Fundamentals of Convective Heat Transfer
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Module-08
Natural Convection -I
Lecture – 28
Natural convection over a vertical plate: Similarity solution of energy equation

Hello everyone. So now, we are studying laminar Natural convection over a vertical plate. In last class, we started with the momentum equation and using similarity transformation method, we converted the PD to OD and we discussed about the boundary conditions. Now, let us consider energy equation and we will use similarity transformation method to convert this PD to OD.

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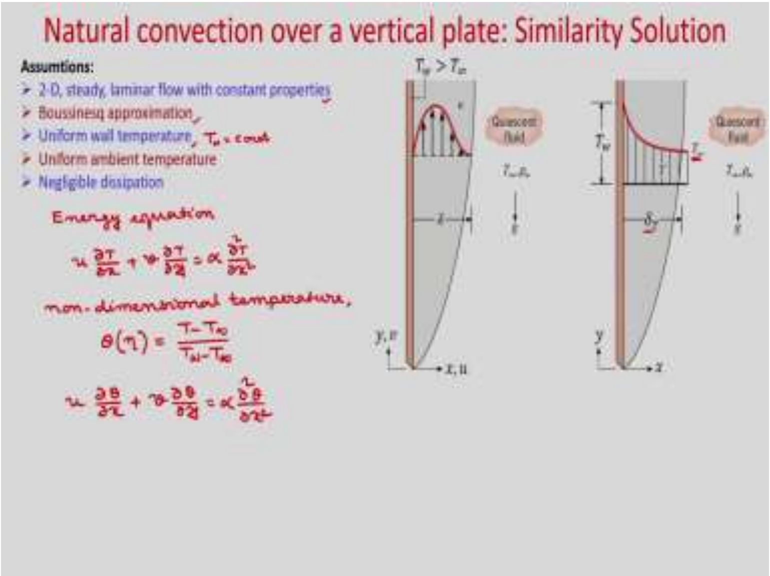

So, these are the assumptions we have already discussed in last class; 2 dimensional steady; laminar flow with constant properties, Boussinesq approximation is valid. We are considering uniform wall temperature; that means T_w is constant and uniform ambient temperature, $T\infty$ is also constant and we are neglecting the viscous dissipation. So, we can write the energy equation as $u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial x^2}$.

So, we are neglecting the viscous dissipation, and this is boundary layer energy equation. And you can see here δ_T is your thermal boundary layer thickness, and how the temperature is varying as $T_w > T_\infty$. So, temperature is maximum at the wall and gradually it is baring to the free stream where you have quiescent fluid temperature T_∞.

Now, let us consider non dimensional temperature $\theta(\eta) = \dfrac{T - T_\infty}{T_w - T_\infty}$. So, now, if you write this equation in terms of non dimensional temperature then you can write as; $u\dfrac{\partial \theta}{\partial x} + v\dfrac{\partial \theta}{\partial y} = \alpha \dfrac{\partial^2 \theta}{\partial x^2}$.

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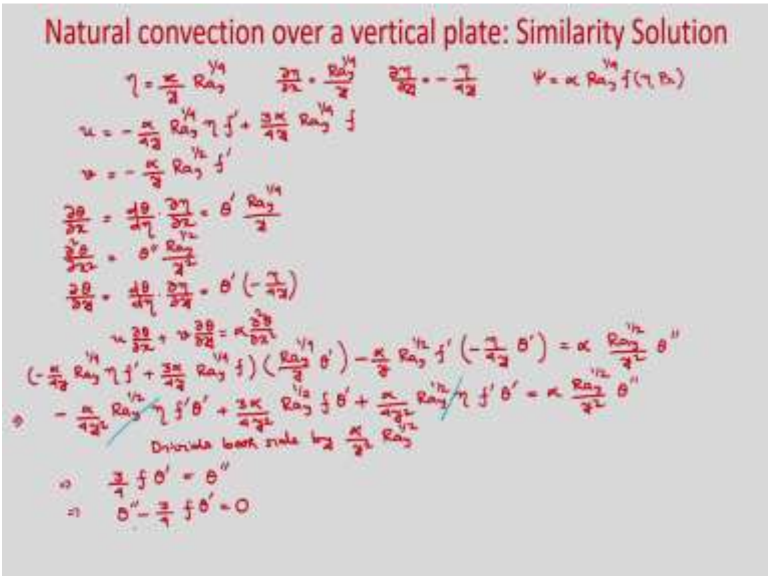

Now we will use the similarity variable eta and will convert this PD to OD.

So, in last class we have already shown that $\eta = \dfrac{x}{y} Ra_y^{1/4}$, and we know $\dfrac{\partial \eta}{\partial x} = \dfrac{Ra_y^{1/4}}{y}$ and $\dfrac{\partial \eta}{\partial y} = -\dfrac{\eta}{4y}$. This already we have derived in last class. And Ψ, the stream function; also we have defined as $\psi = \alpha Ra_y^{1/4} f\left(\eta, \Pr\right)$.

So now, you can write the velocity u and v as $u = -\frac{\alpha}{4y} Ra_y^{1/4} \eta f' + \frac{3\alpha}{4y} Ra_y^{1/4} f$. So, this we have derived in last class and velocity $v = -\frac{\alpha}{y} Ra_y^{1/2} f'$.

Now, let us find the derivative of θ with respect to x and y. So, you can see $\frac{\partial \theta}{\partial x} = \frac{d\theta}{d\eta} \frac{\partial \eta}{\partial x} = \theta' \frac{Ra_y^{1/4}}{y}$.

And $\frac{\partial^2 \theta}{\partial x^2} = \theta'' \frac{Ra_y^{1/2}}{y^2}$. And $\frac{\partial \theta}{\partial y} = \frac{d\theta}{d\eta} \frac{\partial \eta}{\partial y} = \theta' \left(-\frac{\eta}{4y} \right)$.

So now, we have energy equation $u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \alpha \frac{\partial^2 \theta}{\partial x^2}$, if you write $\left(-\frac{\alpha}{4y} Ra_y^{1/4} \eta f' + \frac{3\alpha}{4y} Ra_y^{1/4} f \right) \left(\frac{Ra_y^{1/4}}{y} \theta' \right) - \frac{\alpha}{y} Ra_y^{1/2} f' \left(-\frac{\eta}{4y} \theta' \right) = \alpha \frac{Ra_y^{1/2}}{y^2} \theta''$. So, if you multiply, then you will get $-\frac{\alpha}{4y^2} Ra_y^{1/2} \eta f' \theta' + \frac{3\alpha}{4y^2} Ra_y^{1/2} f \theta' + \frac{\alpha}{4y^2} Ra_y^{1/2} \eta f' \theta' = \alpha \frac{Ra_y^{1/2}}{y^2} \theta''$.

So you can see here, here you can write, so these two terms you can see, this will cancel out. This term and this term will cancel out; Now divide both sides by, $\frac{\alpha}{y^2} Ra_y^{1/2}$; now you will get, $\frac{3}{4} f \theta' = \theta''$. So, you can see this is your equation $\theta'' - \frac{3}{4} f \theta' = 0$.

So you can see, this is ordinary differential equation. This is second order linear ordinary differential equation and you need the velocity distribution from the solution of momentum equation. Because here (Refer Time: 9:49) is there so; obviously, that will be your you can get the solution from the solution of momentum equation. But you can see, these both equations are coupled, because in the momentum equation temperature term is there θ, and in energy equation you have the term f which you will get from the velocity distribution. So, in the coupled way with the proper boundary condition you need to solve.

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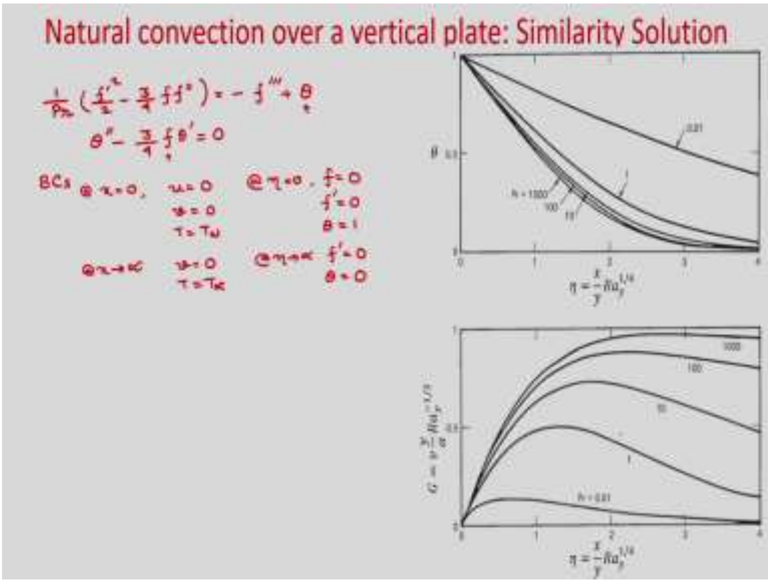

So you can see the whatever we have derived, so we have derived the momentum equation as $\frac{1}{\Pr}\left(\frac{f'^2}{2}-\frac{3}{4}ff''\right)=-f'''+\theta$. So, you will get the velocity profile from this equation and the energy equation is $\theta''-\frac{3}{4}f\theta'=0$.

So you can see, in this equation θ is there and in this equation f is there. So, these are coupled, so together you need to solve and get the velocity distribution from the first equation and temperature distribution from the second equation. And what are the boundary conditions?

So, already we have discussed for velocity at x = 0, you have u = 0; that means, at η=0 you will get f = 0 and v = 0. You will get at η= 0 f'=0. And what is temperature? Temperature T is T_w; so you will get tθas 1. And at x →∞; that means, at the edge of the boundary layer, you will get v = 0.

So, you will get at η→∞, f'=0 and T will be T_∞, so at η→∞, you will get θas 0. So, with suitable numerical integration technique you can solve these two; ordinary differential equation with given boundary condition. And you can find the velocity distribution as well as the temperature distribution. Once you get that, then you will be able to calculate the heat flux, heat transfer coefficient and Nusselt number.

So, you can see the solution of temperature ah, so you can see theta versus eta it is plotted for different Prandtl number. So, you can see as Pr→ ∞. All the temperature profile collapse into a single curve.

You can see here, and this is the solution of G, which actually gives the velocity versus eta. So, you can see that velocity v is actually we have written in terms of G, so this will give the velocity distribution; and here also you can see as η→ ∞, so or you can see that as Prandtl number increases because it is Prandtl number 0.01, 1, 10, 100 and 1000.

So, as Prandtl number increases, the velocity profile extends farther and farther into the isothermal fluid . So, once you know the temperature profile and the velocity profile, you will be able to calculate the heat flux and heat transfer coefficient and hence Nusselt number. So, let us find these parameters.

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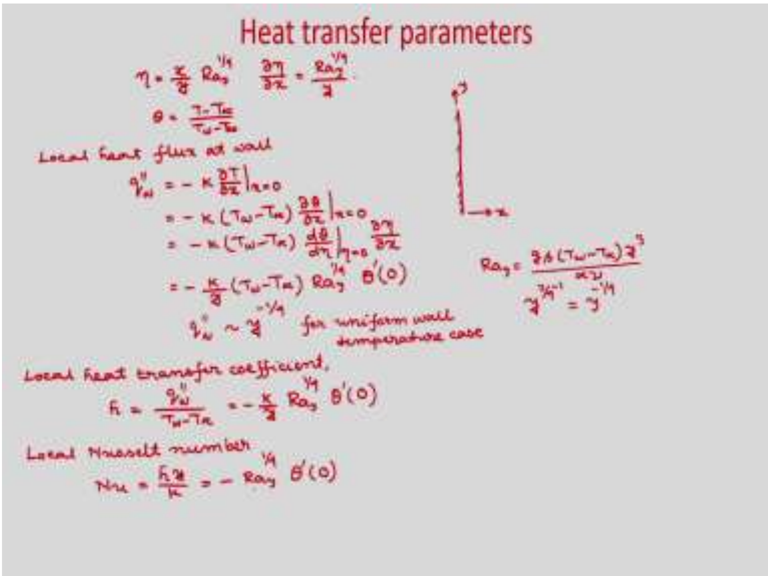

So, we know that $\eta = \frac{x}{y} Ra_y^{1/4}$ and $\frac{\partial \eta}{\partial x} = \frac{Ra_y^{1/4}}{y}$.

And $\theta = \frac{T - T_\infty}{T_w - T_\infty}$. So, now, you can see, we can write the local heat flux. So $q_w^{"}$, local heat flux at wall. So, $q_w^{"} = -K \frac{\partial T}{\partial x}|_{x=0}$. Because if you have a vertical plate and this is your x and this is your y so; obviously, normal gradient is $\frac{\partial T}{\partial x}|_{x=0}$ you need to find the heat flux.

So, now you can write $-K\left(T_w - T_\infty\right)\frac{\partial \theta}{\partial x}|_{x=0}$. So, now we can write $-K\left(T_w - T_\infty\right)\frac{d\theta}{d\eta}|_{\eta=0}\frac{\partial \eta}{\partial x}$ and $\frac{\partial \eta}{\partial x}$ this you can write. So you can write $-\frac{K}{y}\left(T_w - T_\infty\right)Ra_y^{1/4}\theta'(0)$.

So, can you tell how this heat flux varies with y. So you can see from this equation. So, in the Rayleigh number expression, what is the Rayleigh number expression? $Ra_y = \frac{g\beta\left(T_w - T_\infty\right)y^3}{\alpha\nu}$. So, in the Rayleigh number you see y^3, so $Ra_y^{1/4}$ means $y^{3/4}$.

And, here in the denominator y is there. So, it will be -1. So, you can see it will be $y^{-1/4}$. So, $q_w^{''} \sim y^{-1/4}$ for uniform wall temperature case.

Now, once you know the heat flux at the wall, you will able to calculate the local heat transfer coefficient because local heat transfer coefficient, you can calculate from the Newton's law of cooling and h will be your q double prime w divided by the temperature difference.

So, local heat transfer coefficient . So, from Newton's law of cooling you can write $h = \frac{q_w^{''}}{T_w - T_\infty}$. So, from this expression directly you can write $-\frac{K}{y}Ra_y^{1/4}\theta'(0)$.

So once you know the temperature gradient at the wall, you will be able to calculate the heat transfer coefficient. Now, you can calculate the local Nusselt number. So, $Nu = \frac{hy}{K}$. So, from this expression you can see it will be $-Ra_y^{1/4}\theta'(0)$.

So, from the solution of the ordinary differential equation, once you get the temperature gradient at the wall which is function of Prandtl number as well, then you will be able to write the Nusselt number.

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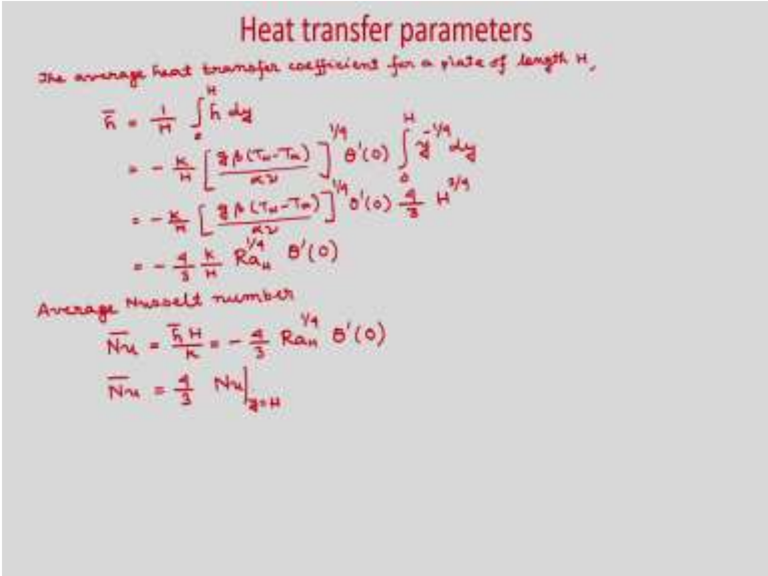

Now, similarly you can calculate the average heat transfer coefficient and average Nusselt number. So, the average heat transfer coefficient for a plate of length H, here vertical plate so the height of the plate is H.

You can write $\bar{h} = \frac{1}{H}\int_0^H h dy$. Where h is the; local heat transfer coefficient; so that we have already found. So, you can write $-\frac{K}{H}\left[\frac{g\beta(T_w - T_\infty)}{\alpha\nu}\right]^{1/4}\theta'(0)\int_0^H y^{-1/4}dy$.

So, if you integrate it then you will get $-\frac{K}{H}\left[\frac{g\beta(T_w - T_\infty)}{\alpha\nu}\right]^{1/4}\theta'(0)\frac{4}{3}H^{3/4}$.

So, this H^3 you insert in this bracket, so that you can get back the Rayleigh number. So, if you write that, then you can write $-\frac{4}{3}\frac{K}{H}Ra_H^{1/4}\theta'(0)$.

So, hence you can write average Nusselt number. So, average Nusselt number you can write as $\overline{Nu} = \frac{\bar{h}H}{K}$.

So, you can write $-\frac{4}{3}Ra_H^{1/4}\theta'(0)$. So, in this case you can see, the $\overline{Nu} = \frac{4}{3}Nu\big|_{y=H}$.

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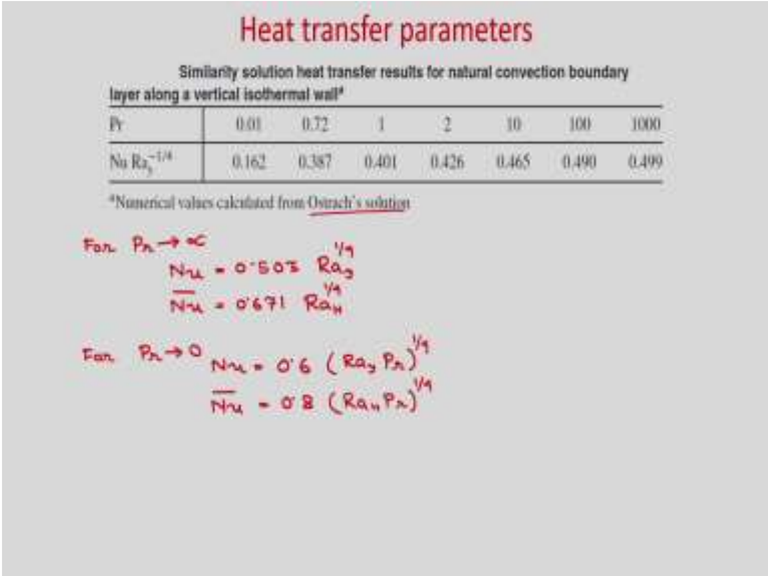

So, you can see the similarity solution heat transfer results for natural convection, boundary layer along a vertical isothermal wall. So, $NuRa_y^{-1/4}$, where Nusselt number is the local Nusselt number. So, for different Prandtl number, these are the values.

So, this is obtained from the Ostrach's solution. So, once you know these values then you will be able to calculate the Nusselt number.

So, at the extreme case for Pr $\to$ ∞, you can write Nusselt number, $Nu = 0.503 Ra_y^{1/4}$ and $\overline{Nu} = 0.671 Ra_H^{1/4}$.

And for Pr $\to$ 0 you can write $Nu = 0.6\left(Ra_y \Pr\right)^{1/4}$, and $\overline{Nu} = 0.8\left(Ra_H \Pr\right)^{1/4}$.

So you can see here, that in this case you have for Pr $\to$ ∞; that means, the case where Prandtl number is very high so; obviously, from the scale analysis we have shown that $Nu \sim Ra_y^{1/4}$.

And, when Pr < 1 then we have shown from the scale analysis that Rayleigh that $\left(Ra_y \Pr\right)^{1/4}$ and from this expression you can see that.

So, till now we considered uniform wall temperature case so; that means, T_w is constant. We can also consider where in some applications you can have the uniform wall heat flux condition. Like, when the solar radiation falls on a body then you will get uniform wall temperature case and in industrial application also we can have use of this uniform wall heat flux case.

So, the simplified case we will take. So we will consider again, vertical flat plate keeping the other assumptions constant, only we will assume that heat flux at the wall will be kept constant.

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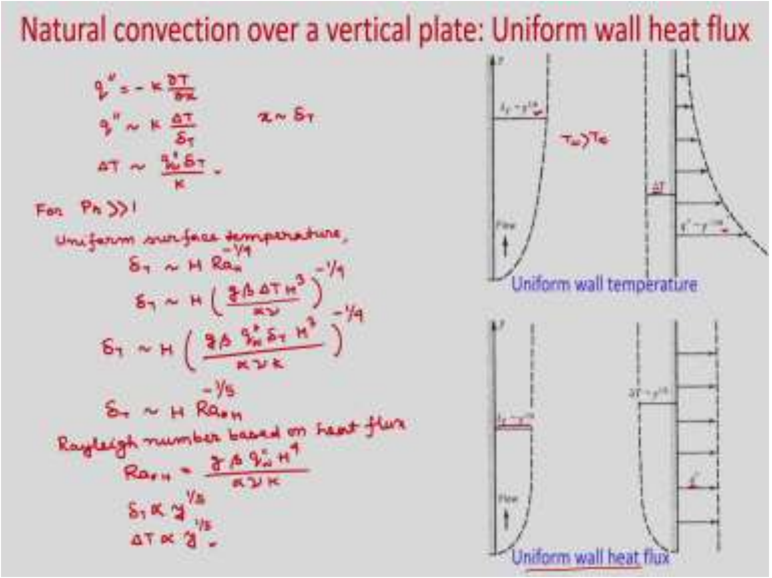

So, you can see that from the uniform wall temperature case, already we have shown $\delta_T \sim y^{1/4}$. So, it is a vertical wall, $T_w > T_\infty$ so; obviously, you will get the thermal boundary layer like this and your $\delta_T \sim y^{1/4}$. And, also we have shown that as your $\Delta T = T_w - T_\infty$, we are keeping constant $q'' \sim y^{1/4}$. So this also we have shown today.

Now, for uniform wall heat flux case what is the heat flux? Generally, we tell $q'' = -K\dfrac{\partial T}{\partial x}$. So, what will be the order? $q'' \sim K\dfrac{\Delta T}{\delta_T}$, because x ~$\delta_T$.

So, from here we can get the $\Delta T \sim \dfrac{q_w'' \delta_T}{K}$. Obviously, your q'' at the wall we are considering. So, this is your at the wall.

So, from here, you can see we can find for Pr> 1, we have already derived for uniform surface temperature case. What is the δ_T? $\delta_T \sim HRa_H^{-1/4}$.

So, we can write $\delta_T \sim H\left[\frac{g\beta\Delta TH^3}{\alpha\nu}\right]^{-1/4}$. So, this we have derived for the uniform wall temperature case. Now, in this expression you see the temperature difference Δ T. We will put $\Delta T \sim \frac{q_w^{"}\delta_T}{K}$.

So, you will get $\delta_T \sim H\left[\frac{g\beta q_w^{"}\delta_T H^3}{\alpha\nu K}\right]^{-1/4}$. So, from here you can see if you bring this δ_T in this side, so now, you can write after rearrangement, $\delta_T \sim HRa_{*H}^{-1/5}$.

So, you do the rearrangement and here, this Rayleigh number based on $q_w^{"}$ is defined as, $Ra_{*H} = \frac{g\beta q_w^{"}H^4}{\alpha\nu K}$.

So it is non dimensional number right. So, from this expression you can see your δ_T. So, δ_T what is the order? So, it will be $\delta_T \propto y^{1/5}$.

And what is about your δ_T. So, $\Delta T \sim \frac{q_w^{"}\delta_T}{K}$. So, $\Delta T \propto y^{1/5}$. So, you can see in this figure, for uniform wall heat flux case, when flow is taking place you can see your thermal boundary layer thickness will vary as $y^{1/5}$ whereas, for uniform wall temperature it is $y^{1/4}$.

And as it is uniform wall heat flux condition. Your $\Delta T \propto y^{1/5}$. So, that we have. So now, and for uniform wall temperature ΔT is constant and $q" \sim y^{1/4}$. So, now, we can see; what is the order of Nusselt number in case of uniform wall heat flux case. So, δ_T we have already found.

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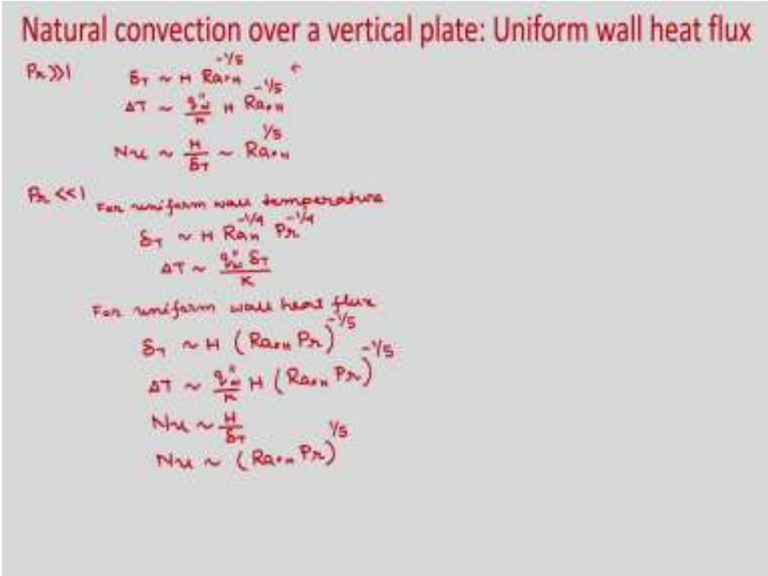

So, $\delta_T \sim HRa_{*H}^{-1/5}$ and this is the case for Pr>> 1 and $\Delta T \sim \frac{q_w^{"}}{K} HRa_{*H}^{-1/5}$.

And we know, we have shown earlier that $Nu \sim \frac{H}{\delta_T}$. So from this expression, the first expression you can write $Nu \sim Ra_{*H}^{1/5}$. So here, you remember that this Rayleigh number is modified Rayleigh number based on wall heat flux.

Similarly, now for Pr<<1 . We can write, for uniform wall temperature case. Already we have found $\delta_T \sim HRa_H^{-1/4} \Pr^{-1/4}$. And $\Delta T \sim \frac{q_w^{"}\delta_T}{K}$.

So, if you put these ΔT in the expression of Rayleigh number where Δ T is there, then for uniform wall heat flux case, you can write $\delta_T \sim H\left(Ra_{*H} \Pr\right)^{-1/5}$.

And Nusselt number and $\Delta T \sim \frac{q_w^{"}}{K} H\left(Ra_{*H} \Pr\right)^{-1/5}$ and $Nu \sim \frac{H}{\delta_T}$. So, $Nu \sim \left(Ra_{*H} \Pr\right)^{1/5}$.

So, we have shown the $Nu \sim \left(Ra_{*H} \Pr\right)^{1/5}$ for Pr << 1 and for high Prandtl number fluids it depends on Rayleigh number. So, $Ra_H^{1/5}$.

So, using similarity method you can solve this problem, but we will not go into details, just we will show some results for uniform wall heat flux case using similarity variable.

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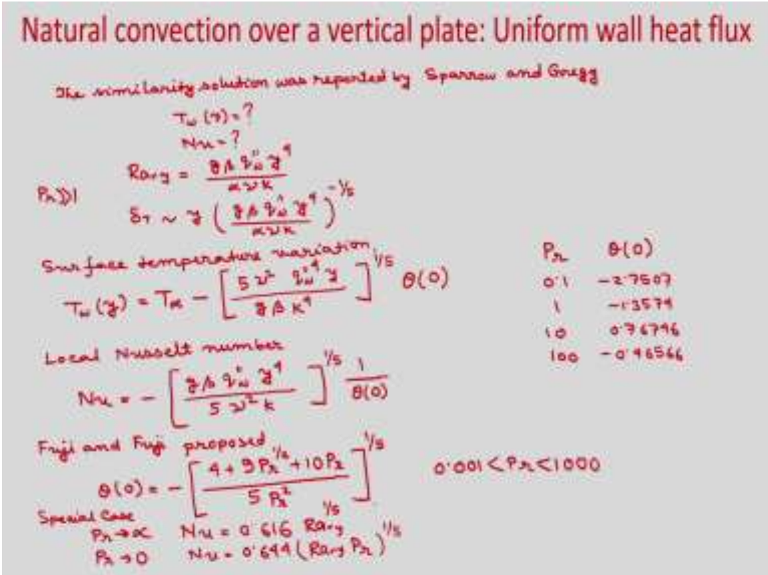

So, the similarity solution was reported by Sparrow and Gregg; So, what we need to find? We need to find T_W as a function of y and Nusselt number right. So, from here, you can see that in this case $Ra_{*y} = \dfrac{g\beta q_w^{"} y^4}{\alpha \nu K}$.

And δ_T for high Prandtl number case, we have defined as $\delta_T \sim y\left[\dfrac{g\beta q_w^{"} y^4}{\alpha \nu K}\right]^{-1/5}$.

So, from the solution the surface temperature variation is given as follows. $T_w(y) = T_\infty - \left[\dfrac{5\nu^2 q_w^{"4} y}{g\beta K^4}\right]^{1/5} \theta(0)$. $\theta(0)$ is the temperature, at η= 0. So these depend on Prandtl number. And you can see, Prandtl number for different Prandtl number what is the $\theta(0)$. So, at 0.1 you have - 2.7507, 1 it is - 1.3574, 10 it is 0.76746 and for 100 it is - 0.46566. And local Nusselt number is given by, $Nu = -\left[\dfrac{g\beta q_w^{"} y^4}{5\nu^2 K}\right]^{1/5} \dfrac{1}{\theta(0)}$.

So, in literature you will find many correlations. So, Fuji and Fuji proposed this correlations to find the temperature at θ = 0. So, Fuji and Fuji proposed, $\theta(0) = -\left[\frac{4+9\Pr^{1/2}+10\Pr}{5\Pr^2}\right]^{1/5}$.

So, this is valid in the range of $0.001 < \Pr < 1000$. And as a special case, for Prandtl number special case $\Pr \to \infty$. $Nu = 0.616 Ra_{*y}^{1/5}$ and you can see from the scale analysis also, we have found that $Nu \sim Ra^{1/5}$ and $\Pr \to 0$.

$Nu = 0.644\left(Ra_{*y}\Pr\right)^{1/5}$, this also we have shown for $\Pr < 1$ the Nusselt number varies as $\left(Ra_{*y}\Pr\right)^{1/5}$.

So, these results we have shown from the similarity solution, but using integral solutions also many researchers have proposed correlations we will just write down what Sparrow proposed.

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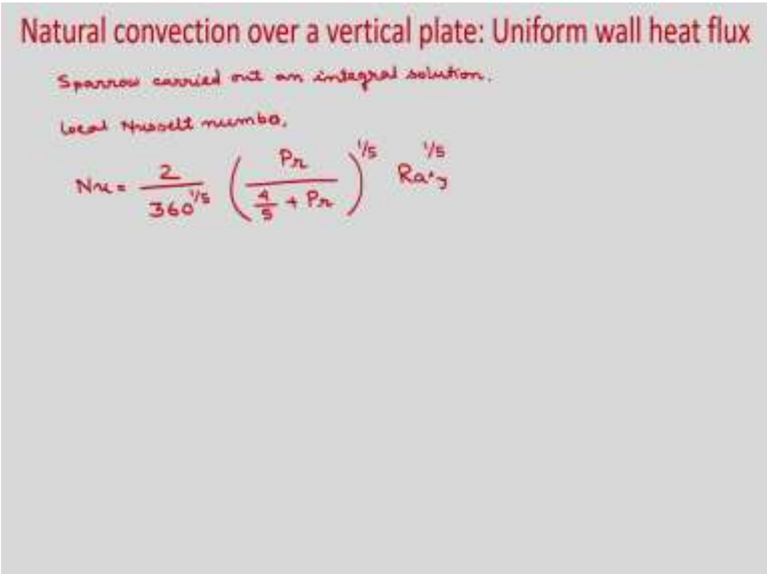

So, Sparrow carried out an integral solution and propose the local Nusselt number as, $Nu = \frac{2}{360^{1/5}}\left(\frac{\Pr}{\frac{4}{5}+\Pr}\right)^{1/5} Ra_{*y}^{1/5}$.

So, in today's class we started with the energy equation and using the similarity variable, we converted this PD to OD and we have shown the boundary conditions for both momentum and energy equations and we have shown the solution of θ and g which is your representation of velocity v versus η. And we have also shown the local heat transfer coefficient and local Nusselt number and average heat transfer coefficient and average Nusselt number for the case of uniform wall temperature case.

Later, we considered uniform wall heat flux case, and from the scale analysis we have found the order of Nusselt number and δ_T. Then we have shown some similarity solution what are available in the literature, and also we have shown the local Nusselt number expression from the integral solution of Sparrow.

Thank you.