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Module - 07
Convection in Internal Flows - III
Lecture - 22

Hydrodynamically developed and thermally developing flow through circular pipe with uniform wall heat flux

Hello everyone. So, till now we considered Hydrodynamically and thermally fully developed flow. So, in analysis it was easier because the non-dimensional temperature phi whatever we defined it was function of r only, so it does not change in the axial direction.

Today, we will consider hydrodynamically developed, but thermally developing laminar fluid flow through circular pipe with uniform wall heat flux. So, you can see that it is thermally developing flow that means, your temperature will change in axial direction and it will vary only inside the thermal boundary layer. However, in the core region it will remain at the inlet temperature.

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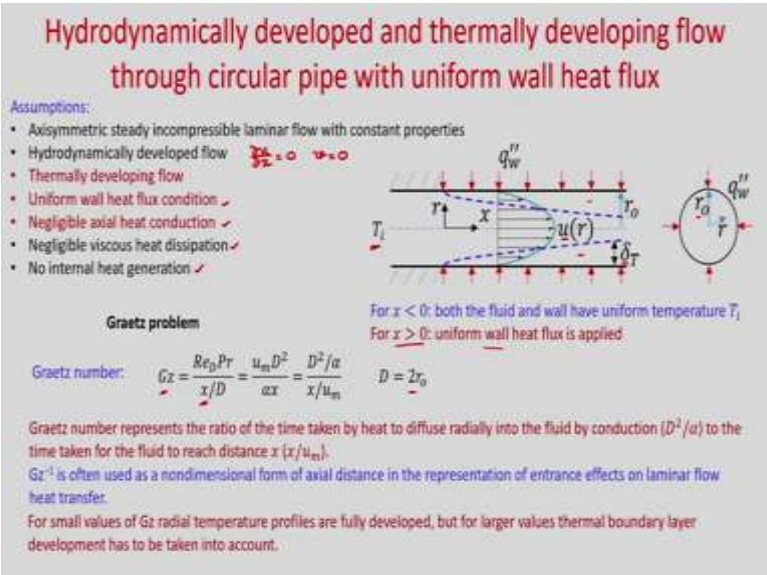

So, let us see. So, this is your circular pipe of radius r 0. We have kept the axis at the center line and at x = 0 we have put in such a way that from there uniform heat flux boundary condition is applied. If x < 0, then both the fluid and wall have uniform temperature T_i. So, after $x \geq 0$, you have uniform wall heat flux, boundary condition and your thermal boundary layer starts growing from x = 0. So, this is your thermal boundary layer thickness.

So, we have considered hydrodynamically fully developed flow. So, you have velocity u which is function of r only and it is fully developed profile parabolic. However, your temperature profile you can see that it will vary inside the thermal boundary layer only. But in the core region it will remain at temperature T_i.

The assumptions for this study are axisymmetric steady incompressible laminar flow with constant properties. You can see that it is geometrically and thermally symmetric and we can have the assumptions of axisymmetric flow. So, there is no variation in of any quantity in circumferential direction. It is hydrodynamically developed flow that means, $\frac{\partial u}{\partial x} = 0$ and radial velocity v = 0. It is thermally developing flow.

We used uniform wall heat flux condition. In this particular case also, we will neglect the axial heat conduction that means, axial heat conduction is we will assume that axial heat conduction is very very small compared to the radial heat conduction. And we will also assume negligible viscous heat dissipation and no internal heat generation.

So, the problem hydrodynamically developed and thermally developing flow inside a circular pipe is known as Graetz problem. So, we will define the Graetz number as $Gz = \frac{\mathrm{Re}_D \Pr}{x/D}$. So, you can see that this is some inverse of non-dimensional form of the axial distance, x is the axial distance and if you put the variables in Reynolds number and Prandtl number you can write it as $Gz = \frac{u_m D^2}{\alpha x} = \frac{D^2/\alpha}{x/u_m}$. So, you can see that in the numerator and denominator both are having the time scale, where D = 2 r_0.

So, Graetz number represents the ratio of the time taken by heat to diffuse radially into the fluid by conduction to the time taken to the fluid to reach distance x. For small values of Graetz number radial temperature profiles are fully developed. So, you can see that if x is very high then your Graetz number will be small.

So, if x is very high it will be thermally fully developed flow, so Graetz number will be small. So, for small values of Graetz number radial temperature profiles are fully developed, but for larger values of thermal boundary layer development has to be taken into account. So, you can see if x is very small then it will be thermally developing region.

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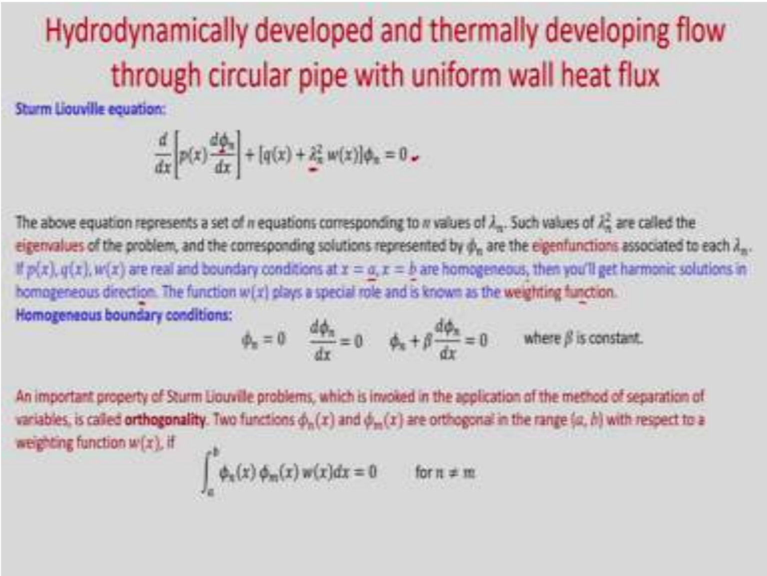

So, before going to the analysis first let us consider this second order ordinary differential equation. So, this is your second order ordinary differential equation $\frac{d}{dx}\left[p(\mathrm{x})\frac{d\phi_n}{dx}\right] + \left[q(x) + \lambda_n^2 w(x)\right]\phi_n = 0$. So, this is known as Sturm-Liouville equation or Sturm-Liouville boundary value problem.

Here λ_n^2 is known as eigen values and ϕ_n is the solution of this ordinary differential equation, and it is the eigen functions associated with each λ_n. So, if p, q, and w are real and boundary condition at x = a and x = b are homogenous then you will get harmonic solution in homogenous direction. The function w(x) plays a special role and is known as the weighting function.

In today's analysis we will use separation of variables method. So, when can we use separation of variables method? If your governing equation is linear and homogenous and in one direction if you have two homogenous boundary conditions and these then you can use the separation of variables method.

And when you will separate the variables then it will be equal to some constant and the sign of constant you need to choose such a way that in homogenous direction you will get harmonic solution. So, we have to choose the λ_n^2 or the constant such a way that in homogenous direction that means, in that direction where you have two homogenous boundary condition, it should give harmonic solution.

So, you can see. So, if you can resemble your governing equation with the Sturm-Liouville equation and p, q, w are real and boundary condition at x = a and x = b are homogenous, then you will get a harmonic solution in the homogenous direction.

One important property of this Sturm-Liouville equation is orthogonality. So, you can see two functions ϕ_n and ϕ_m are orthogonal to each other in the range a, b with respect to weighting function w(x) if $\int_a^b \phi_n(x)\phi_m(x)dx = 0$ for n ≠ m. So, now, for n ≠m these integral will be 0. So, it will be used to find the constant when will use the separation of variables method.

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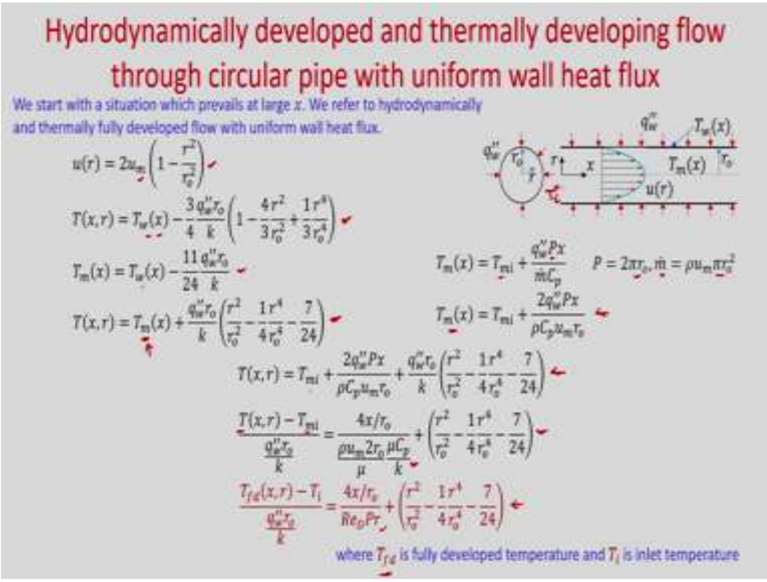

Now, first let us consider hydrodynamically and thermally fully developed flow through the circular pipe with uniform wall heat flux boundary condition. Already we have studied it. You know the solutions. Just we will represent this solution again here.

So, you can see that fully developed velocity profile is this one, where u_m is the mean velocity or average velocity the temperature T(x,r) which is your fully developed temperature profile $T(x,r) = T_w(x) - \frac{3}{4}\frac{q_w'' r_0}{k}\left(1 - \frac{4}{3}\frac{r^2}{r_0^2} + \frac{1}{3}\frac{r^4}{r_0^4}\right)$. So, this already we have derived the fully developed temperature profile in terms of T_w.

Again, we have derived the mean temperature in terms of wall temperature. So, you can see $T_m(x) = T_w(x) - \frac{11}{24}\frac{q_w'' r_0}{k}$. So, if you substitute T_w in this equation then you will get this equation. So, the temperature profile we have written in terms of mean temperature.

Again, from energy valence we have derived this equation you see $T_m(x) = T_{mi} + \frac{q_w'' P x}{\dot{m} C_p}$.

So, you know that P is the perimeter where in this particular case it is $2\pi r_0$ and $\dot{m}$ is the mass flow rate $\rho u_m \pi r_0^2$, so ρu A. Area is πr_0^2.

So, these already we have derived from the energy balance, and where T_{mi} is the at inlet you have the mean temperature. So, these P and $\dot{m}$ these value if you put then you can write in this form. Now, these T_m value you put in this equation, see if you put you will get this equation, $T(x,r) = T_{mi} + \frac{2 q_w'' x}{\rho C_p u_m r_0} + \frac{q_w'' r_0}{k}\left(\frac{r^2}{r_0^2} - \frac{1}{4}\frac{r^4}{r_0^4} - \frac{7}{24}\right)$. So, these T_{mi} if you take in the left hand side and divide by $\frac{q_w'' r_0}{k}$ then you can write in terms of a non-dimensional quantity as, $\frac{T(x,r) - T_{mi}}{\frac{q_w'' r_0}{k}} = \frac{4x/r_0}{\frac{\rho u_m 2 r_0}{\mu}\frac{\mu C_p}{k}} + \left(\frac{r^2}{r_0^2} - \frac{1}{4}\frac{r^4}{r_0^4} - \frac{7}{24}\right)$.

So, you can see that in this particular case it will become $4x/r_0$ and this after rearrangement you will get Reynolds number into Prandtl number and this is also non-dimensional quantity. So, you can see that we have written this T, what is what is this T? T is your fully developed temperature profile.

So, in a fully developed condition fully developed means hydrodynamically and thermally fully developed. So, $\dfrac{T_{fd}(x,r)-T_i}{\dfrac{q_w^{"}r_0}{k}} = \dfrac{4x/r_0}{\mathrm{Re}_D \Pr} + \left(\dfrac{r^2}{r_0^2} - \dfrac{1}{4}\dfrac{r^4}{r_0^4} - \dfrac{7}{24}\right)$. So, this is the temperature profile, in a fully developed flow, and where T_{fd} we have represented as a fully developed temperature and T_i is the inlet temperature. So, this already we have carried out this analysis earlier. Just we have revisited it.

Now, let us consider thermally developing flow. So, in thermally developing flow we will consider a temperature T_d such a way that your temperature profile at any location whether it is in thermally developing region or fully developed region T will be T_{fd} which is your fully developed temperature profile plus T_d.

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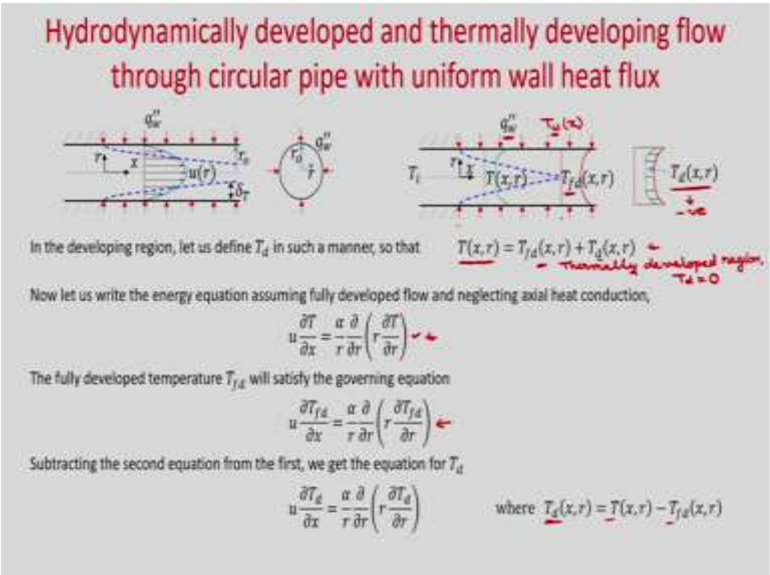

So, you can see here this is our problem. Now, in developing region thermally developing region you can see your thermal boundary layer thickness is growing. So, it is your developing region, thermally developing region. So, the temperature profile it has a T(x), T which is function of x and r. This temperature profile we need to find for thermally developing region. But this T is valid for both in thermally developing region as well as thermally developed region.

And we are defining this temperature as T_{fd} which is your fully developed temperature profile plus some temperature T_d. So, which is we will consider only in the developing region and in fully developed region this T_d will become 0, so in fully developed region thermally fully developed region.

So, in thermally developed region this T_d will be 0. So, we are defining T_d in that way. So, you see we need to find T which is valid in thermally developing region, and thermally developed region these T_{fd} we have already derived which is your temperature profile in thermally developed region.

Now, you can see that this red colored profile is your T_{fd} and this your green colored profile this is your T(x,r). So, obviously, you can see your in axial direction in the core region in a thermally developing region you can see that it will be always T_i because that is the temperature inlet temperature. Only temperature is varying inside the thermally thermal boundary layer.

So, in the core region temperature will remain at T_i. At another location if you consider here also it will be T_i, but once it becomes fully developed region then your core temperature will vary. So, obviously, the temperature profile in the thermally developing region will be lesser than the fully developed region.

This is also true for the wall temperature because at the wall temperature you can see this is a constant wall temperature, this is the constant wall heat flux boundary condition. So, T_w will be function of x and along axial direction your T_w will increase.

So, obviously, when it will come to fully developed region, obviously your T_w will be higher than the T_w at developing region. So, we have represented this green colored temperature profile in the developing region, red colored temperature profile in a fully developed region. So, the difference we are representing with T_d. So, you can see this is actually negative.

This is actually negative, but we are considering T which is your temperature profile at any region whether it is thermally developing region or thermally fully developed region is $T(x,r) = T_{fd}(x,r) + T_d(x,r)$. So, we can see that your T_d will be maximum at x = 0; T_d will be maximum at x = 0. Then, it will start decreasing, decreasing, decreasing; once it becomes thermally fully developed region then your T_d will become 0. So, from the high negative value to 0 it will vary in the developing region.

So, you can see that when it is fully developed region T_d will become 0 and T(x, r) will be just T_{fd} . And in developing region T_d will have some negative value, so these negative value will be directed from T_{fd} and you will get the green color this temperature profile.

So, I hope you have understood that how we have defined a temperature in the developing region T_d which is actually negative quantity and these is having a high value at x = 0 at the entrance region and it will decrease along axial direction in the thermally developing zone. After that once it becomes thermally developed region this T_d will become 0 and your temperature profile T will become T_{fd} . So, these T(x,y) we need to find which is valid in both thermally developing region as well as thermally developed region.

So, now let us write the energy equation in booking all the assumptions. So, you can write this is the energy equation $u\frac{\partial T}{\partial x}=\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$ where ∝ is your thermal diffusivity. So, in fully developed region our temperature profile is T_{fd} . So, obviously, this T_{fd} will satisfy this governing equation, so you can write $u\frac{\partial T_{fd}}{\partial x}=\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_{fd}}{\partial r}\right)$.

So, now if you subtract this equation from this equations. So, what you will get? You will get $u\frac{\partial T_d}{\partial x}=\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_d}{\partial r}\right)$, so where $T_d(x,r)=T(x,r)-T_{fd}(x,r)$. From this definition you can see your $T_d(x,r)=T(x,r)-T_{fd}(x,r)$. So, you can see that the whatever temperature T_d we have introduced in the thermally developing region it also satisfies the energy equation.

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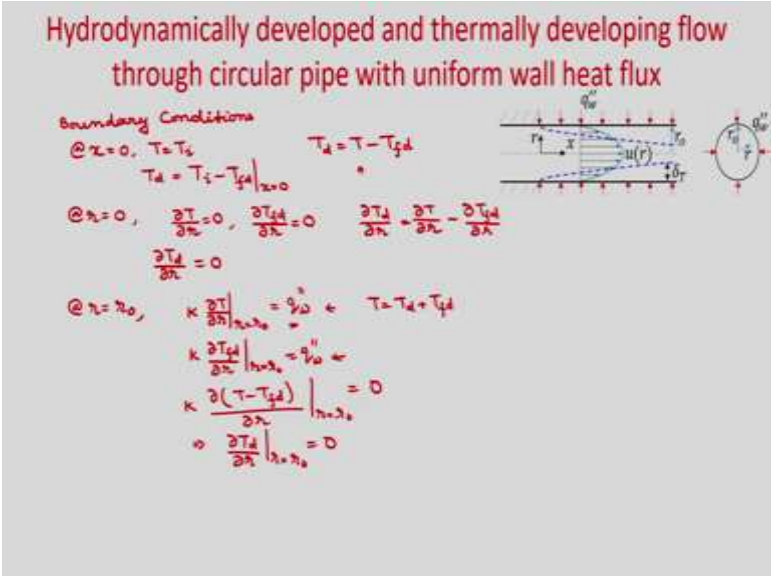

So, now we need to solve this equation first and we will find the T_d. We know already T_{fd}, so the temperature profile $T(x,y) = T_{fd} + T_d$. Now, m is to find the temperature profile T_d.

So, first let us write the boundary conditions for T_d. So, boundary conditions are at x = 0. So, you have $T = T_i$. And we know $T_d = T - T_{fd}$. We have defined this way. So, if at x = 0, $T = T_i$ we will get $T_d = T - T_{fd}$ at x = 0.

Now, in radial direction the boundary conditions are at r = 0, we can see that your due to the axisymmetric condition as it is thermally and geometrically symmetric you will get maximum or minimum temperature at the central line, where r = 0. So, that means, the temperature gradient with respect to radius will be 0.

So, you will get $\frac{\partial T}{\partial r} = 0$ and in fully developed case also you can write $\frac{\partial T_{fd}}{\partial r} = 0$ because already we have imposed this boundary condition. So, you can see from here you can write $\frac{\partial T_d}{\partial r} = \frac{\partial T}{\partial r} - \frac{\partial T_{fd}}{\partial r}$. So, if $\frac{\partial T}{\partial r} = 0$ and $\frac{\partial T_{fd}}{\partial r} = 0$ then obviously, $\frac{\partial T_d}{\partial r} = 0$, so you can write $\frac{\partial T_d}{\partial r} = 0$ at r = 0.

And at the wall at r = r_0, we have the heat flux boundary condition. So, you can see $q_w^{"}$ we have taken in the negative radial direction. So, what will be your $q_w^{"}$? So, it will be $K\frac{\partial T}{\partial r}$, we are not writing minus sign because $q_w^{"}$ sign is in the negative to radial direction that is why it is positive $K\frac{\partial T}{\partial r}|_{r=r_0} = q_w^{"}$. So, this is your for $\frac{\partial T}{\partial r}$.

So, now what is T? T = T_d + T_{fd}. And for a fully developed boundary conditions also we have applied this wall heat plus boundary condition. So, you will get $K\frac{\partial T}{\partial r}|_{r=r_0} = q_w^{"}$. So, now, if you subtract, this two, so what you will get?

If you subtract then you will get in the fully developed case where thermally and hydrodynamically fully developed region your temperature boundary condition at r = r_0 will be $K\frac{\partial T_{fd}}{\partial r}|_{r=r_0} = q_w^{"}$ because already we have solved for these boundary condition, right.

So, now, if you subtract these equation from this equation what you will get? $K\frac{\partial (T-T_{fd})}{\partial r}|_{r=r_0} = 0$, right. And what is T -$T_{fd}$? It is nothing, but T_d. So, that means, you will get $\frac{\partial T_d}{\partial r}|_{r=r_0} = 0$.

So, you see in the radial direction at r = and r = r_0 your temperature gradient of T_d =0, so that means, both the boundary conditions are homogenous, that means, it is the homogenous directions. So, when you will use separation of variables method, we have to choose the sign of the constants such a way that you will get the harmonic solution in r direction.

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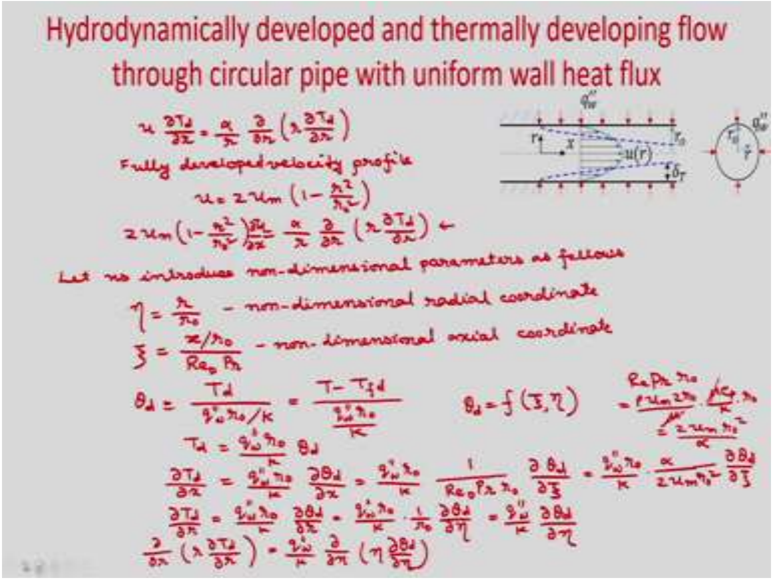

So, now our governing equation is $u\frac{\partial T_d}{\partial x}=\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_d}{\partial r}\right)$, because we need to find this value of T_d, right. Because T_{fd} is known if T_d you can find then the temperature profile in general you can write $T_{fd} + T_d$.

In this case, we are using hydrodynamically developed flow, that means, u you know. So, u you can write $u=2u_m\left(1-\frac{r^2}{r_0^2}\right)$. So, you can write fully developed velocity profile, you know $u=2u_m\left(1-\frac{r^2}{r_0^2}\right)$. So, you put it here, so you will get, $2u_m\left(1-\frac{r^2}{r_0^2}\right)\frac{\partial T_d}{\partial x}=\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_d}{\partial r}\right)$.

So, now let us define some non-dimensional parameters. So, we will use the radial direction $\eta=\frac{r}{r_0}$. So, this is your non-dimensional radial direction and the non-dimensional axial direction $\zeta=\frac{x/r_0}{\mathrm{Re}_D\,\mathrm{Pr}}$.

So, let us introduce non-dimensional parameters as follows. So, $\eta = \frac{r}{r_0}$, so you can see this is your non-dimensional radial coordinate. Then, $\zeta = \frac{x/r_0}{\mathrm{Re}_D \Pr}$. So, this is your non-dimensional axial coordinate. And the non-dimensional temperature $\theta_d = \frac{T_d}{q_w'' r_0 / K}$ which is $\frac{T - T_{fd}}{q_w'' r_0 / K}$.

So, now if you put all this values, so you can see you can write $T_d = \frac{q_w'' r_0}{K} \theta_d$. So, you can write $\frac{\partial T_d}{\partial x} = \frac{q_w'' r_0}{K} \frac{\partial \theta_d}{\partial x}$. So, you see $\theta_d = f(\zeta, \eta)$.

Then, you can write this x coordinate, in terms of ζ. So, you can write $\frac{\partial T_d}{\partial x} = \frac{q_w'' r_0}{K} \frac{1}{\mathrm{Re}_D \Pr r_0} \frac{\partial \theta_d}{\partial \zeta}$. So, $\mathrm{Re}_D \Pr r_0 = \frac{\rho u_m 2 r_0}{\mu} \frac{\mu C_p}{K} r_0$.

So, you can see this μ will get cancel. So, you can write $\frac{\partial T_d}{\partial x} = \frac{q_w'' r_0}{K} \frac{\alpha}{2 u_m r_0^2} \frac{\partial \theta_d}{\partial \zeta}$.

Similarly, you can write $\frac{\partial T_d}{\partial r} = \frac{q_w'' r_0}{K} \frac{\partial \theta_d}{\partial r}$.

So, now, this r will put as r_0 η, so you can write $\frac{\partial T_d}{\partial r} = \frac{q_w'' r_0}{K} \frac{1}{r_0} \frac{\partial \theta_d}{\partial \eta}$. So, this r_0 r_0 if you cancel it, then you will get $\frac{\partial T_d}{\partial r} = \frac{q_w''}{K} \frac{\partial \theta_d}{\partial \eta}$. Similarly, if you write $\frac{\partial}{\partial r}\left(r \frac{\partial T_d}{\partial r} \right) = \frac{q_w''}{K} \frac{\partial}{\partial \eta}\left(\eta \frac{\partial \theta_d}{\partial \eta} \right)$.

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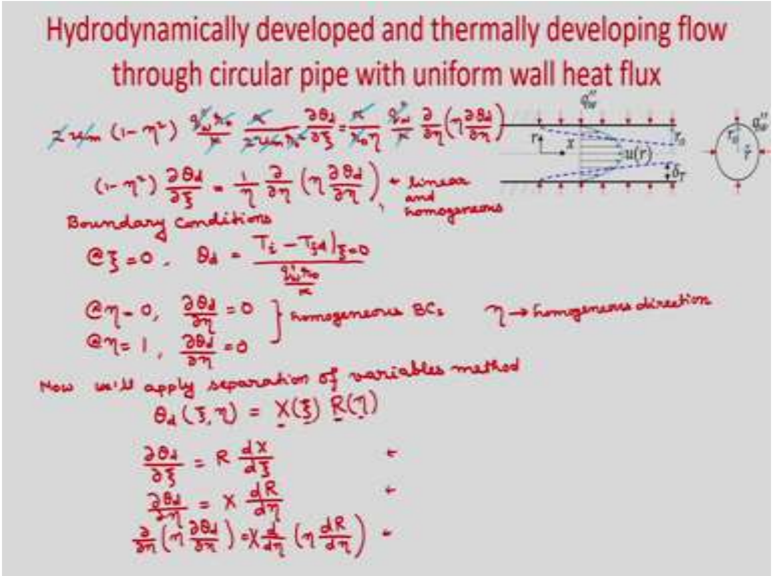

So, all the derivatives we have found. Now, all these you put it in the governing equation. And write it as, $2u_m\left(1-\eta^2\right)\frac{q_w^{"}r_0}{K}\frac{\alpha}{2u_m r_0^2}\frac{\partial\theta_d}{\partial\zeta}=\frac{\alpha}{r_0\eta}\frac{q_w^{"}}{K}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta_d}{\partial\eta}\right)$. After substituting all the terms.

So, now, you can see here, this α, this α you can cancel u_m, u_m, 2, 2, this r_0 here you will get one and another these r_0. So, you can cancel $q_w^{"}$, $q_w^{"}$,K and K. So, you can write the final equation as $\left(1-\eta^2\right)\frac{\partial\theta_d}{\partial\zeta}=\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta_d}{\partial\eta}\right)$.

So, now, we will use separation of variables method. So, what is separation of variables method? We will write the solution of θ_d as a product of two individual solution x and r, where each solution is function of one coordinate only. So, like we are defining θ_d as product of x and r, where x is function of ζ only and r is function of η only.

So, before going to that let us write the boundary conditions. Boundary conditions at ζ=0 that means, x = 0. So, $\theta_d=\frac{(T_i-T_{fd})\big|_{\zeta=0}}{\frac{q_w^{"}r_0}{K}}$ and at η=0 you have $\frac{\partial\theta_d}{\partial\eta}=0$ and at η=1 because r = r_0. So, η=1 and $\frac{\partial\theta_d}{\partial\eta}=0$.

Now, we will apply separation of variables method. So, we will write the solution θ_d which is function of ζ and η as product of two individual solution X which is function of ζ and R which is function of η only. So, you can see you are writing $\theta_d(\zeta,\eta) = X(\zeta)R(\eta)$.

So, when can we use separation of variables method? If the governing equation is linear and homogenous, and in one direction you have two homogenous boundary conditions then you can use separation of variables method. So, you can see our governing equation, these equation is linear and homogenous, and in η direction you have homogenous boundary conditions. That means, η is your homogenous direction. So, η is your homogenous direction.

So, we can use separation of variables (Refer Time: 33:52) and we are writing the solution theta d as product of two individual solution X and R, where X is function of ζ only and R is function of η only. So, now, let us write $\frac{\partial \theta_d}{\partial \zeta} = R\frac{dX}{d\zeta}$. So, now, this is your ordinary derivative were writing because X is function of ζ only.

Similarly, you can write $\frac{\partial \theta_d}{\partial \eta} = X\frac{dR}{d\eta}$. So, similarly if you write , $\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \theta_d}{\partial \eta}\right) = X\frac{d}{d\eta}\left(\eta\frac{dR}{d\eta}\right)$ and one X will be there. So, all these derivatives you put it in this equation.

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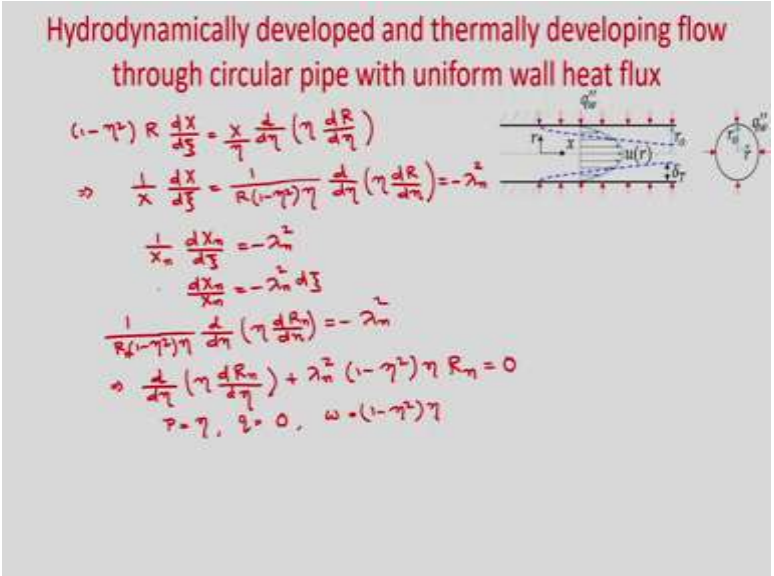

So, what you will get? So, you will get, $\left(1-\eta^2\right)R\frac{dX}{d\zeta}=\frac{X}{\eta}\frac{d}{d\eta}\left(\eta\frac{dR}{d\eta}\right)$. So, you see this equation.

Left hand side X is function of ζ only. That means, the whole term left hand side term is function of ζ only. Right hand side if you see, R is function of η only and, right hand side all the terms are function of η only. So, left hand side is function of ζ, right hand side function of η.

So, this will be equal to some constant. Because left hand side is your function of ζ, right hand side function of η equal to some constant, and that constants sign you have to choose such a way that in η direction you should get the governing equation such a way that you will get the harmonic solution of that governing equation. And obviously, if you can write in R direction the governing equation has Sturm-Liouville equation type, then you will get the harmonic solution in η direction.

So, we will choose the constant as $-\lambda_n^2$, where λ_n^2 is your eigen values. So, you can write $\frac{1}{X_n}\frac{dX_n}{d\zeta}=-\lambda_n^2$. So, you can see its solution will be exponential.

The other term you will get $\frac{1}{R_n\left(1-\eta^2\right)\eta}\frac{d}{d\eta}\left(\eta\frac{dR_n}{d\eta}\right)=-\lambda_n^2$. So, you can see this will be your $\frac{d}{d\eta}\left(\eta\frac{dR_n}{d\eta}\right)+\lambda_n^2\left(1-\eta^2\right)\eta R_n=0$.

Now, you see this second order ordinary differential equation. You compare it with the Sturm-Liouville equation. So, you can see your if you compare with the Sturm-Liouville equation, p =η, q will be 0, and the weighting function, $w=\left(1-\eta^2\right)\eta$. And, ok, these are real and in eta direction you have two homogenous boundary conditions so obviously, you will get the harmonic solution in eta direction.

So, if you see this second order differential equation, it is very difficult to solve. So, you can use some numerical technique to solve this second order ordinary differential equation. Once you get the solution which is at the eigen function of this equation R_n, then you can write the product of this two solution one you will get from the X another solution from R then this product of this two solution will be your the temperature profile θ_T. But for different values of eigen values λ_n^2 you will get different solution X_n and different solution R_n.

So, you need to find what is the value of λ_n, eigen values. Numerically, if you solve these equation, with the proper boundary conditions then you will get the eigen functions of these ordinary differential equation R_n, then you can write the solution θ_d, as summation of all the product of solution X and R because for different values of λ_n, you will get different solution X into R. And as it is a linear solution linear governing equation.

So, you can super impose all the solutions for different values of λ_n, and if you super impose, that means you are adding all the solutions. So, it is possible as you have the governing equation linear. Because you have a linear governing equation, so you can super impose all the solutions.

So, now let us write the final solution θ_d as product of X_n R_n, where X_n is the solution where you will get in the exponential form, R_n you need to find solving this second order ordinary differential equation. Then, you can sum it from n is equal to 1 to ∞ because you will get different values of λ_n and you will get the different solutions.

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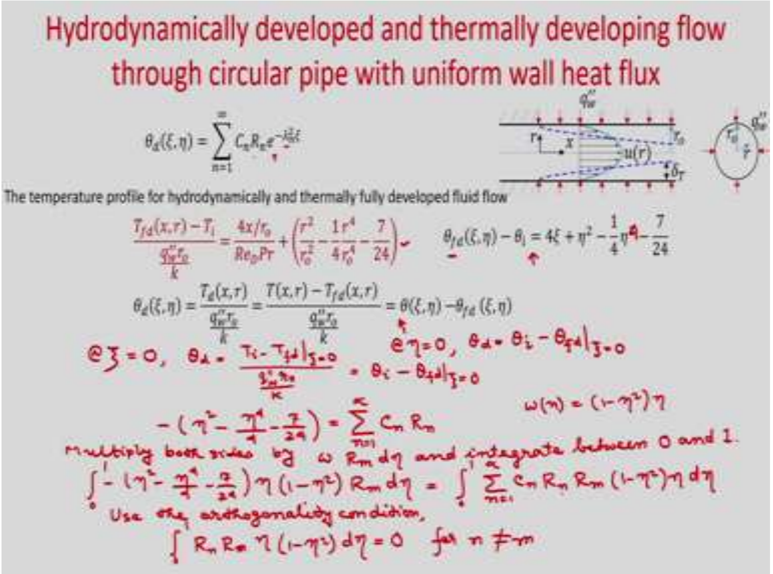

So, assuming that you would solution of this second order differential equation is R_n which are eigen function and λ_n^2 is eigenvalues then you can write the final solution as. So, you can write the final solution $\theta_d(\zeta,\eta)=\sum_{n=1}^{\infty} C_n R_n e^{-\lambda_n^2 \zeta}$ these are the eigen function of this second order differential equation, and this is the solution, this is the solution from the η direction. So, there are first equation you can see.

So, if you solve this equation. So, you will get X_n is equal to some constant into $\theta_d(\zeta,\eta)=\sum_{n=1}^{\infty} C_n R_n e^{-\lambda_n^2 \zeta}$. So, this product if you write R_n if you find and X_n if you find then you will get the final solution θ_d as product of two individual solutions and as it is linear equation you can super impose all the solution n = 1 to ∞ for different values of λ_n. So, R_n you need to find. So, this is the eigen functions of this second order differential equation.

So, now, we need to apply the boundary conditions, to find these constant C n as well as you need to find λ_n^2 because that is also unknown. So, you can see the temperature profile for hydrodynamically and thermally fully developed fluid flow we have written like this, we have already derived today. So, in non-dimensional form if you write, so $\frac{T_{fd}}{\frac{q_w'' r_0}{k}}$ you can write θ_{fd}, and $\frac{T_i}{\frac{q_w'' r_0}{k}}$ you can write θ.

So, $\theta_{fd}(\zeta,\eta)-\theta_i = 4\zeta+\eta^2-\frac{1}{4}\eta^4-\frac{7}{24}$.

So, now theta d we have defined $\theta_{fd}(\zeta,\eta)=\frac{T_d(x,r)}{\frac{q_w'' r_0}{k}}$ and $T_d(x,r)=T(x,r)-T_{fd}(x,r)$.

$\theta_{fd}(\zeta,\eta)=\frac{T(x,r)-T_{fd}(x,r)}{\frac{q_w'' r_0}{k}}$.That means, $\theta_d(\zeta,\eta)=\theta(\zeta,\eta)-\theta_{fd}(\zeta,\eta)$. So, now, apply the boundary condition at η= 0. At η= 0 we know $\theta_d=\frac{T_i-T_{fd}\big|_{\zeta=0}}{\frac{q_w'' r_0}{k}}$. So, this we already we know. So, we can see that.

So, this is your θ_d and you can see here. So, from these; if you put at η=0 that means, this is your at x = 0, right. So, you can write θ_d as; what is this? This is nothing, but $\theta_d=\theta_i-\theta_{fd}\big|_{\zeta=0}$. So, you can write it as $\theta_d=\theta_i-\theta_{fd}\big|_{\zeta=0}$.

So, you can see from this equation, from this equation if ζ=0 if you put and if you reverse it, so you will get this as, $-\left(\eta^2-\frac{\eta^4}{4}-\frac{7}{24}\right)=\sum_{n=1}^{\infty}C_nR_n$ because at ζ=0, x = 0. So, $e^{-\lambda_n^2\zeta}=1$.

So, now we need to find the constant C_n. Now, we will invoke the orthogonality constant. So, now, we will invoke the orthogonality condition what we discussed in the Sturm-Liouville equation. So, what will do now multiply both side by, weighting function $wR_m d\eta$ and integrate between 0 and 1. So, this w is the weighting function.

What is this? You can see here, $\int_0^1 -\left(\eta^2-\frac{\eta^4}{4}-\frac{7}{24}\right)\eta\left(1-\eta^2\right)R_m d\eta=\int_0^1\sum_{n=1}^{\infty}C_nR_nR_m\left(1-\eta^2\right)\eta d\eta$.

So, if you remember the orthogonality condition for n $\neq$m, $\int_0^1 R_n R_m \eta\left(1-\eta^2\right) d\eta = 0$ for n$\neq$m, C_n is constant. So, C_n you take outside. So, obviously, for n = m this will become 0, so only for n = m only one term will remain.

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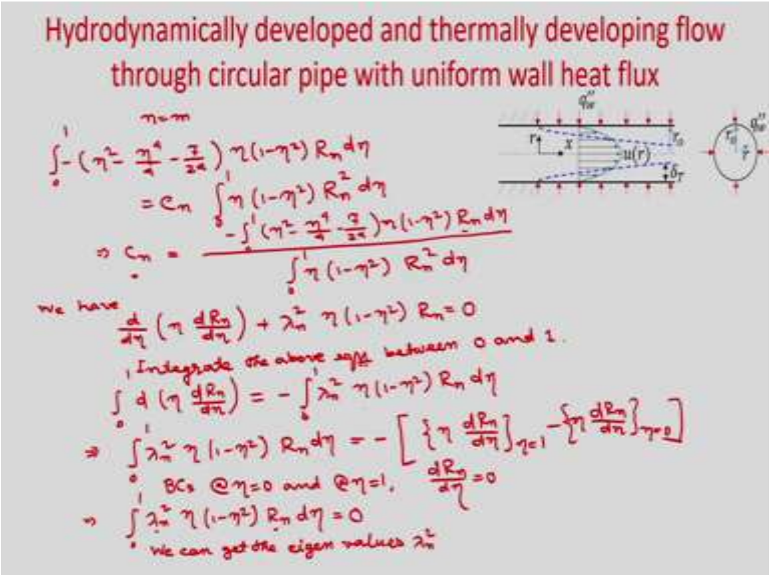

So, if you write for n = m if you keep then you can write $\int_0^1 -\left(\eta^2 - \frac{\eta^4}{4} - \frac{7}{24}\right)\eta\left(1-\eta^2\right) R_n d\eta = C_n \int_0^1 \eta\left(1-\eta^2\right) R_n^2 d\eta$; n = m we have put.

So, now, we can find the constant $C_n = \frac{\int_0^1 -\left(\eta^2 - \frac{\eta^4}{4} - \frac{7}{24}\right)\eta\left(1-\eta^2\right) R_n d\eta}{\int_0^1 \eta\left(1-\eta^2\right) R_n^2 d\eta}$. So, you can see that, if R_n is known, which is the solution, you will get from the second order differential equation then you will be able to integrate this and you can find the constant C_n.

Now, we need to find the value of λ_n. So, first we will start from the governing equation. So, what is your governing equation? We have $\frac{d}{d\eta}\left(\eta \frac{dR_n}{d\eta}\right) + \lambda_n^2 \eta\left(1-\eta^2\right) R_n = 0$. So, this is the equation we have derived you can see. So, this is the equation.

So, now you integrate $\int_0^1 d\left(\eta \frac{dR_n}{d\eta}\right) = -\int_0^1 \lambda_n^2 \eta\left(1-\eta^2\right) R_n d\eta$.

So, this equation you just write in the left hand side, so it will be $\int_0^1 \lambda_n^2 \eta\left(1-\eta^2\right) R_n d\eta = -\left[\left\{\eta \frac{dR_n}{d\eta}\right\}_{\eta=1} - \left\{\eta \frac{dR_n}{d\eta}\right\}_{\eta=0}\right]$.

Now, you recall the boundary conditions. At η=0 and η=1 you have $\frac{\partial \theta_d}{\partial \eta} = 0$. And θ_{fd} is a product of X and R, but R is only function of η. So, your boundary condition at η= 0 ,1, you will get $\frac{dR_n}{d\eta}$=0. So, that means, boundary condition at η= 0 and at η= 1 you will get $\frac{dR_n}{d\eta}$=0. So, you can see that these two terms will become 0. So, hence you will get $\int_0^1 \lambda_n^2 \eta\left(1-\eta^2\right) R_n d\eta = 0$.

So, from here we can get the eigen values λ_n^2 because for you can see this is the condition. So, this R_n if you find, so for different λ_n you will get different R_n. So, R_n if you know then from here it should to satisfy these because, to right hand side will be 0, so to satisfy this you can find the value of λ_n^2 from this equation.

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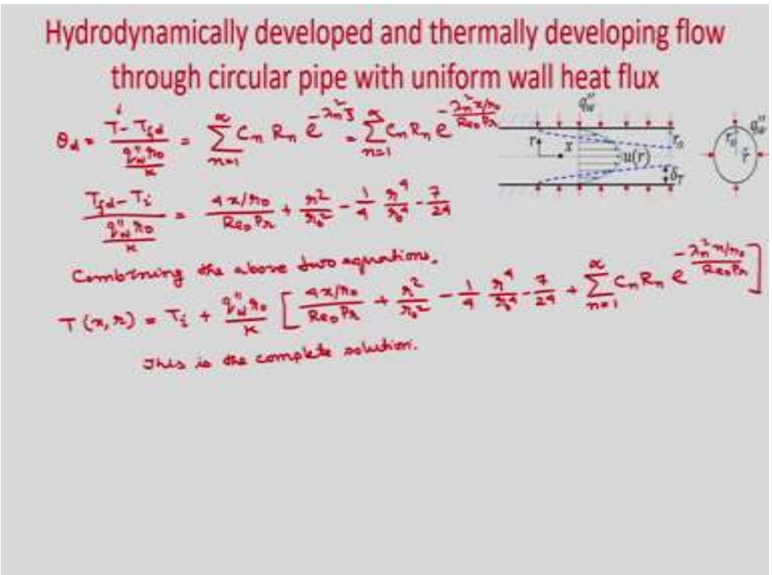

So, now if you put the $\theta_d = \dfrac{T - T_{fd}}{\dfrac{q_w^{"} r_0}{k}} = \sum_{n=1}^{\infty} C_n R_n e^{-\lambda_n^2 \zeta} = \sum_{n=1}^{\infty} C_n R_n e^{-\frac{\lambda_n^2 x/r_0}{\mathrm{Re}_D \Pr}}$. So, you see now we are interested to find theta which is the temperature distribution in general.

So, that now you can write $\dfrac{T_{fd} - T_i}{\dfrac{q_w^{"} r_0}{k}} = \dfrac{4\, x/r_0}{\mathrm{Re}_D \Pr} + \dfrac{r^2}{r_0^2} - \dfrac{1}{4}\dfrac{r^4}{r_0^4} - \dfrac{7}{24}$. So, you combining these two you can see.

So, this if you add it then $T(x,r) = T_i + \dfrac{q_w^{"} r_0}{k}\left[\dfrac{4\, x/r_0}{\mathrm{Re}_D \Pr} + \dfrac{r^2}{r_0^2} - \dfrac{1}{4}\dfrac{r^4}{r_0^4} - \dfrac{7}{24} + \sum_{n=1}^{\infty} C_n R_n e^{-\frac{\lambda_n^2 x/r_0}{\mathrm{Re}_D \Pr}}\right]$.

So, you can see this is the final temperature distribution T(x, r), which we are interested to find and this T(x, r) is value in both thermally developing region as well as fully developed region and you can write in terms of T_i. So, we can see this is the equation and this is the complete solution.

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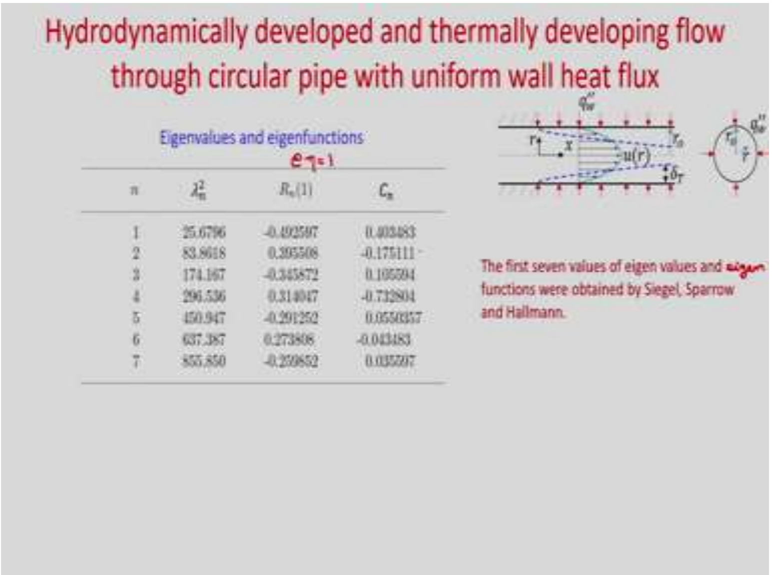

n	λ_n^2	$R_n(1)$	C_n
1	25.6796	-0.492597	0.403483
2	83.8618	0.395508	-0.175111
3	174.167	-0.345872	0.105594
4	296.536	0.314047	-0.732804
5	450.947	-0.291252	0.0550357
6	637.387	0.273808	-0.043483
7	855.850	-0.259852	0.035597

So, now if you see this eigen functions, you will get different eigenfunction, but at η = 1, $R_n(1)$, this is found by this first 7 values of eigen values and functions these are eigen functions, where obtained by Siegel Sparrow and Hallman. So, you can see from this table.

So, for different values of n, n= 1 to 7 these are the λ_n^2 value and $R_n(1)$ that means, at η=1. So, that at means at the boundary, $R_n(1)$, so we can see this is the first value is negative and if R_n is negative then θ_d will be negative then you can see that it is the negative quantity T_d, whatever we defined. So, actually in the developing region you are calculating the temperature distribution which we actually subtracting this T_d from the T_{fd} part, and this is negative coming. And the constant C_n, you can see these are the C_n.

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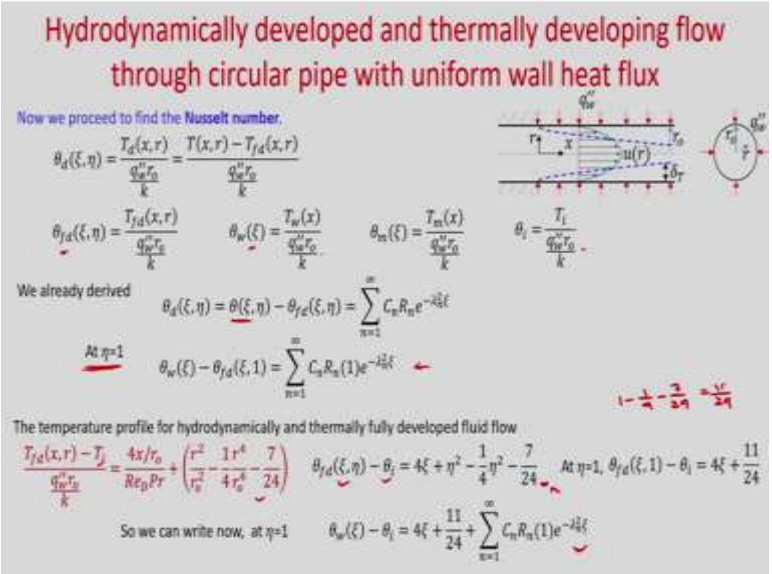

Now, we need to find the Nusselt number. So, you can see that whatever $\theta_d(\zeta,\eta)=\dfrac{T_d(x,r)}{\dfrac{q_w^{"}r_0}{k}}$. And similarly, fully developed region $\theta_{fd}(\zeta,\eta)=\dfrac{T_{fd}(x,r)}{\dfrac{q_w^{"}r_0}{k}}$ and mean temperature $\theta_m(\zeta)=\dfrac{T_m(x)}{\dfrac{q_w^{"}r_0}{k}}$. So, we already derived this θ_d, right. So, this is just

$$\sum_{n=1}^{\infty} C_n R_n e^{-\lambda_n^2\zeta} .$$

Now, at η=1, at the boundary what is that? θwill be θ_w. So, $\theta_w(\zeta)-\theta_{fd}(\zeta,1)=\sum_{n=1}^{\infty} C_n R_n(1) e^{-\lambda_n^2\zeta}$ .So, and also we have found the temperature profile for fully developed flow. So, this is the temperature profile.

So, you can write $\theta_{fd}(\zeta,\eta)-\theta_i$ in terms of non-dimensional coordinate, η and ζ. So, at η=1 if you put, so it will get $\dfrac{T_{fd}-T_i}{\dfrac{q_w^{"}r_0}{k}} = \dfrac{4\,{x}/{r_0}}{\mathrm{Re}_D\,\mathrm{Pr}} + \left(\dfrac{r^2}{r_0^2} - \dfrac{1}{4}\dfrac{r^4}{r_0^4} - \dfrac{7}{24}\right)$. So, this you will get $\theta_{fd}(\zeta,1)-\theta_i = 4\zeta + \dfrac{11}{24}$. So, we can write now at η=1 this θ_w. So, you can see this if you add this two equations these equation and this equation. So, it will be $\theta_w(\zeta)-\theta_i = 4\zeta + \dfrac{11}{24} + \sum_{n=1}^{\infty} C_n R_n(1) e^{-\lambda_n^2\zeta}$.

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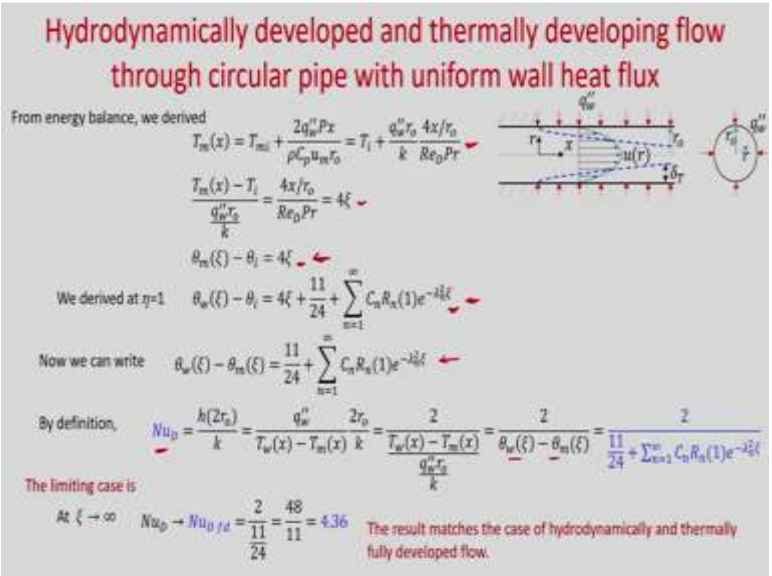

Once you get it, so now, you see the from the energy balance T_m we have found like this. So, $\dfrac{T_m(x)-T_i}{\dfrac{q_w^{"}r_0}{k}} = \dfrac{4\,{x}/{r_0}}{\mathrm{Re}_D\,\mathrm{Pr}} = 4\zeta$. So, from there you can find. So, in non-dimensional form $\theta_m(\zeta)-\theta_i = 4\zeta$.

And we derive that η=1 already in last slide the $\theta_w(\zeta)-\theta_i = 4\zeta + \dfrac{11}{24} + \sum_{n=1}^{\infty} C_n R_n(1) e^{-\lambda_n^2\zeta}$. So, now, we can write, so you can see if you if you subtract the these equation, from this equation subtract this equation from this equation, then you will get, $\theta_w(\zeta)-\theta_m(\zeta)=\frac{11}{24}+\sum_{n=1}^{\infty}C_nR_n(1)e^{-\lambda_n^2\zeta}$.

So, now by definition what is the Nusselt number? Thus, $Nu_D=\frac{h(2r_0)}{k}$. So, $\frac{q_w^{"}}{T_w(x)-T_m(x)}\frac{2r_0}{k}$.

Now, we can write $\frac{2}{\frac{T_w(x)-T_m(x)}{\frac{q_w^{"}r_0}{k}}}$ and this is nothing, but $\theta_w(\zeta)-\theta_m(\zeta)$ and these already we have found. So, Nusselt number for this particular case you can see it is $\frac{2}{\frac{11}{24}+\sum_{n=1}^{\infty}C_nR_n(1)e^{-\lambda_n^2\zeta}}$. So, you can see this is the Nusselt number in general. So, these expression is valid for both thermally developing region and fully developed region.

So, let us check whether it is true for the fully developed region or not. So, when you will get the fully developed region when x $\rightarrow\infty$, that means, $\zeta\rightarrow\infty$. So, at $\zeta\rightarrow\infty$, so Nu_D will become fully developed Nusselt number and you can see that as $\zeta\rightarrow\infty$ this term will become 0.

So, this term will become 0, so you will get $Nu_{Dfd}=\frac{2}{\frac{11}{24}}=\frac{48}{11}=4.36$ and these you have already direct, right for hydrodynamically and thermally fully developed region. So, this is true.

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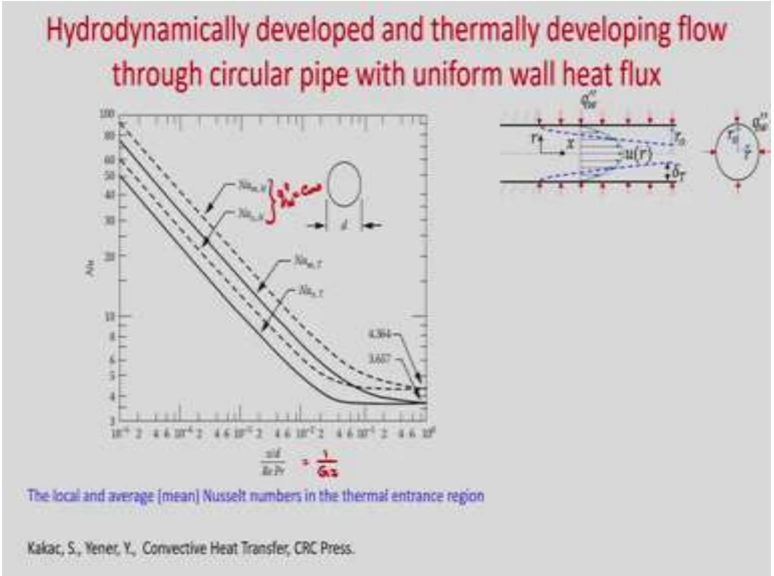

And the temperature profile for this Nusselt number, you can see local and average Nusselt number for thermal intense region. So, this is your $q_w^{"}$ is constant, ok. So, for this case you can see $\frac{x/d}{\mathrm{Re}\,\mathrm{Pr}} = \frac{1}{Gz}$. If you plot, so Nusselt number it will decrease in the developing region and once it becomes fully developed region it will become constant. So, you can see that for high Graetz number you can get the developing region.

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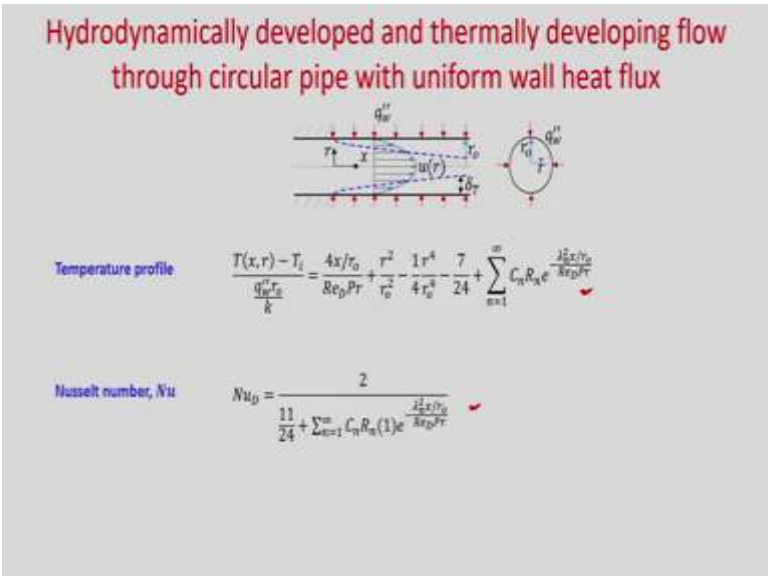

So, let us conclude. So, today we considered hydrodynamically developed and thermally developing flow through circular pipe with uniform wall heat plus boundary condition. And we started with the fully developed boundary condition and we have derived already the temperature profile in that region, and we defined one temperature T_d such a way that the temperature profile $T = T_d + T_{fd}$, and T_d is a quantity negative quantity which you subtract from the fully developed temperature profile.

Then, we used the equation for T_d, and with the boundary condition we got the governing equation and we applied the separation of variables method because the governing equation is linear and homogenous. So, we use the separation of variables method and using separation of variables method we use the orthogonality constant to find the value of C_n.

So, finally, we derived the temperature profile as these which is valid for both in thermal region, thermal developing region and fully developed region and the Nusselt number is these which is also valid for thermally developing region and fully developed region. And you have seen that at $\zeta \to \infty$, that means, in a fully developed region it gives the Nusselt number as $Nu_{Dfd} = \frac{48}{11}$ which you already derived. And it is true for the fully developed region.

Thank you.