Fundamentals of Convective Heat Transfer
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Module - 08
Natural Convection - I
Lecture - 27
Natural convection over a vertical plate: Similarity Solution

Hello everyone. So, in last class, we derived the laminar boundary layer equations for a Natural convection over a vertical plate. We derived continuity equation, y momentum equation and energy equation.

So, we have three partial differential equations. In today's class, we will use similarity method and we will convert these three partial differential equations to ordinary differential equations. Let us write down the governing equations, whatever we derived in last class in non-dimensional form.

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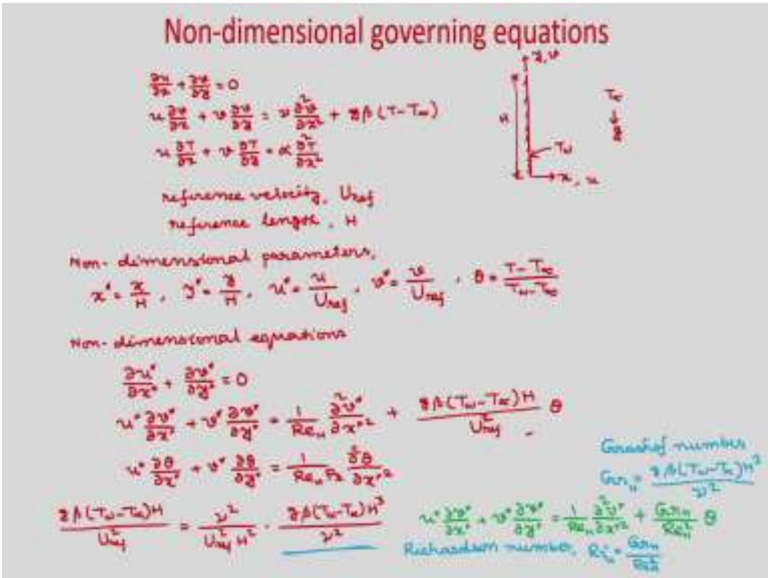

So, if you see that we have derived in last class; continuity equation is $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$; then, we derived the momentum equation as $u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\nu\frac{\partial^2 v}{\partial x^2}+g\beta\left(T-T_\infty\right)$ and also, we have Energy equation $u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial x^2}$.

So, if you know that we have consider vertical plate and this is your x and this is your y. In x direction, we have velocity u and y direction, we have velocity v. Ambient fluid is quiescent and it is having temperature T_∞.

We have gravitational acceleration in negative y direction that is g and this wall is maintained at uniform surface temperature that is your T_w. So, now, we will choose to reference scale, one is for length and one is for velocity. Let us choose that your reference velocity is U_{ref}.

We do not have any free stream velocity in this case. So, we are choosing some reference velocity that is your U_{ref} and reference length, let us choose H; where, H is the height of the fluid or you can choose any other reference length. Now, we will use this non-dimensional parameters, x* is the non-dimensional x coordinate, $x^*=\frac{x}{H}$, $y^*=\frac{y}{H}$, then $u^*=\frac{u}{U_{ref}}$, $v^*=\frac{v}{U_{ref}}$; then temperature, we will take $\theta=\frac{T-T_\infty}{T_w-T_\infty}$.

So, with this now, if you put it here and if you rearrange, you will get the non-dimensional equations. Non-dimensional equations as $\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0$, then we have $u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}=\frac{1}{\mathrm{Re}_H}\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{g\beta\left(T_w-T_\infty\right)H}{U_{ref}^2}\theta$. So, using these non-dimensional parameters, you put all those things in the momentum equation. Then, if you rearrange, you will get the equation in this form; where in the viscous term, you will get 1 by Reynolds number and the buoyancy term you will get like this.

So, we will simplify it later. First let us write the non-dimensional energy equation, $u*\frac{\partial\theta}{\partial x*}+v*\frac{\partial\theta}{\partial y*}=\frac{1}{\mathrm{Re}_H \Pr}\frac{\partial^2\theta}{\partial x*^2}$. So, if you see these term in the buoyancy term, so if you rearrange it, you will get $\frac{g\beta\left(T_w-T_\infty\right)H}{U_{ref}^2}=\frac{\nu^2}{U_{ref}^2H^2}\frac{g\beta\left(T_w-T_\infty\right)H^3}{\nu^2}$.

So, these term is now one another non-dimensional number that is known as Grashof number ; Grashof number.

So, how Grashof number is defined Grashof number is the ratio of buoyancy force to the viscous force because in the numerator, you can see this is coming from the buoyancy force and in the denominator ν is there, so it is coming from the viscous force. So, Grashof number is the ratio of buoyancy force to the viscous force. So, and it is written in this from $Gr_H=\frac{g\beta\left(T_w-T_\infty\right)H^3}{\nu^2}$.

So, Grashof number based on H . So, now, you can write this term. So, momentum equation you can write as $u*\frac{\partial v*}{\partial x*}+v*\frac{\partial v*}{\partial y*}=\frac{1}{\mathrm{Re}_H}\frac{\partial^2 v*}{\partial x*^2}+\frac{Gr_H}{\mathrm{Re}_H^2}\theta$.

So, this is your momentum equation. So, here we will define another non-dimensional number which is your Grashof number by Reynolds number square and it is known as Richardson number.

So, this is known as Richardson number and denoted as, $Ri_H=\frac{Gr_H}{\mathrm{Re}_H^2}$. So, this Richardson number has physical significance that it compares which force is dominant; buoyancy force or your inertia force.

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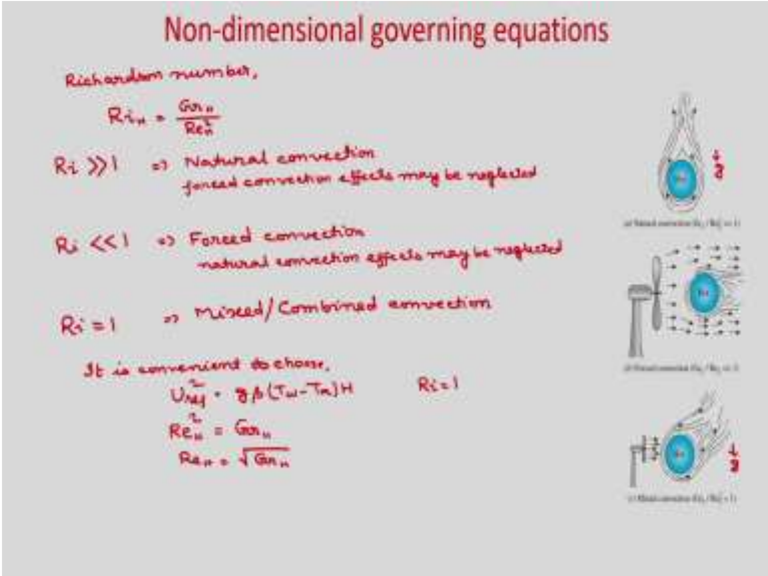

So, Richardson number ok. So, this is $Ri_H = \frac{Gr_H}{\mathrm{Re}_H^2}$. So, you can see Richardson number, so it is kind of buoyancy force to the inertia force . So, if it is so, now you can see three different cases. If Richardson number >> 1, so what will happen?

So, if Richardson number >> 1 , so now, you can see that Richardson number >> 1; that means, $Gr_H \gg \mathrm{Re}_H^2$. So, what does it mean? It means that buoyancy force will dominant.

So, in that case obviously, buoyancy force is dominating, then you can consider that it is natural convection. So that means, forced convection effects may be neglected. If Ri<<1, so in that case you can see inertia force will dominant .

So that means, it will be forced convection. So, in this case, your natural convection effects may be neglected and if Ri = 1, then what will happen? Your buoyancy force is comparable to inertia force. So, you cannot neglect either buoyancy force or inertia force.

So, you will have both natural convection and forced convection and that is known as mixed convection or combined convection. So, for Ri = 1, so you will get mixed convection which is known also as combined convection. So, in this case, you need to consider both forced and natural. So, you can see this example.

So, if natural convection is happening, say you have a sphere hot sphere, then $\frac{Gr_H}{\text{Re}_H^2} = Rz_H$ >> 1. L is the any characteristic length. So, in this case you will have only natural convection. So, due to buoyancy, you can see the you will have a flume and it will go up.

Now, if Ri << 1, then natural convection can be neglected and you have a forced convection. So, you can see this velocity is generated using some fan and over this hot sphere, this flow will takes place and heat transfer will takes place and that is your forced convection.

But if you have Ri = 1, then you will have both natural convection and buoyancy effect will be there. So, in this case, g is there and also in this case g is there. So obviously, you can see you will have buoyancy effect. So, this flume will go up, but you have forced convection also in the x direction. So, you will see the flume is go up in a inclined manner. So, it is known as mixed convection.

Now from here, you can see that if you choose the velocity scale in such a way that the coefficient in the buoyancy term in the θ will become 1. So, in that case, you can see because reference velocity is arbitrary right. So, it can be chosen to simplify the form of the equation.

So, it is convenient to choose; convenient to choose $U_{ref}^2 = g\beta\left(T_w - T_\infty\right)H$ so that your the term with the θ, so you can see this term. So, if $U_{ref}^2 = g\beta\left(T_w - T_\infty\right)H$, then this will become 1. If it becomes 1; then obviously, it is kind of Richardson number will become 1.

And for this case you can see your $\text{Re}_H^2 = Gr_H$ right. That is we have already defined because this is your ratio right. $Rz_H = \frac{Gr_H}{\text{Re}_H^2}$ and $\text{Re}_H = \sqrt{Gr_H}$. So, you can see that Grashof number place the role same role in the natural convection, what Reynolds number place the role in forced convection right.

Now, let us use similarity transformation method and convert this three partial differential equation to ordinary differential equation so that we can have the numerical solution. Now, you can see in last class, we have found scales for different quantities like thermal boundary layer thickness, then velocity v and also, Nusselt number and those are tabulated in this table.

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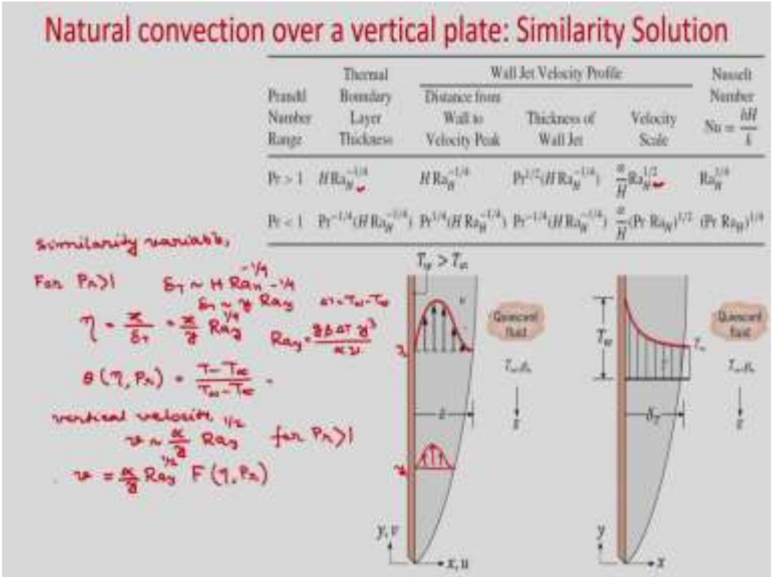

So, you can see for Prandtl number >> 1, the boundary layer thickness $\delta_T \sim HRa_H^{-1/4}$. And also, velocity scale $\dfrac{\alpha}{H}Ra_H^{1/2}$ and $Nu_H \sim Ra_H^{1/4}$ and you can see that the velocity looks like this.

So, because at the wall, you will have 0 velocity and at the edge of the boundary layer, we will have 0 velocity. So, your maximum velocity occur in between this boundary layer and this is the known as wall jet velocity profile.

So, you can see at different location, you will get this type of profile and if you can say if you see at this location; let us say it is y_1 and this is your y_2. So, you will get some profile like this and you can see that if you scale down this velocity profile with a proper scale, then you can bring down to this profile.

So that means, if somehow we can convert this partial differential equation to ordinary differential equation using some scale factor, then your similarity transformation exists and we can write this partial differential equation to ordinary differential equation.

So, now how to choose these similarity variable? So, you here in that table, you can see that we have this $\delta_T \sim HRa_H^{-1/4}$. So, this scale we can take for the similarity variable eta and we have also velocity scale $\frac{\alpha}{H} Ra_H^{1/2}$.

So, what we can do? You can so, this similarity variable η; similarity variable $\eta = \frac{x}{\delta_T}$. So, we can write for Prandtl number > 1 , so thermal boundary layer thickness you can see $\delta_T \sim HRa_H^{-1/4}$.

So, we can write $\eta = \frac{x}{\delta_T}$. So, you can write; so, this $\delta_T \sim yRa_y^{-1/4}$. So, in this case we can write $\eta = \frac{x}{\delta_T} = \frac{x}{y} Ra_y^{1/4}$. So, it is a in the numerator we have written. So, we are equally depend + 1/4.

So, here $Ra_y = \frac{g\beta\Delta T y^3}{\alpha\nu}$. So, now, you can see $\theta(\eta, \Pr) = \frac{T - T_\infty}{T_w - T_\infty}$. So, in this case, you can see this is the temperature profile and this also you can use some scale factor so that it will become same or all the temperature profile will collapse, if you plot it with η.

So, theta you can write this way and we know the vertical velocity scale as $v \sim \frac{\alpha}{y} Ra_y^{1/2}$ for Pr > 1. This we are considering. So, this we can write velocity scale $v = \frac{\alpha}{y} Ra_y^{1/2} F(\eta, \Pr)$. So, this is your dimensionless similarity profile of the wall jet. So, this scale we can use and we can now find what is the value of Ψ from this expression .

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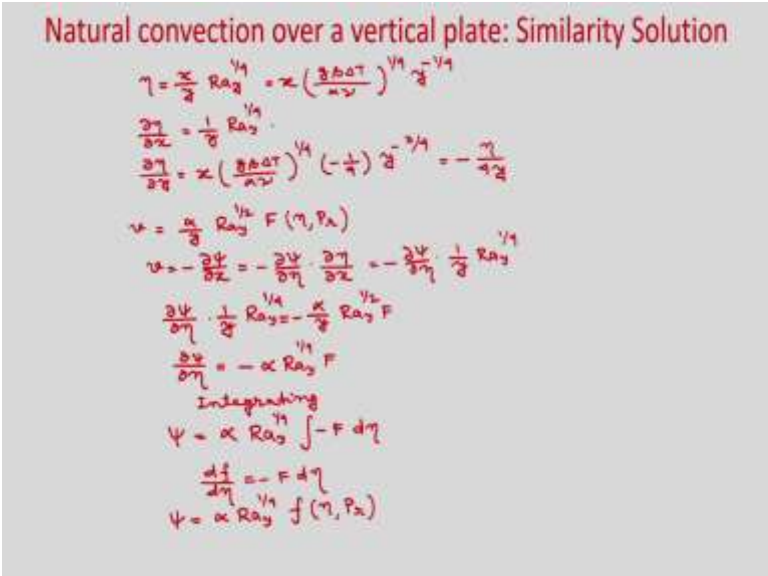

So, now we will use $\eta = \frac{x}{y} Ra_y^{1/4}$ and if you put the value of Ra_y, then you can write in terms of y as , $x\left(\frac{g\beta\Delta T}{\alpha\nu}\right)^{1/4} y^{-1/4}$. So, now, you can find $\frac{\partial\eta}{\partial x} = \frac{1}{y} Ra_y^{1/4}$ and $\frac{\partial\eta}{\partial y} = x\left(\frac{g\beta\Delta T}{\alpha\nu}\right)^{1/4} \left(-1/4\right) y^{-5/4}$. So, now, you rearrange it and then, you can simplify it and you will get $\frac{\partial\eta}{\partial y} = -\frac{\eta}{4y}$ after simplification.

Now, we have v. So, $v = \frac{\alpha}{y} Ra_y^{1/2} F\left(\eta, \Pr\right)$. Now, we will find the stream function from this velocity v and once stream function is known, then you will be able to calculate the velocity u and velocity gradient like $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ and $\frac{\partial^2 v}{\partial x^2}$.

So, now, let us find what is stream function Ψ. So, you can define $v = -\frac{\partial\psi}{\partial x}$. So, if you put it here, you will get; so, this you can write $-\frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial x}$. So, $\frac{\partial\eta}{\partial x}$ already we have found here. So, you can write $-\frac{\partial\psi}{\partial\eta}\frac{1}{y} Ra_y^{1/4}$.

So, now, if you put these. So, you can find $\frac{\partial \psi}{\partial \eta}\frac{1}{y}Ra_y^{1/4} = -\frac{\alpha}{y}Ra_y^{1/2}F$. So, now if it is so, now we can write $\frac{\partial \psi}{\partial \eta} = -\alpha Ra_y^{1/4}F$. Now you can integrate it. So, integrating you can write $\psi = \alpha Ra_y^{1/4}\int -Fd\eta$ and the integration constant is put inside this integral.

So, now, we will define a new variable small f. So, that your $\frac{df}{d\eta} = -Fd\eta$. So, if it is so, now you can write $\psi = \alpha Ra_y^{1/4} f\left(\eta, \text{Pr}\right)$.

So, you can see from here F will be just $-Fd\eta$ integration constant is involved here. So, you can write $f\left(\eta, \text{Pr}\right)$. So, now, we have found the stream function Ψ. So, you will able to calculate the velocity u right.

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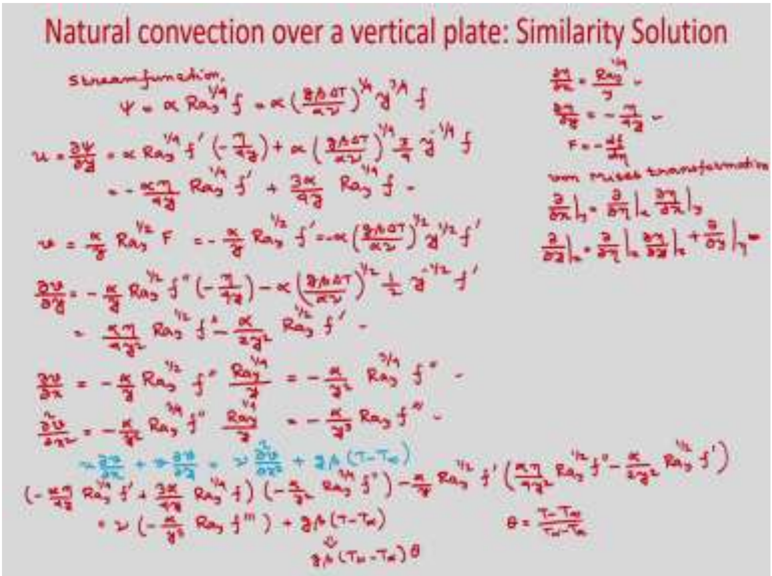

So, here just let us write $\frac{\partial \eta}{\partial x} = \frac{Ra_y^{1/4}}{y}$ and $\frac{\partial \eta}{\partial y} = -\frac{\eta}{4y}$ and we have used $F = -\frac{df}{d\eta}$.

And von Mises transformation, again we will use it. So, let us write these von Mises transformation. So, in this case it is a vertical plate, so we will use $\frac{\partial}{\partial x}\Big|_y = \frac{\partial}{\partial \eta}\Big|_x \frac{\partial \eta}{\partial x}\Big|_y$, we will write as $\frac{\partial}{\partial y}\Big|_x = \frac{\partial}{\partial \eta}\Big|_x \frac{\partial \eta}{\partial y}\Big|_x + \frac{\partial}{\partial y}\Big|_\eta$.

So, we will use these a transformation while finding the derivative. So, stream function you have found $\psi = \alpha Ra_y^{1/4} f$ and in terms of y if you write, then you will get $\alpha\left(\frac{g\beta\Delta T}{\alpha\nu}\right)^{1/4} y^{3/4} f$. So, that we have written outside.

So, now velocity $u = \frac{\partial \psi}{\partial y}$ and $\frac{\partial \psi}{\partial y}$ if you write, then this transformation you have to use. So, if you use these transformation, you can see so $\frac{\partial \psi}{\partial y} = \alpha Ra_y^{1/4} f'+\left(-\frac{\eta}{4y}\right)+\alpha\left(\frac{g\beta\Delta T}{\alpha\nu}\right)^{1/4}\frac{3}{4}y^{-1/4} f$.

So, if you rearrange it. So, you will get $-\frac{\alpha\eta}{4y}Ra_y^{1/4} f'+\frac{3\alpha}{4y}Ra_y^{1/4} f$. So, we have found the velocity u.

Now, let us write the velocity v in terms of f. So, in terms of capital F already we have written. So, you know that we have written $v = \frac{\alpha}{y}Ra_y^{1/2}F$ and this you can see $F = \frac{d\alpha}{d\eta}$.

So, you can write $-\frac{\alpha}{y}Ra_y^{1/2} f'$ and in terms of y if you take $-\alpha\left(\frac{g\beta\Delta T}{\alpha\nu}\right)^{1/2} y^{1/2} f'$.

So, we are writing in this form because we need to find the derivative with respect to y that is why we have taken y outside. So, that it will be easiest to integrate. Now, you can see in the momentum equation, we have the velocity gradient $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ and $\frac{\partial^2 v}{\partial x^2}$. So, that let us find from this expression.

So, if you use $\frac{\partial v}{\partial y}$. So, again this von Mises transformation this one we will use. So, you can see, it will be $\frac{\partial v}{\partial y} = -\frac{\alpha}{y} Ra_y^{1/2} f''\left(-\frac{\eta}{4y}\right) - \alpha\left(\frac{g\beta\Delta T}{\alpha\nu}\right)^{1/2} \frac{1}{2} y^{-1/2} f'$. Now, let us find the derivative of v with respect to x.

So that means, $\frac{\partial v}{\partial x} = -\frac{\alpha}{y} Ra_y^{1/2} f'' \frac{Ra_y^{1/4}}{y}$.

So, if you simplify it, you will get $-\frac{\alpha}{y^2} Ra_y^{3/4} f''$ and again, $\frac{\partial^2 v}{\partial x^2} = -\frac{\alpha}{y^2} Ra_y^{3/4} f''' \frac{Ra_y^{1/4}}{y}$.

So, if you simplify it, you will get $-\frac{\alpha}{y^3} Ra_y f'''$. So, now let us put all these in the momentum equation. What is your momentum equation? $u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 v}{\partial x^2} + g\beta\left(T - T_\infty\right)$.

So, now, if you put the value here. So, first u; So, this if u if you put, so you will get $\left(-\frac{\alpha\eta}{4y} Ra_y^{1/4} f' + \frac{3\alpha}{4y} Ra_y^{1/4} f\right)\left(-\frac{\alpha}{y^2} Ra_y^{3/4} f'\right) - \frac{\alpha}{y} Ra_y^{1/2} f'\left(\frac{\alpha\eta}{4y^2} Ra_y^{1/2} f'' - \frac{\alpha}{2y^2} Ra_y^{1/2} f'\right)$

So, left hand side, the inertia terms we have written; now in the right hand side, you write the viscous terms $\nu\left(-\frac{\alpha}{y^3} Ra_y f'''\right) + g\beta\left(T - T_\infty\right)$.

So, now, we have $\theta = \frac{T - T_\infty}{T_w - T_\infty}$. So, this term $g\beta\left(T_w - T_\infty\right)\theta$. So, you multiply it first, then simplify it.

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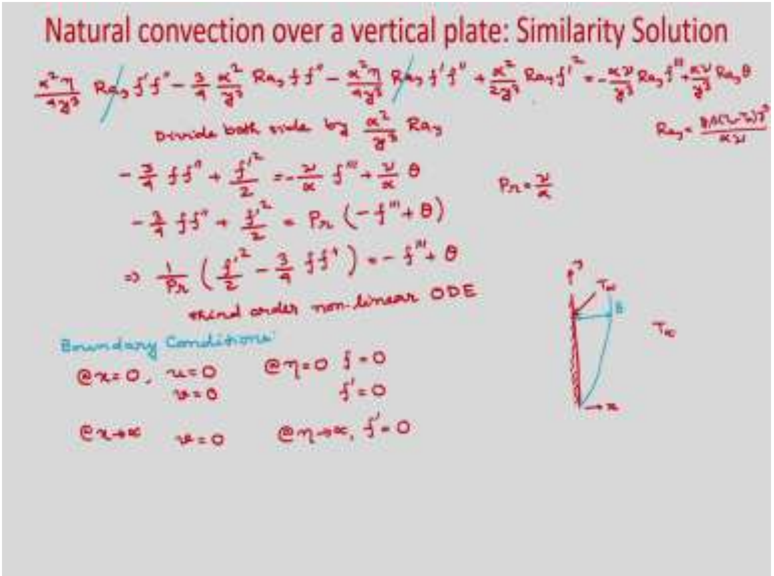

So, this if you do, then you will get $\frac{\alpha^2\eta}{4y^3} Ra_y f' f'' - \frac{3}{4}\frac{\alpha^2}{y^3} Ra_y ff'' - \frac{\alpha^2\eta}{4y^3} Ra_y f' f'' + \frac{\alpha^2}{2y^3} Ra_y f'^2$.

Then, we have the viscous term, $-\frac{\alpha\nu}{y^3} Ra_y f''' + \frac{\alpha\nu}{y^3} Ra_y \theta$. You see $Ra_y = \frac{g\beta(T_w - T_\infty)y^3}{\alpha\nu}$. So, now, you can see this two terms will get cancelled; this term and this term. Now, you can write this equation; divide both side by $\frac{\alpha^2}{y^3} Ra_y$.

So, if you divide by these, then first term what you can write? So, you can see, it will be $-\frac{3}{4} ff'' + \frac{f'^2}{2} = -\frac{\nu}{\alpha} f''' + \frac{\nu}{\alpha}\theta$ and you know the definition of Prandtl number, $\Pr = \frac{\nu}{\alpha}$. So, if you put it here, so you can see it will be $-\frac{3}{4} ff'' + \frac{f'^2}{2} = \Pr(-f''' + \theta)$.

So, if you write in this way, so it will be $\frac{1}{\Pr}\left(\frac{f'^2}{2} - \frac{3}{4} ff''\right) = -f''' + \theta$. So, you can see that choosing the similarity variable, we could convert the three partial differential equations; continuity equation; momentum equation and energy equation to ordinary differential equation.

So, if you see that using the similarity approach, we could convert the momentum equation which is your partial differential equation to ordinary differential equation. You can see this is the third order differential equation and it is non-linear because you can see this is f''', so this is third order and you have ff'' that means, it is non-linear ordinary differential equation.

But here you can see in this ordinary differential equation, in the momentum equation also you have term θ which is your temperature . So that means, this equation also will depend on the temperature profile. So, what are the boundary conditions now?

So, for a vertical plate if you see, so this is your x, this is your y right and I am going to write T_∞ and this is your T_w which is uniform and there will be hydrodynamic boundary layer like this, which is your δ; at any location, you can find what is the δ. So, now, you can see what are the boundary conditions.

So, we can see at x = 0, you have no slip boundary condition; that means, u = 0 and v= 0. So that means, at η=0, if you put u = 0, so you can see the u_s scale. So, you can see from here. So, here you can see that if u = 0, then $f' = 0$.

So, let us say that at η=0, so first find v = 0. So, if v = 0, then you can see from this expression. If v =0 from this expression, $f' = 0$; from this expression, $f' = 0$. So, you can see it will be v = 0, it will be $f' = 0$.

So, if $f' = 0$, then for u = 0. So, u is the this is the expression. So, f' is already 0. So, f will be 0. So, this is the stream function actually. So, you are putting stream function value as 0 because it is a vertical plate, so u_0 will give you the stream function and vertical plate, you can see it is a streamline and along a streamline your stream function will be constant; so, f = 0.

So, and x →∞. So, near to this edge of the boundary layer. Obviously, you will get the velocity 0. So that means, again you have v = 0. So, you have v = 0. So, at η→∞ you will get f' as 0.

Now, you can solve this ordinary differential equation using these boundary conditions, using some numerical technique and you can find the velocity distribution . But you cannot solve it alone because along with these you need to solve the energy equation, because theta is involved.

So, you have to solve in coupled way. So, you have to apply boundary condition for velocity in this equation and when you solve the energy equation, you need to invoke the temperature boundary condition and together, numerical you need to solve these two equations and you will find the velocity distribution as well as the temperature distribution.

So, if you see that when we wrote the governing equations for flow over vertical plate when with uniform temperature boundary condition, what are the assumptions we took? Obviously, it is two-dimensional steady and laminar flow and we have invoke the bossiness approximation and you can see the properties like thermal conductivity, specific heat and the viscosity are constant and the ambient temperature T_∞ is also constant and T_w which is your wall temperature that is also constant.

So, in the next class, we will solve the energy equation and we will convert this partial differential equation to ordinary differential equation, using the same similarity variable. So, in today's class, first we wrote the non-dimensional form of the boundary layer equations for natural convection over a vertical plate.

So, we have introduced with two non dimensional numbers; one is Grashof number which is the ratio of buoyancy force to the viscous force and another non-dimensional number that is Richardson number which is defined as Grashof number divided by Reynolds number square. This actually gives the ratio of buoyancy force to the inertia force. So, depending on the value of Richardson number, you can see which is the dominant force.

So, for Richardson number >> 1; obviously, you can neglect the forced convection and it will be purely natural convection. If Richardson number << 1, then you can see that inertia force will be dominant and it you can have the forced convection and effect of natural convection you can neglect.

But when Ri = 1, then both are significant natural convection as well as forced convection. So, this is known as mixed or combined convection. Then, we have used the similarity transformation approach and using the similarity variable eta and we have defined a velocity v from the scale of velocity v, we converted this momentum equation which is your partial differential equation to ordinary differential equation.

So, first we found the stream function from the velocity v and once you know the stream function; from there, we found the velocity u and the velocity gradient $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ and $\frac{\partial^2 v}{\partial x^2}$.

So, all those terms if you put in the momentum equation, then you could get third order non-linear ordinary differential equations and the boundary conditions already we have discussed that at the wall you have velocity v = 0 which will give f' is 0 and at the wall again u is 0 and it will give stream function as 0. So, f = 0 and as $\eta \to \infty$; that means, v will be 0 and f' will be 0.

So, in the next class, we will derive the ordinary differential equation for the energy equation.

Thank you.