation in terms of enthalpy, then we will write in terms of temperature.

So, you know that enthalpy from thermodynamics, you can know that enthalpy, $h = i + \frac{p}{\rho}$, where p is the thermodynamic pressure and ρ is the fluid density. So, you can write $\rho i = \rho h - p$. Now this ρi you substitute here.

So, if you substitute here you will get this equation, $\frac{\partial(\rho h)}{\partial t} + \frac{\partial(\rho h u_j)}{\partial x_j} - \frac{\partial \rho}{\partial t} - p \frac{\partial u_j}{\partial x_j} = Q^{'''} - \frac{\partial q_j^{''}}{\partial x_j} - p \frac{\partial u_k}{\partial x_k} + \mu \phi$.

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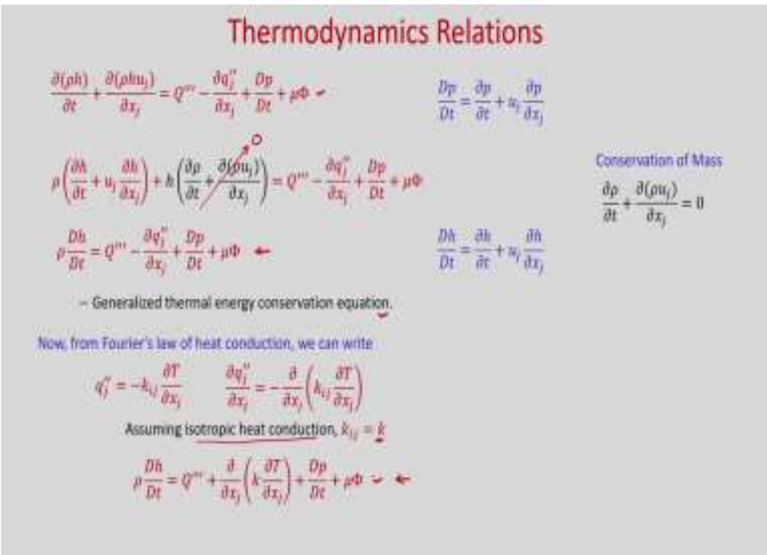

So, now you can write. So, this is in conservative form we have written $\frac{\partial(\rho h)}{\partial t} + \frac{\partial(\rho h u_j)}{\partial x_j} = Q^{'''} - \frac{\partial q_j^{''}}{\partial x_j} + \frac{Dp}{Dt} + \mu \phi$ and now if you write these two terms. So,

$\rho \frac{\partial h}{\partial t} + h \frac{\partial \rho}{\partial t}$ and here if you write $\rho u_j \frac{\partial h}{\partial x_j} + h \frac{\partial (\rho u_j)}{\partial x_j}$ then you can write in terms of non conservative form.

And here you can see this is your conservation of mass $\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_j)}{\partial x_j}$ = 0. So, this is your 0 right. So, we have written this term. So, this is your generalized thermal energy conservation equation we have written in terms of enthalpy.

So, this enthalpy now we have to write in terms of temperature and we have this heat flux $q^{"}$ and that we need to write in terms of temperature. So, first let us write this heat flux using Fourier's law of heat conduction. So, from Fourier's law of heat conduction you know $q_j^{"} = -k_{ij} \frac{\partial T}{\partial x_j}$ and if you write $\frac{\partial q_j^{"}}{\partial x_j} = -\frac{\partial}{\partial x_j}(k_{ij} \frac{\partial T}{\partial x_j})$.

So, now, you assume isotropic heat conduction so; that means, this thermal conductivity will be independent of the directions. So, k_{ij}=k we can write. So, if you write k then in this expression if you put then you can write $\rho \frac{Dh}{Dt} + Q^{'''} + \frac{\partial}{\partial x_j}(k \frac{\partial T}{\partial x_j}) + \frac{D\rho}{Dt} + \mu\phi$.

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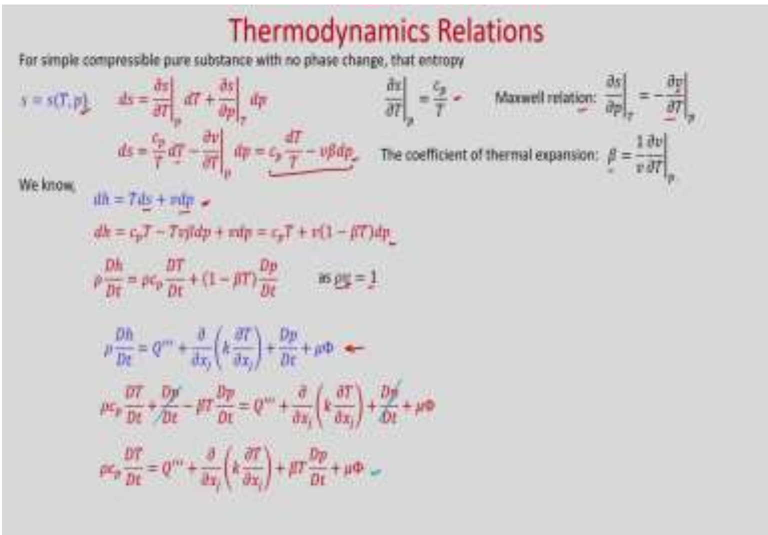

So, now let us write the enthalpy in terms of temperature. So, for that we will use some thermodynamic relations which we have already studied in your thermodynamics course, for simple compressible pure substance with no phase change entropy you can write $s = s(T, p)$.

So, we can write s as a function of two variable. So, we can write $ds = \frac{\partial s}{\partial T}\Big|_p dT + \frac{\partial s}{\partial p}\Big|_T dp$. So, now, you can see that you know $\frac{\partial s}{\partial T}\Big|_p = \frac{c_p}{T}$. So, this is your specific heat at constant pressure.

And from Maxwell relations you can see $\frac{\partial s}{\partial p}\Big|_T = -\frac{\partial v}{\partial T}\Big|_p$ where v is your specific volume.

So, if you put these $\frac{\partial s}{\partial T}\Big|_p$.

So, $\frac{c_p}{T}dT$ and $\frac{\partial s}{\partial p}$ you put this expression. So, $-\frac{\partial v}{\partial T}\Big|_p dp$ and now you use the coefficient of thermal expansion expression. So, β you know that $\frac{1}{v}\frac{\partial v}{\partial T}\Big|_p$. So, you can see $\frac{\partial v}{\partial T}\Big|_p$ you can write vβ and this expression $ds = c_p \frac{dT}{T} - v\beta dp$.

Now again you can write the enthalpy, . So, now, $dh = c_p T - Tv\beta dp + vdp$. So, it will be c_pT and v if you take outside. So, (1- βT)dp. So, now, if you write in terms of material derivative so, $\rho\frac{Dh}{Dt} = \rho c_p \frac{DT}{Dt} + (1-\beta T)\frac{Dp}{Dt}$.

So, this is your material derivative $\frac{Dp}{Dt}$ and ρv=1 because v is specific volume. So, ρ v will be 1. So, this I have put 1. So, now, we can see this $\rho\frac{Dh}{Dt}$ now if you put in this expression then you can write. So, $\rho\frac{Dh}{Dt} = Q''' + \frac{\partial}{\partial x_j}(k\frac{\partial T}{\partial x_j}) + \frac{Dp}{Dt} + \mu\phi$, this we have already derived. So, if you put this expression in the left hand side then you can get these.

Now you see in left hand side you have $\frac{Dp}{Dt}$, right hand side also you have $\frac{Dp}{Dt}$. So, these you can cancel. So, if you now simplify it you will get

$$\rho c_p \frac{DT}{Dt} = Q''' + \frac{\partial}{\partial x_j}(k\frac{\partial T}{\partial x_j}) + \beta T \frac{Dp}{Dt} + \mu\phi \, .$$

So, now you can see that we have written the energy equation in terms of temperature and left hand side we have written in terms of material derivative $\frac{DT}{Dt}$.

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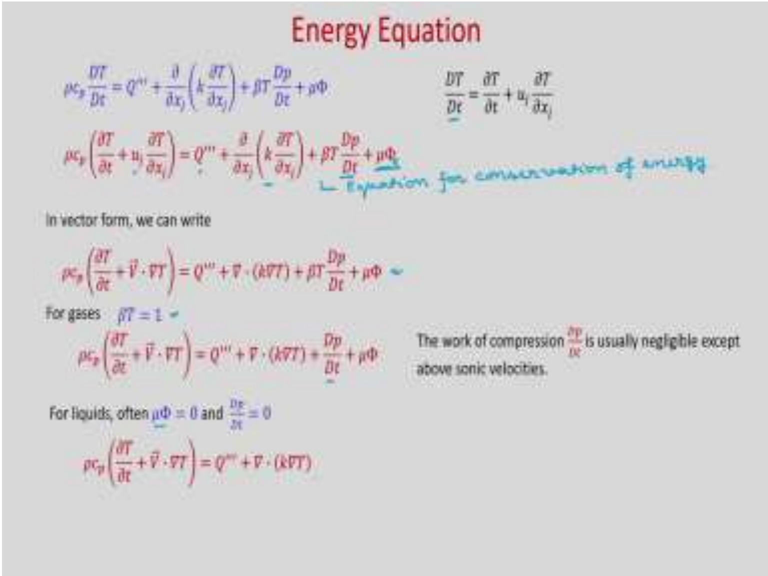

So, now if you write this material derivative of T as $\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j}$ then you can see $\rho c_p (\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j}) = Q''' + \frac{\partial}{\partial x_j}(k\frac{\partial T}{\partial x_j}) + \beta T \frac{Dp}{Dt} + \mu\phi$. So, this is the equation for conservation of energy.

So, you see left hand side term, ρ is density, c_p is the specific heat this is the temporal variation of this temperature $\frac{\partial T}{\partial t}$ and this is the convective term $u_j \frac{\partial T}{\partial x_j}$, because your these scalar temperature is convicted by this velocity u_j right and right hand side this term is internal heat generation per unit volume and this is the term you can see this is coming from your heat conduction.

So, in the energy equation this is the term while deriving this equation you have seen it is coming from the heat conduction right, but in general if k is constant you can take it outside and you can write $\frac{\partial T}{\partial x_j}$ whole you can write $(\frac{\partial T}{\partial x_j})^2$.

So, this is the term which is coming from the Fourier's law of heat conduction right and substituting that in the expression we have got this term which is known as diffusion term, it is due to the heat conduction right plus beta T$\frac{Dp}{Dt}$. So, $\frac{Dp}{Dt}$ which is your material derivative of p. So, these term having the significance at high velocity. So, above the sonic velocity if you have the flow then this term will have significant contribution.

And μΦ, what is μΦ? μΦ is the viscous dissipation rate right per unit volume. So, and phi is the dissipation function that we have written in terms of velocity gradient.

So, you can see when μ is having very high value then this term will have the significance and if you have a high velocity then due to high velocity there will be contribution from the velocity gradient in the dissipation function Φ then also this term will have some significance. So, you can see this is for high viscosity or high velocity you can consider this term and this term you can consider when you have very high velocity above sonic velocity.

So, in vector form this equation now if you write $\rho c_p(\frac{\partial T}{\partial t} + \vec{V}.\nabla T) = Q''' + \nabla.(k\nabla T) + \beta T\frac{Dp}{Dt} + \mu\phi$ and these if you assume that your thermal conductivity is constant then you can take it outside otherwise you keep it inside. So, most of the gases you know for gases you know that β=$\frac{1}{T}$; that means, βT=1.

So, if you put βT=1 then this expression here it will be just $\frac{Dp}{Dt}$ and this the work of compression $\frac{Dp}{Dt}$ is usually negligible except above sonic velocities and for liquids open you will get μΦ = 0 and $\frac{Dp}{Dt}$ = 0. Then you can express this equation in the right hand side only two terms will be there $Q''' + \nabla.(k\nabla T)$, but if there is a significant contribution from μΦ then you can consider in this equation.

So, today we have started with the Reynolds transport equation to derive the equation for conservation of energy. So, for a control volume we have written the total energy as internal energy plus kinetic energy because potential energy we consider in the body force term from first law of thermodynamics we have written $\frac{DE}{Dt}\Big|_{sys} = \dot{Q}_{CV} - \dot{W}_{CV}$, where we are taking that rate of heat transfer to the system $\dot{Q}_{CV}$ is positive and rate of work done by the system $\dot{W}_{CV}$ is positive.

Then we have expressed this $\dot{Q}_{CV}$ and $\dot{W}_{CV}$ considering the volumetric volume heating as well as surface heating, for the rate of heat transfer calculation and for the rate of work done calculation we have considered two different forces body force as well as surface force. Then we have expressed these terms and finally, we have got one term $\tau_{ij}\frac{\partial u_i}{\partial x_j}$ which is your the term which is contributing from the friction because you have a shear stress between the two fluid elements and these frictional force is converting to internal energy.

Then that we have expressed in terms of velocity gradients and finally, these total energy we have written in terms of internal energy plus kinetic energy. So, to get the equation for thermal energy we have subtracted the kinetic energy considering the Navier's equation. Then we have written the energy equation in terms of internal energy and later we have converted this internal energy in terms of enthalpy then in terms of a measurable quantity temperature.

And using some thermodynamics relations we have written the equation for this internal energy in terms of temperature which is your equation for conservation of energy. We have also discussed that the viscous dissipation term μΦ is significant when you have high viscosity fluid or you have high velocities.

And $\frac{Dp}{Dt}$ is significant above the sonic velocities and finally, we have written this equation in terms of vectorial form for gases assuming $\beta = \frac{1}{T}$ and finally, we have written for liquids where most of the time we consider negligible viscous heating and $\frac{Dp}{Dt}$ as 0.

Thank you.