Fundamentals of Convective Heat Transfer

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Module – 09
Natural Convection – II
Lecture – 29
Natural convection over a vertical plate: Integral solution

Hello everyone. So, in today's class we will solve natural convection over a vertical flat plate using integral method. We have already learned this integral method in module 4, lecture 1. We have derived the momentum integral equation, in module 4, lecture 2 we have derived the energy integral equation.

So, please refer these derivations. In today's class we will use the momentum integral equation and energy integral equation with some modification, because we have a buoyancy term in the boundary layer equations for natural convection flow and we will solve for the unknown variables δ.

So, if you remember in post convection we have two unknowns; one is hydro (Refer Time: 01:31) boundary layer thickness δ and thermal boundary layer thickness δ_T and we had two integral equations and we solve for δ and δ_T. In natural convection will assume that $\delta = \delta_T$. For these we solve this equations and we will find the unknown variable δ and another unknown variable velocity profile.

Although, we are assuming that $\delta = \delta_T$, but later will show that the solutions; whatever we will be deriving using this integral analysis it will be valid for wide range of Prandtl number.

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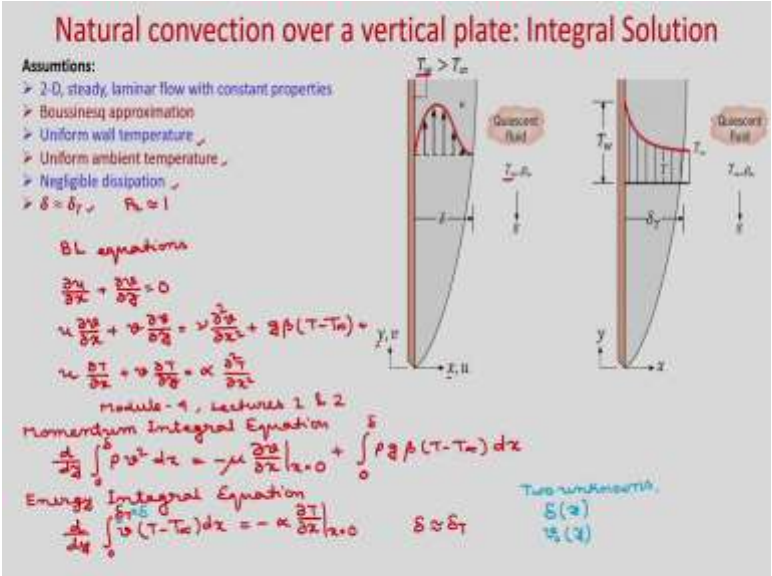

So, let us consider natural convection over a vertical flat plate. The plate is maintained at uniform wall temperature T_w and the quiescent medium temperature is T_∞ and that is also maintained at constant temperature. Here, we have these assumptions 2 dimensional steady laminar flow with constant properties, we have Boussinesq approximation valid and T_w is constant, T_∞ is constant and we are neglecting the viscous dissipation.

So, in this integral solution we will assume $\delta \approx \delta_T$; that means, Pr ≈1, but we will write the solution in terms of Prandtl number and we will show that the solution is reasonable reasonably valid for wide range of Prandtl number.

So, first let us write the boundary layer equations for a natural convection over a flat plate. So, we have already derived these equations, boundary layer equations. So, continuity equation, the momentum equation is $u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}=\nu\dfrac{\partial^2 v}{\partial x^2}+g\beta\left(T-T_\infty\right)$. So, in this case x is perpendicular to the wall and y is along the vertical wall and energy equation, $u\dfrac{\partial T}{\partial x}+v\dfrac{\partial T}{\partial y}=\alpha\dfrac{\partial^2 T}{\partial x^2}$.

So, you refer module 4, lecture 1 and 2 for derivation of these integral equation. So, if you derive you will get momentum integral equation as; so, using the continuity equation you can derive the momentum integral equation from this momentum equation as

$\frac{d}{dy}\int_0^{\delta} \rho v^2 dx = -\mu \frac{\partial v}{\partial x}|_{x=0} + \int_0^{\delta} \rho g \beta \left(T - T_\infty\right) dx$ and energy integral equation you can write as $\frac{d}{dy}\int_0^{\delta_T} v\left(T - T_\infty\right) dx = -\alpha \frac{\partial T}{\partial x}|_{x=0}$. So, you can see $-K\frac{\partial T}{\partial x}|_{x=0}$ will give you wall heat flux and here alpha is thermal diffusivity.

So, now, we will assume $\delta \approx \delta_T$. So, here the integral, we will integrate up to δ. We will solve for two unknowns; one is δ which is function of y and another is unknown velocity profile. So, some will derive later, some velocity v_0, so which will be function of y. So, these are the two unknowns we will use this two integral equations and will solve for δ(y) and v_0(y).

So, what is the next step while using the integral method? So, you have to assume the velocity profile and temperature profile, then once you get the velocity and temperature profile you have to invoke those in the integral equations.

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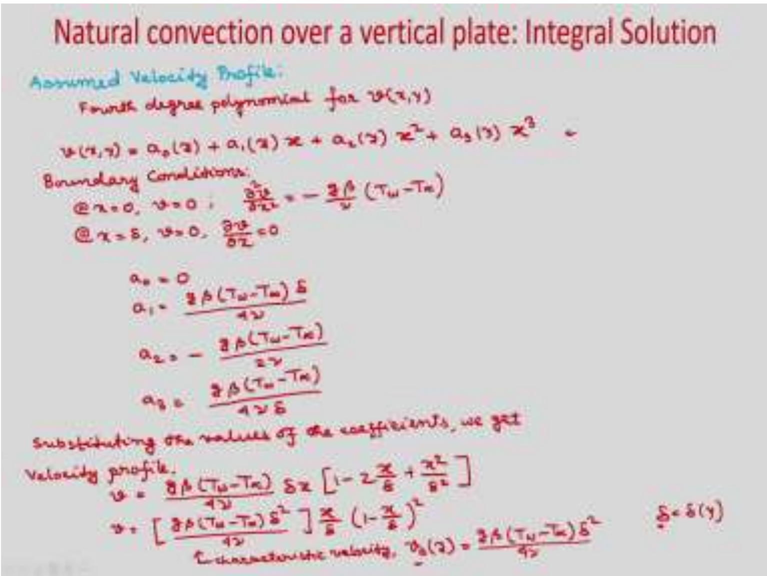

So, first let us see what is the assumed velocity profile. So, we will use here fourth degree polynomial for velocity. So, will use,

$$v\left(\mathrm{x}, y\right) = a_0\left(y\right) + a_1\left(y\right)x + a_2\left(y\right)x^2 + a_3\left(y\right)x^3.$$

So, we have to find these coefficients a_0, a_1, a_2, and a_3. So, there are four coefficients. So, how many boundary conditions we need? So, we need four boundary conditions, two boundary condition you know easily; one is the wall you have velocity 0 at x →∞ you have velocity 0 and also at x →∞ you have velocity gradient is 0, because it is a quiescent medium. Another boundary condition you have to derive; so, from the momentum equation. So, that is the derived boundary condition, these already we have discussed in module 4.

So, boundary conditions at x = 0, you have v = 0, at x = δ, you have v = 0, also the velocity gradient is 0, because you have a quiescent medium. So, $\frac{\partial v}{\partial x}$ =0 and another boundary condition that is derived from the momentum equation. So, at x = 0 you can write $\frac{\partial^2 v}{\partial x^2} = -\frac{g\beta}{\nu}\left(T_w - T_\infty\right)$.

So, in a momentum equation the inertia term will become 0, because u ,v at 0. So, this is your viscous term and at wall you have T = T_w; so, you can write $\frac{\partial^2 v}{\partial x^2} = -\frac{g\beta}{\nu}\left(T_w - T_\infty\right)$.

So, if you use invoke this boundary condition and find the coefficients you will get $a_0 = 0$ $a_1 = \frac{g\beta\left(T_w - T_\infty\right)\delta}{4\nu}$, $a_2 = -\frac{g\beta\left(T_w - T_\infty\right)}{2\nu}$ and $a_3 = -\frac{g\beta\left(T_w - T_\infty\right)}{4\nu\delta}$.

So, now this coefficient if you substitute in the assumed velocity profile, then you will get the velocity profile. So, if you see substituting the values of the coefficient we get; so, if you substitute here, so we will get the velocity profile as $v = \frac{g\beta\left(T_w - T_\infty\right)}{4\nu}\delta x\left[1 - 2\frac{x}{\delta} + \frac{x^2}{\delta^2}\right]$.

If you rearrange it, you will get $v = \left[\frac{g\beta\left(T_w - T_\infty\right)\delta^2}{4\nu}\right]\frac{x}{\delta}\left(1 - \frac{x}{\delta}\right)^2$. So, you can see the first term in the right hand side in the inside the bracket. So, these term can form the characteristic velocity.

Now, we will say that it is the characteristic velocity and v_0 which is function of y and this is to be found from the solution. So, this is another unknown $v_0(y)=\frac{g\beta\left(T_w-T_\infty\right)\delta^2}{4\nu}$. So, you can see δ is function of y. So, v_0 is also function of y. So, this is the second unknown, first unknown is the δ we have to find and another unknown v_0 we have to find .

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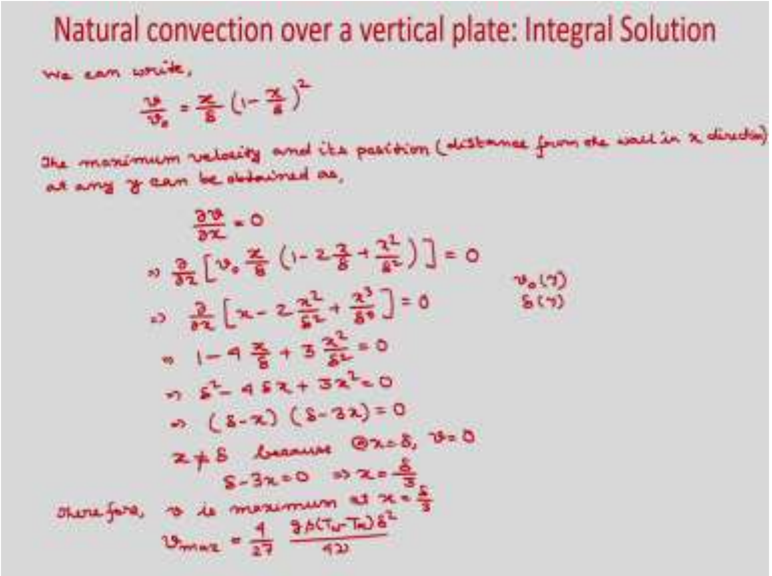

So, now you can write the velocity profile as $\frac{v}{v_0}=\frac{x}{\delta}\left(1-\frac{x}{\delta}\right)^2$. Now, you know the velocity profile; assume velocity profile and that from that we can find the maximum velocity location. So, at which location you will get the maximum velocity. So, if you take the derivative of velocity v, then make it 0, then we will be able to find at which location you will get the maximum velocity, at which x location right.

So, the maximum velocity and its position, distance from the wall in x direction at any y can be obtained as. So, you can write $\frac{\partial v}{\partial x}$ =0 so; that means, $\frac{\partial}{\partial x}\left[v_0\frac{x}{\delta}\left(1-2\frac{x}{\delta}+\frac{x^2}{\delta^2}\right)\right]=0$. So, you see v_0 is function of y and δ is also function of y. So, you can take it outside the derivative.

So, you can write $\frac{\partial}{\partial x}\left[x-2\frac{x^2}{\delta^2}+\frac{x^3}{\delta^3}\right]=0$, because v_0 is function of y and δ is function of y right and from here if you see; so, you will get, $1-4\frac{x}{\delta}+3\frac{x^2}{\delta^2}=0$.

If you rearrange it you will get $\delta^2-4\delta x+3x^2=0$ and you will get $(\delta-x)(\delta-3x)=0$. You can see x = δ you have velocity,v= 0.

So, there will not be maximum velocity. So, x ≠δ, because at x = δ you have v = 0 right. So, you have $\delta-3x$=0; that means, $x=\frac{\delta}{3}$. So, you can see at $x=\frac{\delta}{3}$ you will get the maximum velocity. So, therefore, v is maximum at $x=\frac{\delta}{3}$ and its value if you find, it will be v_{max} , after simplification I am writing $v_{max}=\frac{4}{27}\frac{g\beta(T_w-T_\infty)\delta^2}{4\nu}$.

So, we have found the velocity distribution now, let us find the temperature profile, assume temperature profile. So, for that also we will use a third degree polynomial.

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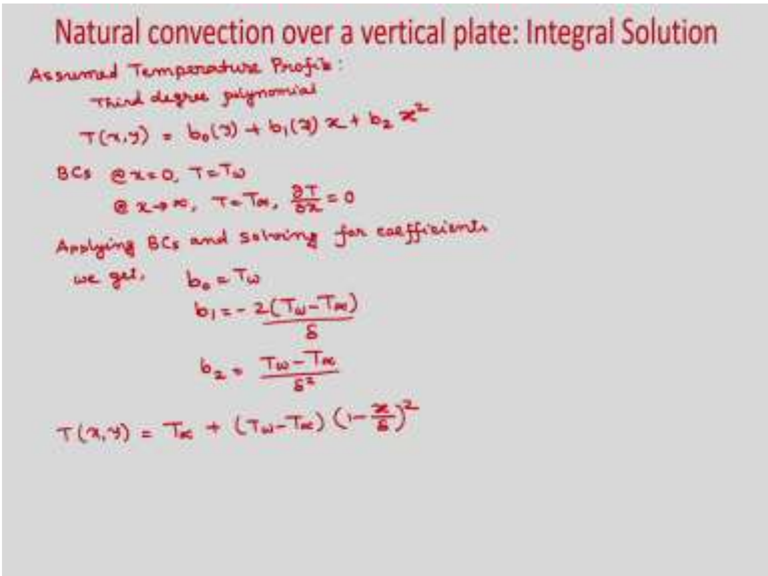

So, assumed temperature profile we will use third degree polynomial. So, if it is so then you can write $T(x,y)=b_0(y)+b_1(y)x+b_2x^2$.

So, now you need three boundary conditions to find the three unknowns b_0 , b_1 and b_2. So, you know at wall you have temperature T_w, at $x \to \infty$ you have temperature T_∞ which is the quiescent free temperature, as well as $x \to \infty$ your temperature gradient is 0. So, you can write boundary conditions at x = 0 you have $T = T_w$, at $x \to \infty$ you have $T = T_\infty$ and also the temperature gradient is 0.

So, applying boundary conditions and solving for the coefficients we get, applying boundary conditions and solving for coefficients we get $b_0 = T_w$, $b_1 = -\frac{2(T_w - T_\infty)}{\delta}$ and $b_2 = \frac{(T_w - T_\infty)}{\delta^2}$. So, if you put these values in the assumed temperature profile, then you will get $T(x, y) = T_\infty + (T_w - T_\infty)\left(1 - \frac{x}{\delta}\right)^2$.

So, now, we have assumed the velocity profile as well as the temperature profile. Now, you want to put this profiles into the integral equations. So, because you have the momentum equation where you have temperature profile as well as you have the velocity profile. So, in the momentum integral equation you need the temperature profile as well as the velocity profile.

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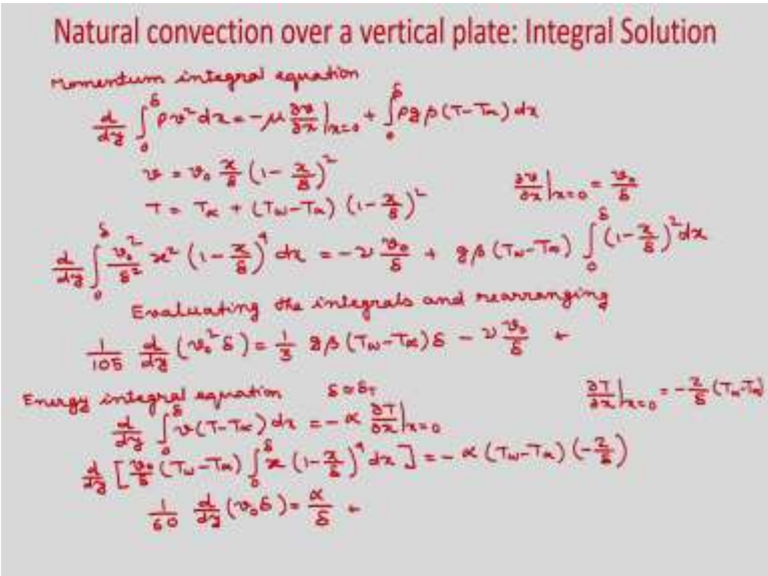

So, if you put there; so, you have momentum integral equation so that is your, $\frac{d}{dy}\int_0^\delta \rho v^2 dx = -\mu \frac{\partial v}{\partial x}|_{x=0} + \int_0^\delta \rho g \beta (T - T_\infty) dx$. Now, put the velocity profile and temperature profile. So, you have $v = v_0 \frac{x}{\delta}\left(1-\frac{x}{\delta}\right)^2$ and temperature profile $T = T_\infty + \left(T_w - T_\infty\right)\left(1-\frac{x}{\delta}\right)^2$.

So, here you can see you have T -T_∞. These equation if you divide both sides by ρ then if you rearrange you will get $\frac{d}{dy}\int_0^\delta \frac{v_0^2}{\delta^2}x^2\left(1-\frac{x}{\delta}\right)^4 dx = -\nu\frac{v_0}{\delta} + g\beta\left(T_w - T_\infty\right)\int_0^\delta \left(1-\frac{x}{\delta}\right)^2 dx$.

So, now, evaluating the integrals and rearranging, you will get, $\frac{1}{105}\frac{d}{dy}\left(v_0^2\delta\right) = \frac{1}{3}g\beta\left(T_w - T_\infty\right)\delta - \nu\frac{v_0}{\delta}$.

Now, similarly you put the value of temperature profile in the energy integral equation. So, energy integral equation if you remember. So, it has $\frac{d}{dy}\int_0^\delta v\left(T - T_\infty\right)dx = -\alpha\frac{\partial T}{\partial x}\Big|_{x=0}$.

So, put the velocity profile and the temperature profile here. So, you will get $\frac{d}{dy}\left[\frac{v_0}{\delta}\left(T_w - T_\infty\right)\int_0^\delta x\left(1-\frac{x}{\delta}\right)^4 dx\right] = -\alpha\left(T_w - T_\infty\right)\left(-\frac{2}{\delta}\right)$. So, from here you can see, $\frac{\partial T}{\partial x}\Big|_{x=0} = \left(-\frac{2}{\delta}\right)\left(T_w - T_\infty\right)$. So, just rearranging, you will get $\frac{1}{60}\frac{d}{dy}\left(v_0\delta\right) = \frac{\alpha}{\delta}$. So, you can see we have got first order ordinary differential equation. So, this is one equation and this is another equation. These are first order ordinary differential equation and we need to find v_0 and δ from here.

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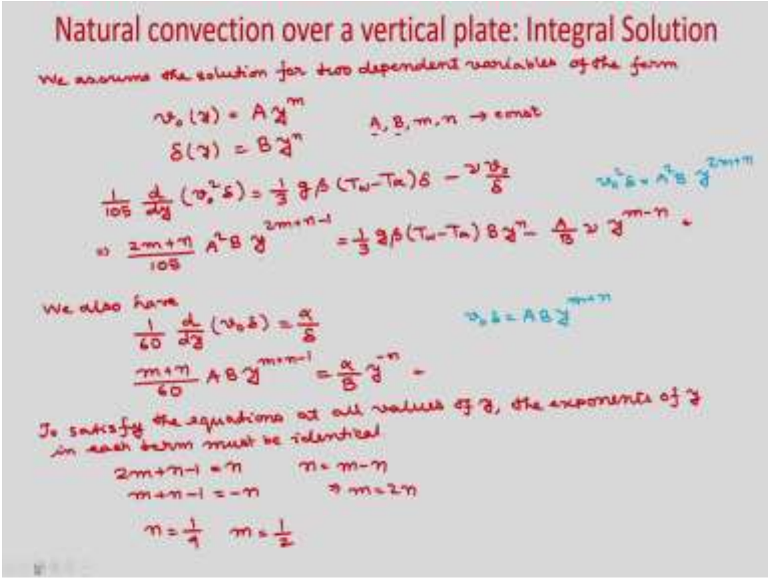

So, we assume the solution for two dependent variables of the form v_0 which is function of y. So, we will use $v_0\left(y\right) = Ay^m$, because it is function of y and $\delta\left(y\right) = By^n$ where A, B, m, n, are constants .

So, now if you put these values in the ordinary differential equation; so for momentum equation we got this ordinary differential equation and find the value of A, B and m and n. So, you will get $\frac{1}{105}\frac{d}{dy}\left(v_0^2\delta\right) = \frac{1}{3}g\beta\left(T_w - T_\infty\right)\delta - \nu\frac{v_0}{\delta}$.

So, put the v_0 and δ expression here. So, you will get $v_0^2\delta = A^2By^{2m+n}$. So, if you take the derivative with respect to y, then you will get, $\frac{2m+n}{105}A^2By^{2m+n-1} = \frac{1}{3}g\beta\left(T_w - T_\infty\right)By^n - \frac{A}{B}\nu y^{m-n}$ and another ordinary differential equation we have, so that is your $\frac{1}{60}\frac{d}{dy}\left(v_0\delta\right) = \frac{\alpha}{\delta}$. So, if you see $v_0\delta = ABy^{m+n}$. So, you can take the derivative with respect to y.

So, you will get $\frac{m+n}{60}ABy^{m+n-1} = \frac{\alpha}{B}y^{-n}$. So, you can see, we have these equation and we have these equation and to satisfy the equations at all values of y, the exponents of y in each term must be identical.

So, to satisfy the equations at all values of y the exponents of y in each term must be identical. So, if it is so then you can write $2m+n-1=n$ and also you can write n =m - n and you can write $m+n-1=-n$.

So, from here you can find so, you can see from here you can find m = 2 n and if you put it here. So, you will get n = 1 /4 and m =1/2. So, now, you can see we have found the exponent m and n, m =1/2 and n = 1 /4. So, now, we have to find other two constants that is A and B.

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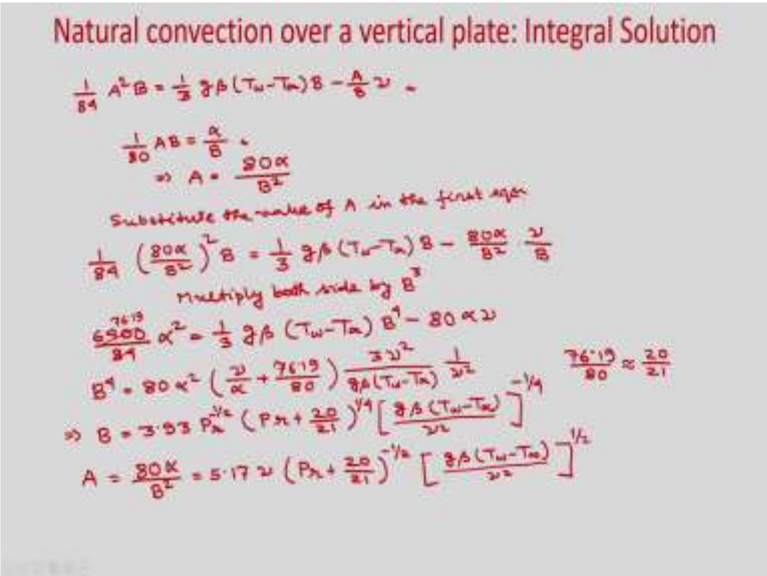

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So, now these exponents you put in those equations. After simplification you will get $\frac{1}{84}A^2B=\frac{1}{3}g\beta\left(T_w-T_\infty\right)B-\frac{A}{B}\nu$ and if you simplify it, so you will get; so, this is one equation you will get an another equation you will get $\frac{1}{80}AB=\frac{\alpha}{B}$. So, what we are doing? So, we are substituting the value of m and n in this equation and in this equation.

So, from the first equation you are getting this and from the second equation you are getting this. So, from here you can write $A=\frac{80\alpha}{B^2}$ and this you substituted in the first equation. So, what you will get? So, substitute the value of A in the first equation, top equation. So, what you will get? $\frac{1}{84}\left(\frac{80\alpha}{B^2}\right)^2 B=\frac{1}{3}g\beta\left(T_w-T_\infty\right)B-\frac{80\alpha}{B^2}\frac{\nu}{B}$.

So, what you do? You multiply both side by B^3. So, if you rearrange it, so you will get $\frac{6400}{84}\alpha^2 = \frac{1}{3} g\beta(T_w - T_\infty) B^4 - 80\alpha\nu$. So, from here you will find that B^4; so, if you see it will come almost 76.19.

So, we are writing 76.19. So, $B^4 = 80\alpha^2 \left(\frac{\nu}{\alpha} + \frac{76.19}{80}\right) \frac{3\nu^2}{g\beta(T_w - T_\infty)} \frac{1}{\nu^2}$. After rearrangement you will get these and this $\frac{76.19}{80} \approx \frac{20}{21}$.

So, we can write the value of $B = 3.93 \Pr^{-1/2} \left(\Pr + \frac{20}{21}\right)^{1/4} \left[\frac{g\beta(T_w - T_\infty)}{\nu^2}\right]^{-1/4}$. So, now, you will be able to find the value of $A = \frac{80\alpha}{B^2}$.So, B value substitute and rearrange you will get $A = 5.17\nu \left(\Pr + \frac{20}{21}\right)^{-1/2} \left[\frac{g\beta(T_w - T_\infty)}{\nu^2}\right]^{1/2}$. So, now you know the value of A and B, m and n; so, you will be able to write the velocity profile and the boundary layer thickness δ.

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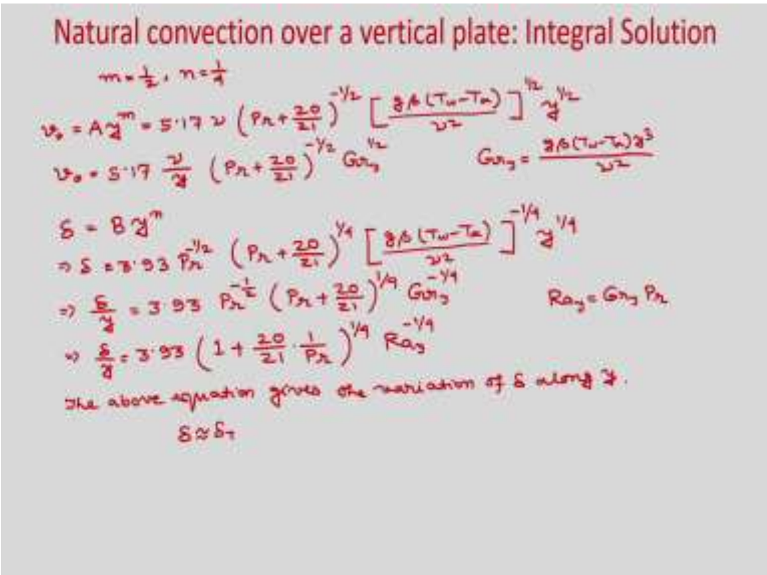

We have m =1/2, n =1/4 the expression of A, B. So, you will be able to find the velocity, $v_0 = Ay^m$. So, if you put all these values and you will get, $5.17\nu\left(\Pr+\frac{20}{21}\right)^{-1/2}\left[\frac{g\beta(T_w-T_\infty)}{\nu^2}\right]^{1/2} y^{1/2}$.

So, now what we will do? You can see if we put inside here y^3 then it will represent the Grashof number $Gr_y = \frac{g\beta(T_w-T_\infty)y^3}{\nu^2}$, so that will give you the non dimensional number Grashof number. So, we will put inside y^3, so it will be and we will subtract from here. So, after rearrangement you will get $v_0 = 5.17\frac{\nu}{y}\left(\Pr+\frac{20}{21}\right)^{-1/2} Gr_y^{1/2}$.

Where $Gr_y = \frac{g\beta(T_w-T_\infty)y^3}{\nu^2}$ and we have $\delta = By^n$.

So, $\delta = 3.93\Pr^{-1/2}\left(\Pr+\frac{20}{21}\right)^{1/4}\left[\frac{g\beta(T_w-T_\infty)}{\nu^2}\right]^{-1/4} y^{1/4}$.

And similar way here we will also put inside y^3 and we will rearrange and we will get $\frac{\delta}{y} = 3.93\Pr^{-1/2}\left(\Pr+\frac{20}{21}\right)^{1/4} Gr_y^{-1/4}$. So, Prandtl number and into Grashof number what it will give? It will give Rayleigh number . So, $Ra_y = Gr_y\Pr$. So, if you write it you will get, $\frac{\delta}{y} = 3.93\left(1+\frac{20}{21}\frac{1}{\Pr}\right)^{1/4} Ra_y^{-1/4}$.

So, you can see this equation will give you the boundary layer thickness along Y. So, the above equation gives the variation of δ along Y and if you remember we have assumed $\delta \approx \delta_T$, but we have written the expression in terms of Prandtl number .

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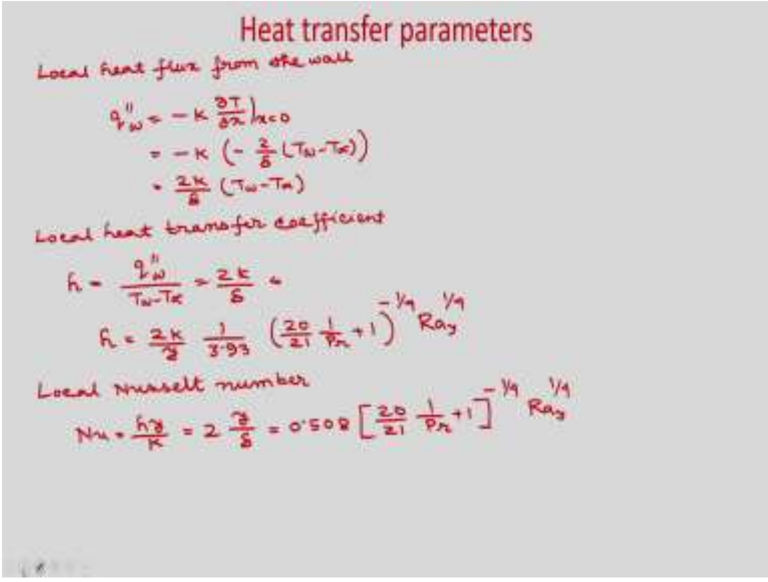

So, now we will find the heat transfer parameter. So, we will find the local heat flux then heat transfer coefficient and the Nusselt number. So, let us write the local heat flux from the wall.

So, we have $q_w^{"} = -K \frac{\partial T}{\partial x}|_{x=0}$ and we have shown that you will get, $q_w^{"} = -K \frac{\partial T}{\partial x}|_{x=0} = -K\left(-\frac{2}{\delta}(T_w - T_\infty)\right)$. So, this we will get $\frac{2K}{\delta}(T_w - T_\infty)$. So, the local heat transfer coefficient you can write $h = \frac{q_w^{"}}{T_w - T_\infty}$. So, this is from Newton's law of cooling. So, you will get h= $\frac{2K}{\delta}$.

So, δ expression you know. So, if you put the value then you will get $h = \frac{2K}{y}\frac{1}{3.93}\left(\frac{20}{21}\frac{1}{\Pr}+1\right)^{-1/4} Ra_y^{1/4}$. So, this we are writing from this expression. So, you can see we have δ(y). So, this expression, from this expression we are writing the value of heat transfer coefficient.

Now, you can write the local Nusselt number . So, you will get $Nu = \frac{hy}{K}$ so, it will be $2\frac{y}{\delta}$. So, if you put this value of $\frac{\delta}{y}$, then you will get $Nu = 0.508\left(\frac{20}{21}\frac{1}{\Pr}+1\right)^{-1/4} Ra_y^{1/4}$.

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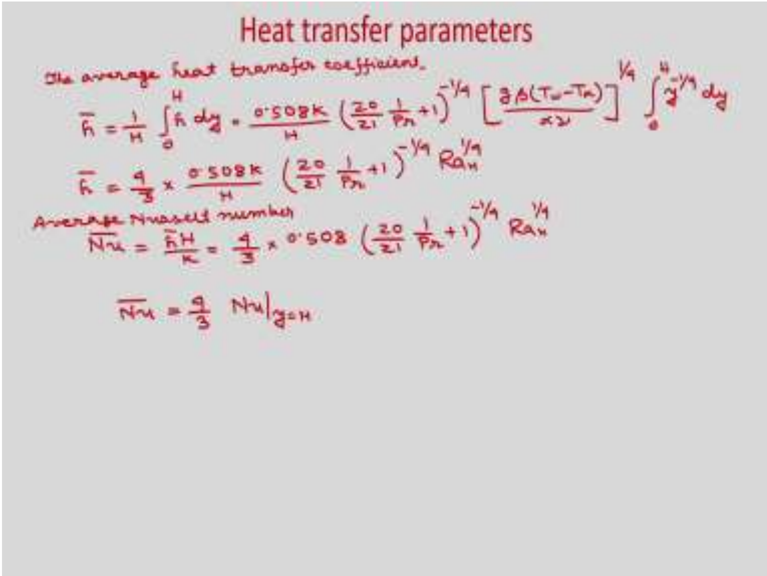

So, now the average heat transfer coefficient; the average heat transfer coefficient you can write as $\bar{h} = \frac{1}{H}\int_0^H h dy$ and put the values then you will get,

$$\frac{0.508K}{H}\left(\frac{20}{21}\frac{1}{\Pr}+1\right)^{-1/4}\left[\frac{g\beta\left(T_w - T_\infty\right)}{\alpha\nu}\right]^{1/4}\int y^{-1/4}dy .$$

So, if you integrate it and rearrange you will get $\bar{h} = \frac{4}{3}\times\frac{0.508K}{H}\left(\frac{20}{21}\frac{1}{\Pr}+1\right)^{-1/4} Ra_H^{1/4}$. So, now, Nusselt number, average Nusselt number you can write $\overline{Nu} = \frac{\bar{h}H}{K}$ so, you will get, $\frac{4}{3}\times 0.508\left(\frac{20}{21}\frac{1}{\Pr}+1\right)^{-1/4} Ra_H^{1/4}$. From here you can see the $\overline{Nu} = \frac{4}{3}Nu\big|_{y=H}$.

So, from here you can see that your $\overline{Nu} = \frac{4}{3}Nu\big|_{y=H}$. So, now we will discuss about two limiting cases; one is the $\Pr \to 0$ and another is $\Pr \to \infty$. You know when $\Pr << 1$ your

Nusselt number will be function of Rayleigh number and Prandtl number and when $Pr \to \infty$ that means when you have a high Prandtl number fluids then your Nusselt number will be function of only Rayleigh number. So, already we have found the value of $\frac{\delta}{y}$ first let us write that K expression.

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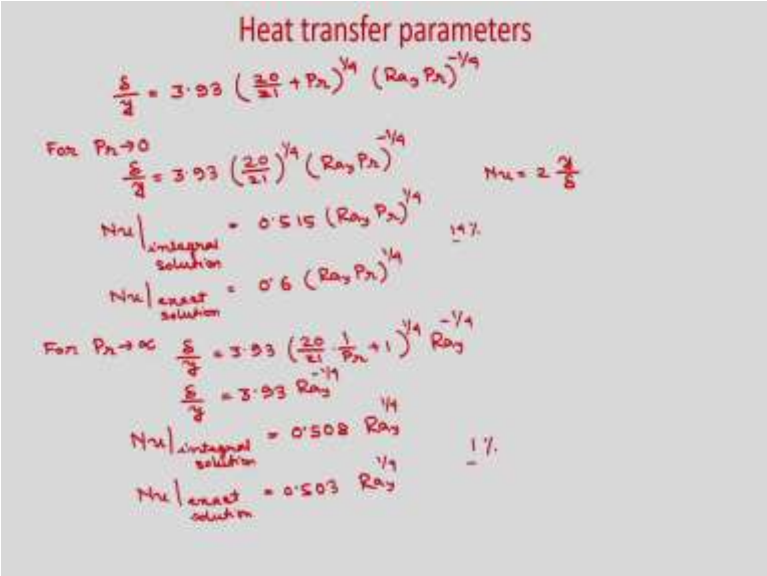

So, $\frac{\delta}{y} = 3.93\left(\frac{20}{21} + \Pr\right)^{1/4} \left(Ra_y \Pr\right)^{-1/4}$ So, now, consider the limiting case for $\Pr \to 0$. So, if $\Pr \to 0$, you can see this term you can put 0.

So, you can write $\frac{\delta}{y} = 3.93\left(\frac{20}{21}\right)^{1/4} \left(Ra_y \Pr\right)^{-1/4}$. So, from here now you can write the expression of Nusselt number .You know $Nu = 2\frac{y}{\delta}$. So, these expression if you put local Nusselt number now, you can write from the integral solution. After rearranging, you will get $Nu\big|_{\substack{integral\\solution}} = 0.515\left(Ra_y \Pr\right)^{1/4}$.

And if you remember these Nusselt number from the exact solution from the similarity solution we have written $Nu\big|_{\substack{exact\\solution}} = 0.6\left(Ra_y \Pr\right)^{1/4}$ and for $\Pr \to \infty$ another limiting case we can write the expression of $\frac{\delta}{y} = 3.93\left(\frac{20}{21}\frac{1}{\Pr}+1\right)^{1/4} Ra_y^{-1/4}$. So, as Pr→ ∞, you can see these term will become 0, because 1 /Pr is there.

So, we can write $\frac{\delta}{y} = 3.93 Ra_y^{-1/4}$; obviously, now $Nu = 2\frac{y}{\delta}$. So, from integral solution, if you write it will get $Nu\big|_{\substack{integral\\ solution}} = 0.508 Ra_y^{1/4}$ and you can see for high Prandtl number fluids, it depends on only the Rayleigh number and if you remember we have written the exact solution from the similarity solution as $Nu\big|_{\substack{exact\\ solution}} = 0.503 Ra_y^{1/4}$.

So, you can see here and this is almost 1 % variation, for Pr→ ∞there is a 1 % variation and maximum variation occurs as Pr→ 0; so, these around 14 % you will get. So, in these two limiting cases you can see these variation of this Nusselt number between the integral solution and the exact solution varies between 1 % to 14 %.

So, you can see although, we have assumed that δ ~ δ_T, but still this gives a reasonable good solution for the different Prandtl numbers.

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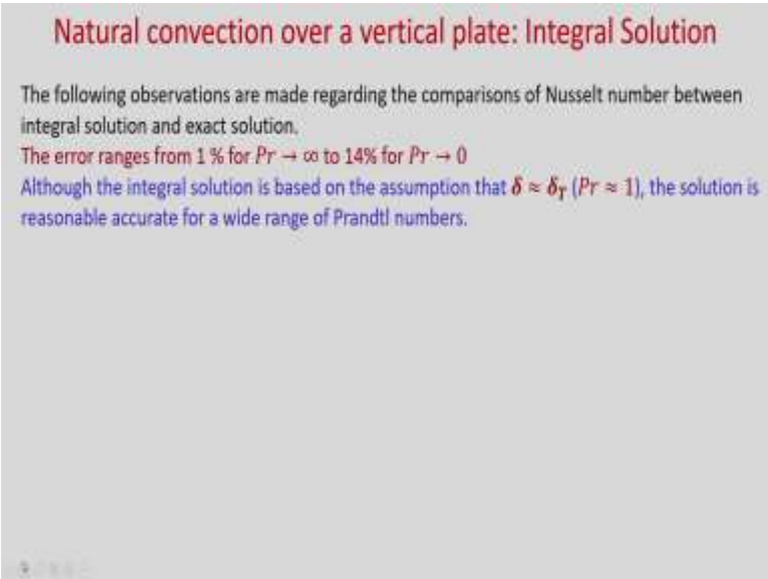

So, the observation whatever we have made you can see the error ranges from 1 % for Pr→ ∞ to 14 % for Pr→ 0 and although the integral solution is based on the assumptions that δ ≈ δ_T the solution is reasonable, accurate for a wide range of Prandtl numbers.

So, in today's class we have solved the boundary layer equations using integral method. So, first we integrated the momentum equation and the energy equation and we have written the momentum integral equation and energy integral equation, then we have assumed the velocity profile and temperature profile and invoking those expression in the integral equation.

We have written two ordinary differential equations then we have written the expression for δ and v_0 and we have put in the ordinary differential equation, and we have found the value of $\frac{\delta}{y}$ and from there we have found the heat transfer parameters like heat transfer coefficient Nusselt number.

And we have also shown the two limiting cases where $Pr \to 0$ and $Pr \to \infty$ and we have shown that in these limiting cases your error of integral solution compared to the exact solution varies between 1 % to 14 %.

Thank you.