Computational Continuum Mechanics

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Lecture – 09
Worked Examples - 1

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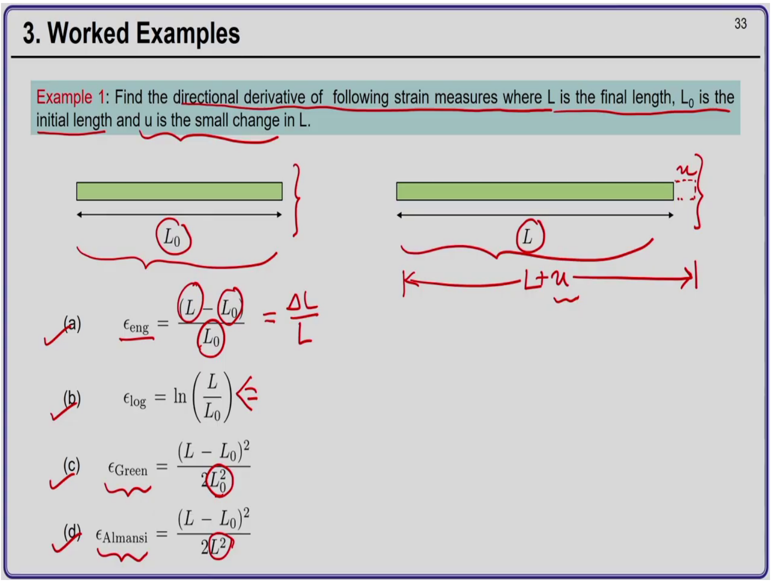

So, for the remaining part of this module, we will do some Worked out Examples ok. So, these examples will help you understand the theory that we have already discussed. And also we will derive some results, which will be used later during the later part of the course ok. So, we will first start with the following example ok. Consider that you have a rod of initial length L 0 ok. And under the action of external loads the rod elongates and the current length of the rod is L ok.

So, now, you can define different kind of strains ok. Without going deeper into how these strains are actually derived that or arrived at ok, we will just look into these four strain measures. So, this is a uni axial case.

So, there is no tensor or vector involved. The first definition of strain is called the engineering strain ok. This is the engineering strain which is nothing but the change in length ok, which is nothing but the final length minus the initial length divided by the initial length. So, this is like delta L by L ok. That is the change in length divided by initial length. That is called the engineering strain ok.

The next strain we can define is logarithmic strain ok. So, this is nothing but log of final length divided by initial length ok. So, this strain is also called the true strain ok. The third strain measure ok, it is called the Green strain ok, which is written here, the subscript is green. This is defined as the change of length square divided by 2 times of square of the original length and the last strain measure we have is the Almansi strain measure ok.

So, Almansi strain measure defines the change in length square divided by twice of square of the final length. So, now, once we have the initial rod which is extended to this final length then, what we have been asked is to find out the directional derivative of these strain measures, where L is the final length and L 0 is the initial length and for a small change u in the length L ok.

So, if there is a small change u ok. So, now, what is the change in the each of the strain measures ok? What we are doing is we are changing length from L to L plus u. And we want to know, what will be the change in the strain measures. Each of these four strain measures, because of this much change u change in the final length of the bar.

So, you would and I mean you would recall from the concept of directional derivative that the final value of the quantity, if you expand using Taylor series will be the initial value plus the change ok. So, now, we actually want to compute that change which is nothing, but the directional derivative.

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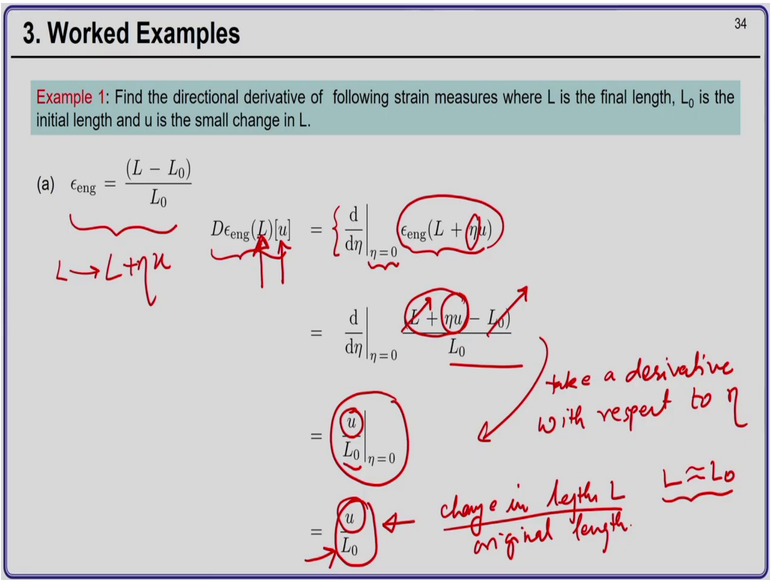

So, we will start with the first strain measure, which is the engineering strain ok. And then the directional derivative of the engineering strain at length L ok. So, the engineering strain depends on length L ok. L 0 is fixed, it is a constant quantity ok.

Only thing that will change is L. So, the directional directive of engineering strain at length L in the direction of change u. And the direction is here in the same direction as the length L will be given by and you will recall from our previous discussion on how to compute the directional derivative. It is given by d by d eta d by d eta of the engineering strain evaluated at L plus eta u and then substituting eta equal to 0.

So, now, what we do? In our definition of the engineering strain which is over here, we will substitute L by L plus eta u ok. And then that is what we have done here. We put L as L plus eta u, that is what this expression means ok. We have to compute engineering strain at L plus eta u. So, we substitute L as L plus eta u. And then the usual definition of engineering strain follows L plus eta u minus L 0 divided by L 0 ok.

Then, the next thing we do is take a derivative with respect to eta with respect to eta ok. So, if you take derivative with respect to eta.

So, L does not depend on eta ok; obviously, there is only one eta term which is over here. And L 0 does not depend on eta so that also goes away. So, the only term we are left with is u ok, because d by d eta of eta u will be u divided by L 0 and this has to be evaluated at eta equal to 0.

Now, since we do not have any eta term in this expression. So, this eta equal to 0 does not have any meaning ok. So, finally, we arrive at following value of the directional derivative of the engineering strain at length L in the direction u and this is given by u by L 0 ok. So, this is nothing but the change in length L divided by original ok.

So, now, if L was very close to L 0 if L was nearly equal to L 0 as would happen in elasticity problem small deformation problem. Then, in that case the directional derivative of this engineering strain is nothing, but your usual definition of strain true strain that you take, which is change in length divided by original length. That is the true strain that you take, but this is only when you have small deformation problem ok, where elasticity assumptions hold ok.

So, this is the. So, this final values shows the change in the value of engineering strain, when you change the length from L to L plus u ok. So, u by L 0 will be the change in the value of engineering strain in the direction of u ok. So, this is the directional derivative of the engineering strain ok.

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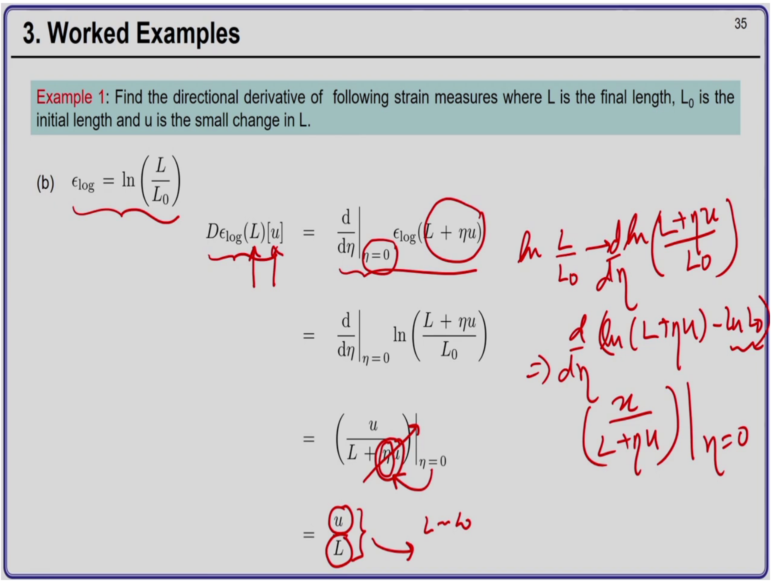

Next, we come to what is called the logarithmic strain or the actual true strain. So, this is given by natural log of the ratio of final length to initial length ok. And to compute the directional derivative of logarithmic strain at L length L in the direction of u will follow the usual procedure, which is given here. So, D epsilon log ok. So, logarithmic strain at L in the direction of u will be equal to d by d eta of logarithmic strain evaluated at L plus eta u and then substituting eta equal to 0 ok.

So, now, in our definition of logarithmic strain which is log of L by L 0, we will substitute L equal to L plus eta u L plus eta u divided by L 0. And then we have to take the derivative with respect to eta ok. So, this is nothing, but d by d eta of l n L plus eta u minus l n L 0 ok. So, the second term does not have any eta term.

So, it goes away and the derivative of first time will be u by L plus eta u evaluated at eta equal to 0. So, next thing is once you have taken the derivative with respect to eta, you substitute eta equal to 0. Remember eta is only an instrument for carrying out the derivative ok.

So, unlike in engineering strain wherever the where there was no eta present ok, here we have a term for eta ok. So, naturally when you substitute eta equal to 0 the second term in the denominator goes away it goes away. And what you actually get is u by L. So, this is the change in the value of length L divided by length L itself ok. Sorry in this previous slide I said this is the true strain sorry this is your u by L 0 is your usual expression for engineering strain ok. Change in length divided by original length ok and then, this in case of elasticity this u by capital L, which is the final length. In case of elasticity, because L is very close to L 0, this is your actual definition of true strain change in length divided by current length.

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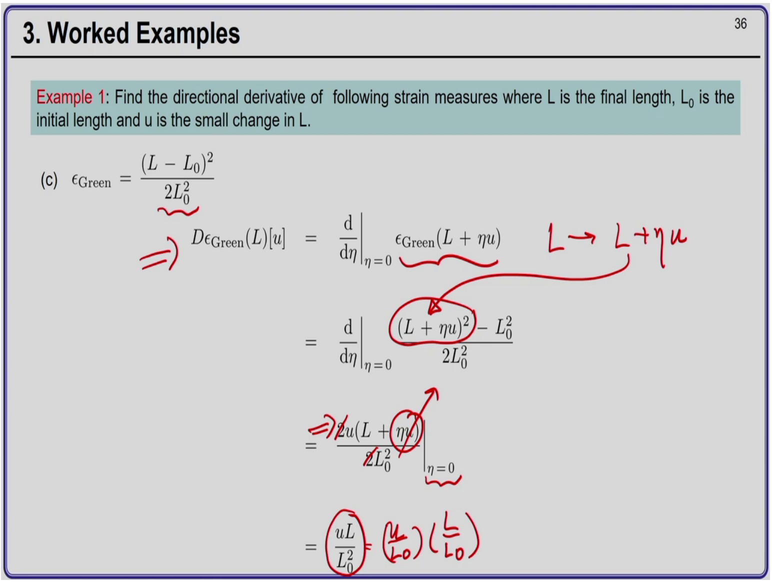

Next, we go to our third strain measure which is nothing, but Green strain. And this is defined as L minus L 0 the whole square divided by twice of L 0 square ok.

So, taking the directional derivative of the Green strain at length L in the direction u will be given by following expressions ok. So, the way you compute is, you compute substitute in the expression of green strain L as L plus eta u ok. So, you substitute L as L plus eta u ok. And then, that is what you get here ok. L has been substituted with L plus eta u and this minus L 0 divided by twice of L 0 square ok.

So, once you take the derivative, what do you get? It is only the first time here has eta ok. So, when you take derivative with respect to eta you will get twice of L plus eta u into u ok, which is here ok. And then the second step for computing the directional derivative is, you substitute eta equal to 0 ok. So, which means the second term over here will go away and this 2 will cancel out ok.

So, what you are left with is u L divided by L 0 square ok. So, this is u by L 0 you can see this is like this L by L 0 ok. So, now, u by L 0 you would recall was your the directional derivative of engineering strain and the second term is the ratio of the lengths ok. So, this is the u L by L 0 square is the change in the value of Green strain ok, because of a change u in the value of length L ok. So, once we have done this, the last strain measure that we had is called the Almansi strain measure ok.

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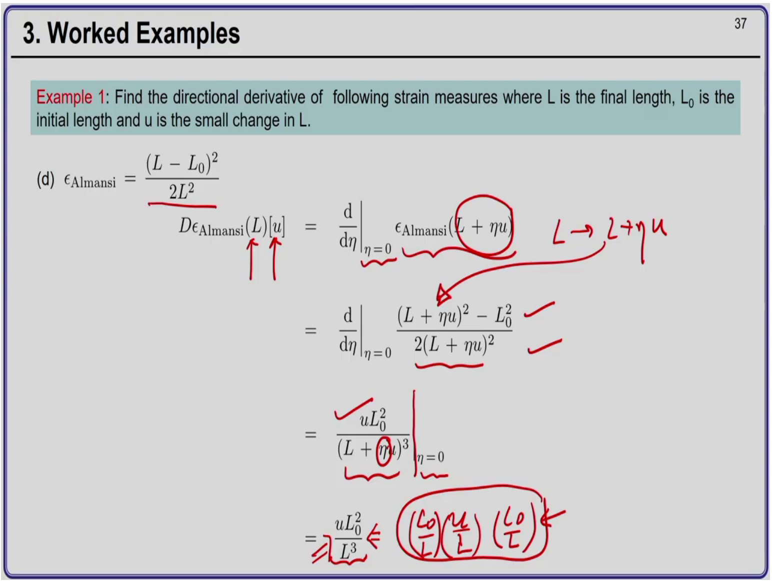

We will look into detail of all these strain measures once we go into kinematics ok. So, the Almansi strain measure is given by L minus L 0 the whole square divided by twice of L square ok. So, to compute the directional derivative of the Almansi strain measure at length L in the direction u is d by d eta of Almansi strain measure evaluated at L plus eta u and then substituting eta equal to 0 ok.

So, in the next step what you do? Substitute L by L plus eta u in the expression for Almansi strain ok. So, that is what we have done here minus L 0 square divided by 2 plus L plus eta u the whole square ok.

So, taking the derivative ok, now, you note both the numerator and the denominator have eta ok. So, you can take the derivative of this expression ok, which is a standard exercise. Finally, you will get u by L 0 square divided by L plus eta u the whole cube and then you have to calculate this at eta equal to 0. So, only in the denominator you have eta ok.

So, when you substitute eat equal to 0 you get u L 0 square divided by L cube ok. So, this is nothing but u by L L 0 by L and this is L 0 by L ok. So, we will come to it later when we go to multidimensional case. This is a 1D case, but if we go to multidimensional case the same form will come, but in the multidimensional sense ok.

So, I leave it at this point. All you need to care about right now is that the change in the Almansi strain at length L in the direction u is nothing but u by L 0 square divided by L cube ok. So, for this simple case you can now, see how we compute the directional derivative ok.

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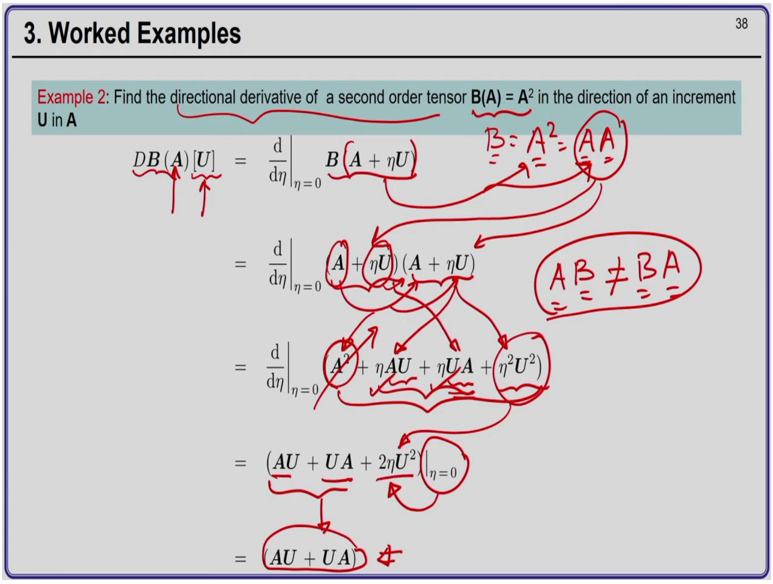

So, next we go to computing the directional derivative where second order tensors are present and consider this expression ok.

So, here B is A second order tensor which depends on another second order tensor which is A and how it depends? B is A square which means A into A ok. So, once we have A into A we can write it as A square. See, in direct notation, you can write power, but when you are writing in indicial notation I told if you remember that you cannot write that power ok. You have to break into terms of single degree ok.

So, now, we wish to compute the directional derivative of the second order tensor B in the direction of an increment U in A ok. What it means is, suppose I change A in the second order tensor from A to A plus U. So, because B depends on A therefore, there must be some change in the value of B, because I have changed A itself ok. So, what will be that change in the value of B at A in the direction of U ok.

So, to do that we have to take the directional derivative and from the concept of directional derivative ok. So, the directional derivative of B at A in the direction of U will be d by d eta of B evaluated at A plus eta U ok. So, this bracket over here does not mean that you have to multiply B by A ok. It means B is the function of A plus eta U ok. So, the directional derivative is d by d eta of B plus eta U B function of A plus eta U evaluated at eta equal to 0 ok.

So, now, what we do. In our expression for B we substitute instead of A we write A plus eta U and we do that in this second expression ok. So, here if you substitute A as A plus eta U this is what you are going to get. The first this is the first A which is replaced by A plus eta U, this is the second A replaced by A plus eta U ok. So, now, what we need to do is. We open up the brackets and then simplify the expressions ok.

Remember the order of multiplication for two tensors is very important. What it means is A multiplication of two tensors A and B will not necessarily be equal to B into A ok. So, we have to maintain that order ok. So, we have to see here. So, there is a first term multiply by these two terms ok. So, A into A is a square then, you have A into eta U which gives you eta into A U ok. Then the second term eta into U multiply it by this first term ok. So, gives you eta into U A. And then, the second term multiply by the second term on the second expression gives you eta square U square.

So, now, you have seen. We have A U and U A ok. I cannot write this also as U A A U ok. It will be wrong to do it ok, because of this property ok. Now, what we do. We have to take the derivative of this expression in the bracket with respect to eta ok. So, now, you have a constant term ok, you have a linear term in eta and you have a quadratic term in eta.

Once you take the derivative. The first term goes away because there is no eta involved. The second and the third term you have eta. So, derivative with respect to eta gives you A U plus U A. And the derivative of this last term over here will be twice of eta into U square ok.

And then, we have to compute this expression at eta equal to 0 ok. So, once you substitute eta equal to 0 this third term over here goes away, because two times 0 into U square will give you 0. And we are left with the first two terms and that is your answer ok.

So, the directional derivative of tensor B at A in the direction U is nothing, but A U plus U A. So, if you change A from A to A plus U, the change in B will be A U plus U A ok.

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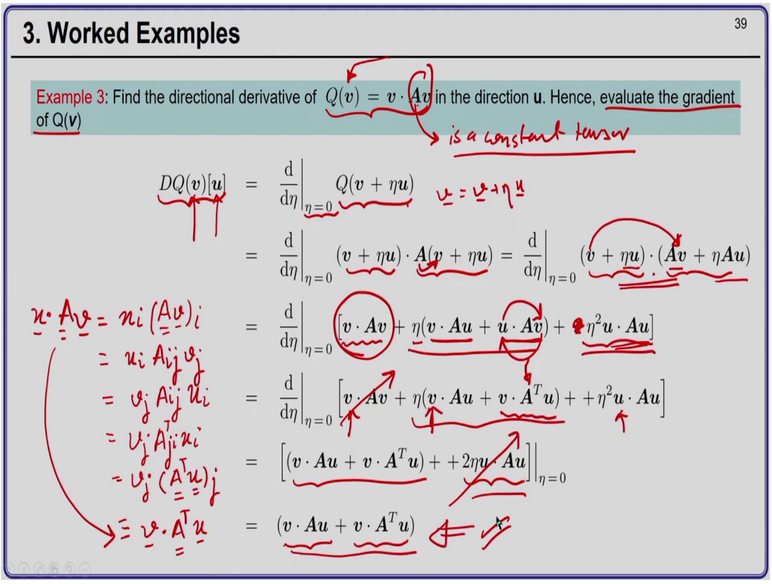

Next, we move to a little more complicated case ok. And we have encountered this particular expression in our previous lectures. So, we have to compute the directional derivative of this quantity Q v equal to v dot A v, where A is a constant tensor constant second order tensor ok. So, what it means is, you have this quadratic function ok.

It is a scalar value Q is a scalar value ok, because A v will be a vector, because a second order tensor operates on a vector v to give you another vector. And then you take a dot product with the vector itself. So, finally, you will have a scalar ok.

Now, what we want to know is if we change the vector v from v to v plus u ok. So, what will be the change in the value of Q in the direction of u?

Now, you are given that A is constant tensor. Finally, we are all also interested in evaluating the gradient of Q ok. So, how do we calculate the gradient of Q? So, the way to do is the directional derivative of Q at v in the direction u will be d by d eta of Q evaluated at v plus eta u and then substituting eta equal to 0.

So, this will give you the value of the directional derivative of Q at v in the direction u ok. So, the next step is we substitute v equal to v plus eta u in our expression of Q which is v dot A v. So, that is what we get. v plus eta u dot A into v plus eta u remember A is a constant tensor second order tensor ok. So, we can take A inside the bracket ok. That is what we have done ok.

So, we have v plus eta u dotted with A v plus eta A u ok. And then we can again take the dot product of these two brackets. We can carry out this expression simplify this expression. The first term will be v dot A v, which is here ok. The second term will be v dot eta A u. So, eta times v dot A u. Then, the next term will be eta u dotted with A v. So, eta u dot A v plus eta square u dot A u ok. Sorry there was one more plus eta square u dot A u ok.

Now, we have to take the derivative with respect to eta ok. So, our first term over here does not have any eta. We have a linear term in eta and we have a quadratic term in eta, but before that we proceed further, we can see. We have v dot A u here and we have u dot A v here ok. So, we can bring v on this side and u on this side by following procedure ok. That will be easier for us to understand and later on to compute the gradient ok.

So, how I have written u dot A v ok. So, the way I have to write is I can write indicial notation. This is u i and A v i ok, because a is a vector ok. So, u i A v i. now, this is u i A i j v j ok. So, this I can write as v j A i j u i. And this I can write as v j A transpose A j i u i ok. Once we swap the indices, I have to put a transpose ok.

So, now this I can write v j A transpose u u j ok. And then equivalently I can write in direct notation A transpose ok. So, that is how u dot A v is same as v dot A transpose u. And this is what we use here to write u dot A v as v dot A transpose u ok. Now, we have v everywhere we have v and we have u here ok. So, we have v at the first position.

Now, we take derivative with respect to eta ok. So, the first term goes away. This term will be v dot A u plus v dot A transpose u plus 2 eta u dot A u ok. And then we substitute eta equal to 0, which means this term this last term goes to 0. And we are left with v dot A u plus v dot A transpose u ok. So, if you change v to v plus u. So, the change in Q will be v dot A u plus v dot A transpose u ok.

So, you can very clearly see, it was a non-linear expression. So, you just changed v to v plus u. But this change over here which we get in the value of Q is not very clear at the very start unless until, we have computed the directional derivative explicitly ok.

So, this is a directional derivative. Now, we have been asked to compute the gradient ok.

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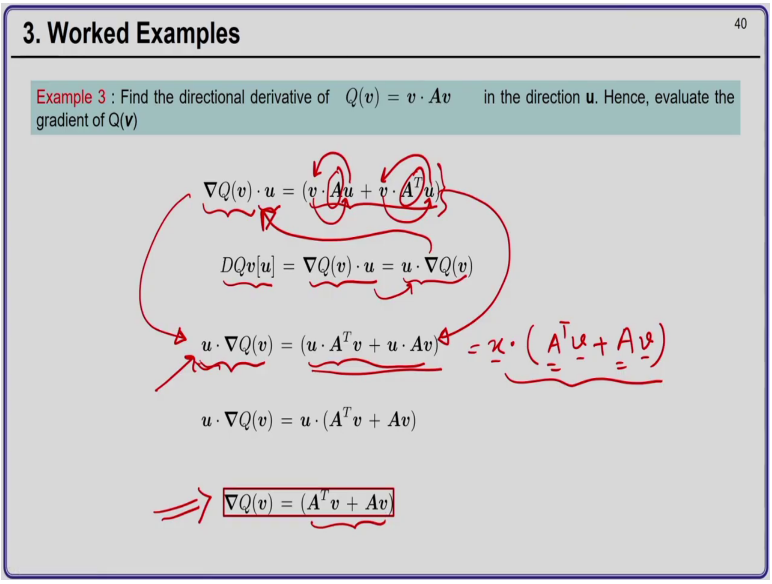

So, to compute the gradient ok, we note that the gradient of Q with respect to v gradient of Q at v in the direction u ok. This is nothing, but your directional derivative this is nothing, but v dot A u plus v dot A transpose u ok. And then our directional directive is also equal to gradient of Q dot u. And, because A dot B is same as B dot A, I can write this as u dot gradient of Q ok.

So, I can substitute this on the left hand side and I will get and then I will get u dot gradient of Q will be equal to u dot A transpose v plus u dot A v remember. So, what I have done over here as I have shifted u from the right hand side to the left hand side ok.

So, over here if I shift u here and v here, I will get a transpose on A. If I get u here and v here, I will get a transpose of this transpose which is nothing but the tensor itself ok. So, I will have u dot A transpose v plus u dot A v ok. And then I can take out u from the left hand side and I can write a transpose v plus A ok. And now, I can compare the left hand side to the right hand side.

So, you have u dot gradient of Q and on the right hand side you have u dot this particular expression in the bracket ok. Therefore, the gradient of Q is nothing, but A transpose v plus A v ok. That is how you compute the gradient of this scalar function Q ok.