Computational Continuum Mechanics
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Introduction to Tensors - 1
Lecture - 05
Tensor and Tensor Algebra – 2

So, the next concept that we have to look is eigenvalues and eigenvectors of a second order Tensor ok.

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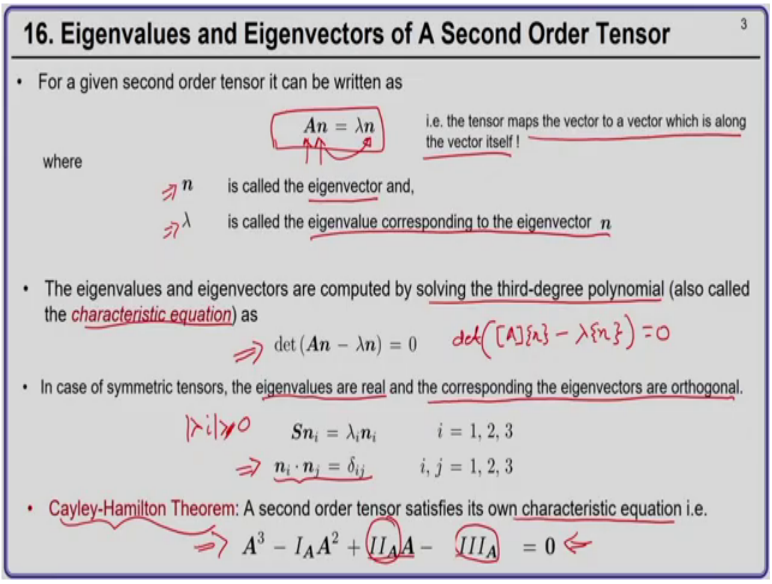

So, for a given second order tensor A, can we say that there is a vector which gets mapped to itself or some factor of itself? So, if there is such a vector which exists, that vector is called a eigenvector and the scalar multiple by which its projected is called the eigen value. So, as you can see here this equation A n equal to lambda n ok. So, here the second order tensor A is mapping this vector n to a scalar multiple of itself ok; scalar multiple is lambda and then you have n on the right hand side ok.

So, this tensor A maps, the vector to a vector which is along the direction of the vector itself ok. So, in this case n is called the eigenvector of A and lambda is called the eigenvalue corresponding to the eigen vector n ok. So, how do you find the eigenvalues and eigenvectors of a second order tensor? Ok. So, this you can find using solving a third-degree polynomial which is also called the characteristic equation and this characteristic equation is obtained, when you take the determinant of A n minus lambda n and set it to equal to 0 ok.

So, if I write using matrix notation, this would be determinant of A into vector n minus lambda vector n equal to 0 ok. So, this is just like you are finding eigenvalues and eigenvectors of a matrix A ok. In the previous lectures, we saw that you can write a second order tensor as a 3 by 3 matrix ok. So, to find a eigenvalue and eigenvectors of a second order tensor, you have to first express that tensor as a 3 by 3 matrix and then, you have to find the eigenvalues and eigenvector of the matrix.

Now, there are special class of second order tensor which are symmetric and symmetric tensors play the huge role in continuum mechanics. So, an important property for these symmetric tensors is that the eigenvalues of these tensors are real and that the corresponding eigenvectors are all orthogonal ok. So, this means that all lambda i are greater than equal to ok, sorry are greater than 0; they are real ok.

Sorry, mod of this. So, they are all real numbers and then, n i dot n j ok, if i is not equal to j will be delta ij, which means if i is equal to j, so n 1 dot n 1 will be equal to 1. Because, the eigenvalues are orthonormal and then n 1 dot n 2 for example: will be equal to 0 because the eigenvectors are orthogonal ok. So, the another important thing to note is this Cayley Hamilton theorem ok.

What it states is that a second order tensor satisfies its own characteristic equation which is given by A cube minus I A; I A is the first invariant of tensor A times A square plus I 2 into A ok; I 2 into A. So, II A; so, this II A is your second invariant of the tensor A minus III A into

A. So, this III A is basically third invariant which is nothing but the determinant into A. So, a second order tensor will satisfy its own characteristic equation ok. This is called the Cayley Hamilton theorem ok.

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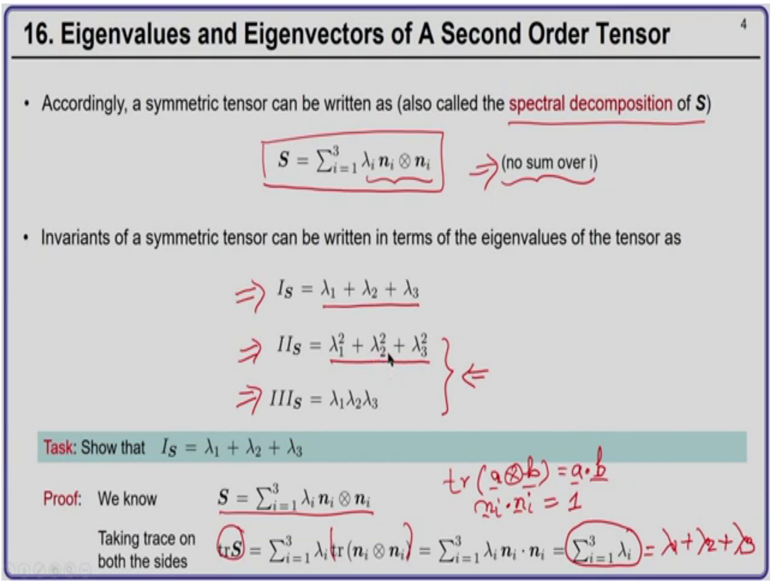

So, once you have the eigenvalues and eigenvectors, you can express a symmetric tensor S in terms of its eigenvalues and eigenvectors and this is called the spectral decomposition of the tensor symmetric tensor S and this is given by following equation ok. As you can see here s is given by summation over i equal to 1 to 3 lambda i n i tensor product n i ok.

Now, there are you would notice that there are three i's here in this expression. So, we have to explicitly write no sum over i. Because earlier, when we are using indicial notation, we denoted that if the indices is repeated, a summation over is implied; but here, we do not want summation.

So, we have to explicitly write here no sum over i ok. So, for a second order tensor which is symmetric, the three invariants of a second order tensor can be written as in terms of the eigenvalues of the second order tensor S. So, the first invariant is nothing but the sum of the eigenvalues; the second invariant is nothing but the sum of square of the eigenvalues and the third invariant is nothing but the product of the three eigenvalues ok.

So, now, suppose, we want to show that indeed the first invariant of the tensor, symmetric tensor is the sum of the eigenvalues. So, how can we show that? This we can show when we you can start with the spectral decomposition expression ok. So, we start with this spectral decomposition of S and then we take trace on both the sides.

So, now we have taken trace on both the sides. So, trace on the left hand side should be equal to trace of the right hand side. Now, we can take the trace inside the bracket that is here and then, we note that and this property, we discussed in the previous slides that trace of a tensor product b ok, will be equal to a dot b. So, here trace of ni tensor product n i will be n i dot will be n i dot ni ok. And we know from the property of eigenvectors of a symmetric tensor that the eigenvectors of a symmetric tensors are orthogonal and they are also orthonormal.

So, ni dot ni will be equal to 1 ok. Once you have trace of ni tensor product ni is 1, you are left with only summation over lambda i's which is nothing but lambda 1, plus lambda 2 plus lambda 3 and that is what we have we wanted to show ok. So, similarly, you can show the other two expressions here ok. So, you start with the expression for second invariant and the third invariant and you can definitely show that these properties hold and these properties will be use later on during hyper elasticity ok.

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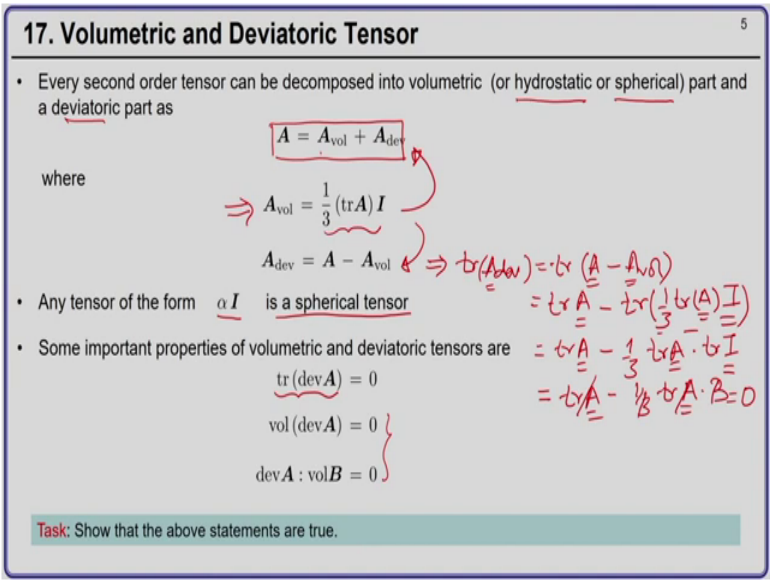

Next, we come to what is meant by volumetric and deviatoric tensors ok. So, every second order tensor A can be decomposed into a volumetric part ok, which is also called the hydrostatic part or the spherical part and a deviatoric part ok. So, as you can see here from this equation, a second order tensor a can be written as a volumetric part A subscript vol; vol stand for volumetric and plus A subscript dev which is short for deviatoric ok. So, now, the volumetric part is given by 1 by 3 trace of A into second order identity tensor and then, using this expression over here, we can get the expression for the deviatoric part which is given by A minus the volumetric part ok.

So, one thing to note is any tensor of the form alpha I, where alpha I is a real number will be a volumetric tensor or a hydrostatic tensor or a spherical tensor ok. So, some of the important properties of the volumetric and deviatoric tensors are given here. So, the trace of the deviatoric part of the tensor A will be equal to 0 ok.

So, this we can prove, suppose we want to prove. So, we can start from the expression. So, this is nothing but the deviatoric part of A, if you take the trace ok. Trace of A deviatoric will be trace of A minus A volumetric which is nothing but trace of A minus trace of A volumetric ok.

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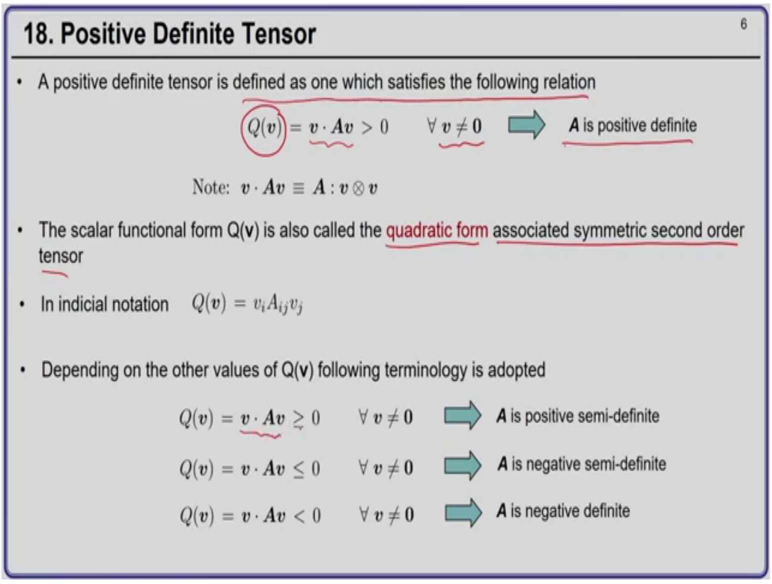

Where, the volumetric part is 1 by 3 trace of A into I ok. Now, 1 by 3 and trace of a these are scalar quantities. So, they can be taken out. So, trace of A minus 1 by 3 trace of A into trace of second order identity tensor. So, what is the trace of second order identity tensor? That is nothing but it is equal to 3 ok. So, you have 1 by 3 trace of A into 3 to trace of A. So, this 3 gets canceled out and then, this trace of A and trace of A canceled out, which gives you 0 and that is what we wanted to prove ok.

Similarly, you can show the other two relations and this, I leave it to you as a task ok. You can do it and if you have any problem, as usual you can always contact me. The next important concept is the concept of positive definite tensor ok. So, what is a positive definite tensor? So, positive definite tensor is defined as one which satisfies the following relation. So, you are given a tensor A and for any given vector v which is not equal to the 0 vector, if v dot A v is greater than 0, then that tensor A will be called a positive definite tensor ok.

So, this quantity Q v is also called the quadratic form associated with the symmetric second order tensor A ok. Now, there can be other terminologies for the quadratic form Q v associated with the values of v dot Av ok. So, say v dot Av is greater than equal to 0. See earlier, it was only greater than 0. Now, if you allow also values to go to 0 ok, for all v s which are not equal to 0; then, A will be called positive semi definite ok.

Now, if v dot Av is less than equal to 0 ok, for all values of v which are not equal to 0 vector, then A is called the negative semi definite and if v dot A v is always less than 0, whatever with the value of v ok. Then, A is called the negative definite tensor.

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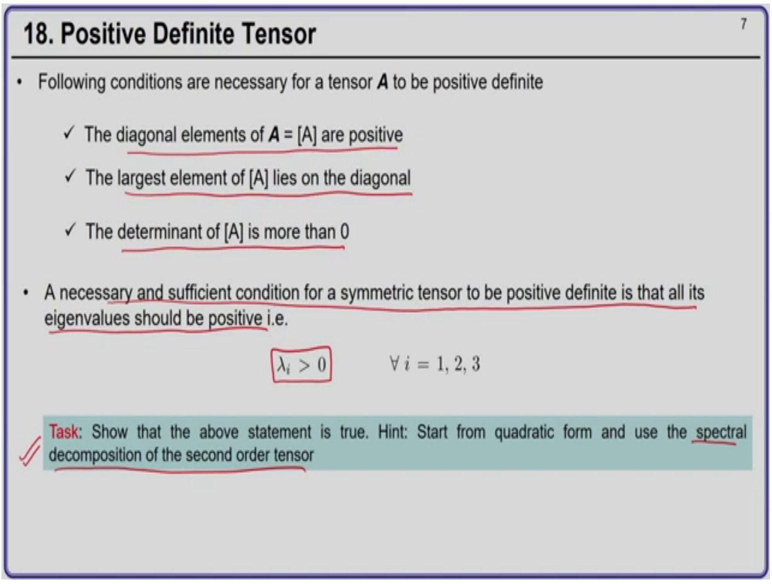

So, what are the conditions under which a tensor can be positive definite? So, I have stated three conditions here; the first one is the diagonal elements of A that is the matrix formed by the second order tensor A are all positive, they all should be positive, that largest element of A should lie along the diagonal of the matrix and the determinant of A should be more than 0 ok. If these properties are satisfied, then the tensor will be a positive definite.

So, you need not do all the way v dot A v. If these three properties hold, then that tensor will be a positive definite tensor ok. A necessary and sufficient condition for a symmetric tensor to be positive definite is that all its eigenvalues are positive ok; as you can see here, all the eigenvalues have to be positive ok. They have to be real and they have to be positive ok.

Now, your task is to show that this statement that the for a symmetric second order tensor to be positive definite, all is eigenvalues are greater than 0; you have to prove that statement and

I have provided hint also. So, again, you try it yourself and if you have any problem, you can always contact me and I will give you some hints on how to solve this.

So, written here, you have to start from the spectral decomposition of a second order tensor. You can also look into some linear algebra book because a second order tensor can be written as a 3 by 3 matrix. So, it also implies that is for a symmetric matrix, its eigenvalues are always greater than 0 ok. So, in linear algebra book, you can find this proof; the same proof applies for a symmetry second order tensor ok.

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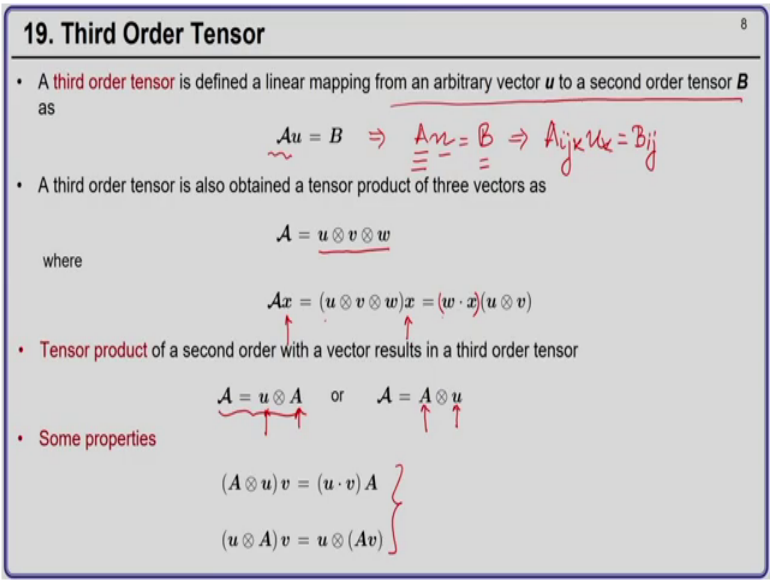

Now, we come to leave second order tensor and now, we come to higher order tensor ok. Before we go to fourth order tensor, it is good that we spent some time with what are called the third order tensor. We will not encounter third order tensor much, but before making the jump from a second order tensor to a fourth order tensor, definitely there will be a question are there some third order tensor. So, we will discuss this. So, a third order tensor is defined as a linear mapping from an arbitrary vector u to a second order tensor B ok.

So, you see the symbol here ok, it is a calligraphic A that is how you we represent higher order tensors Third order or fourth order, we will use the calligraphic bold symbol. So, when we are writing will always write A and then, will put three under bars that is how I prefer, you put 3 under bars to show that it is a third order tensor. So, A u is equal to B ok.

So, a third order tensor operates on a vector and gives you a second order tensor ok. So, just like a second order tensor operated on a vector to give you, another vector; a third order tensor operates on a vector to give you a second order tensor. So, in indicial notation, this we can write as Aijk u k equal to Bij ok.

Now, a third order tensor can also be obtained using the tensor product of three vectors say you have three vectors u, v and w. So, if you take u tensor product v tensor product w, so the resulting result in quantity that you get is a third order tensor A and the way it functions is if you take an arbitrary vector x, then u tensor product v tensor product w operating on x is defined as w dot x into u tensor product v ok.

So, there are other ways to obtain a third order tensor. So, one way is you take the tensor product of a vector with a second order tensor; u tensor product A gives you a third order tensor A; tensor product of a second order tensor A with vector u again gives you a second order tensor, a third order tensor A. So, some of the properties of some of the properties of a third order tensor are mentioned here ok. And this, you can show it yourself ok. So, you have to remember these properties.

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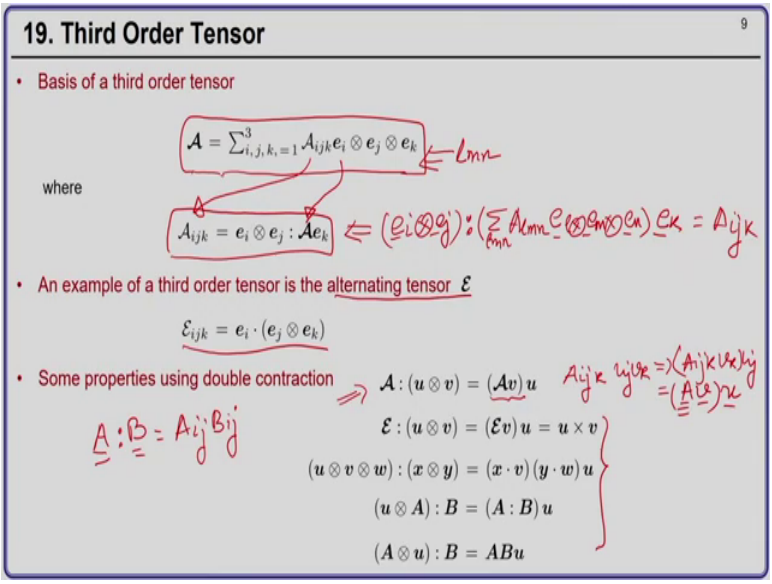

Now, just like we had defined basis for vectors and second order tensor, similarly a basis for a third order tensor is the tensor product of the base vectors ei, ej, ek and its written here; as third order tensor A will be triple summation over i, j, k Aijk ei tensor product ej tensor product ek ok, where the component of this tensor a that is Aijk can be obtained by following relation ok; ei tensor product ej double contraction with A ek ok.

Now, you can show this also. It is not very difficult. Just notice that i, j, k in this expression of free indices. So, all you need to do is substitute this expression of A here and then, instead of using ijk in this expression, just use some other symbol say l, m and n ok. So, all you need to do is start from ei tensor product ej double contraction with summation over lmn, A lmn e l tensor product e m tensor product e n operating on e k ok.

And then, use the properties which are mentioned in the previous slides and you can simply show that you will get Aijk. So, one of the examples of a second order or a third order tensor is the alternating tensor e that is do your permutation symbol. So, epsilon ijk is given by ei dot ej tensor product ek ok. This also again you can show, we already had discuss ok.

How to get the component of a our third order tensor? You can use the expression and show that indeed epsilon ijk is ei dot ej tensor ek ok. Now, the trouble contraction that we define between two second order tensor, you can recall that A tensor product B was Aij Bij, that is how a second order, the double contraction between two second order tensors was defined. Similarly, we can extend the definition to higher order tensor.

So, for example, a third order tensor double contracted with a dyad u tensor product v will be nothing but A v into u ok. So, this you can easily show. I will write using index notation to be very fast Aij k uj vk which is nothing but Aijk vk uj and this is nothing but indirect notation a v into u which is the right hand side.

Similarly, there are other properties associated with the double contraction applied to third order tensor and are mentioned here. You can obviously, prove them using indicial notation or using operational approach.

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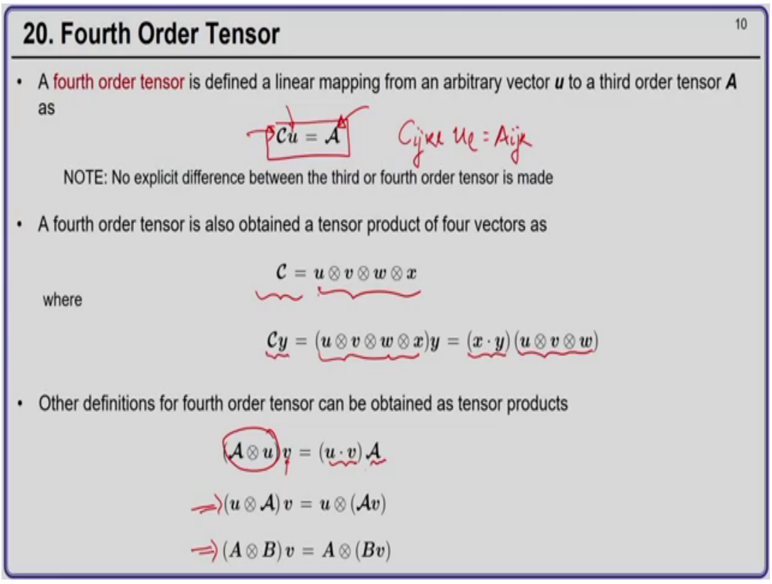

Next, we move to what are called fourth order tensor ok. So, a fourth order tensor will operate on a vector to give you what is called a third order tensor ok. So, notice let calligraphic C, that bold calligraphic C here represents a fourth order tensor. It operates on a vector u and it gives you a third order tensor A. So, an indicial notation I can write Cij kl u l equal to Aijk.

So, note that we do not make any explicit differentiation difference between third order or fourth order tensor ok. They are both written in the same way and then, from the context of your problem you will be able to judge which one is the fourth order tensor which one is the third order tensor. So, at this stage, there should not be any confusion ok. You will know from the context of your problem.

Now, as was as in the case for a second order tensor and third order tensor, you can take the dyadic of a 4 vectors to get a fourth order tensor. So, if you have vectors u, v, w and x; so, the tensor product of u with v, v with w and w with x. So, u tensor product v tensor product w tensor product x, this will give you a fourth order tensor C and when then, C operates on a vector from the definition; it should give you a third order tensor.

So, u tensor product v tensor product w tensor product x operating on v will be x dot y and u tensor product v tensor product w. So, there are some other definitions of fourth order tensor that can be obtained as tensor product. So, you can take tensor product of a third order tensor with a vector to get a fourth order tensor and this when operates on a vector v will give you u dot v into the third order tensor A ok.

So, tensor product of a vector with a third order tensor also gives you a fourth order tensor and tensor product of two second order tensor gives you a fourth order tensor.

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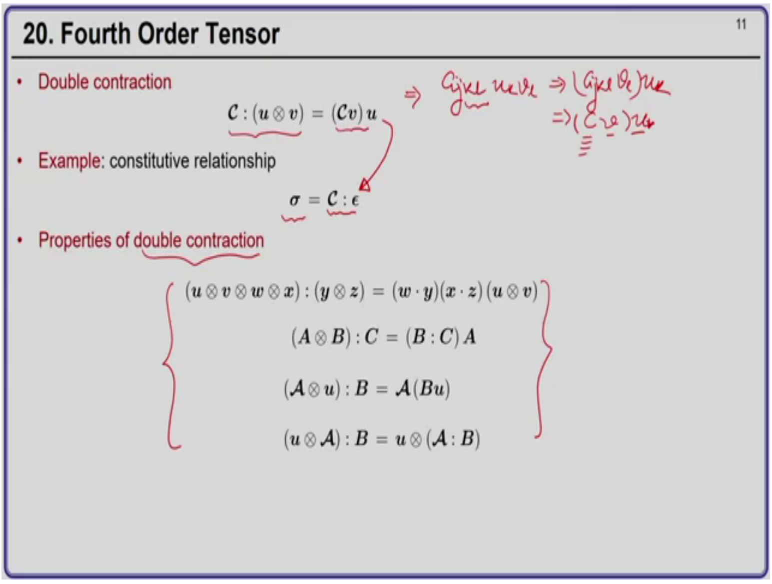

Now, the double contraction between a fourth order tensor and a second order tensor ok. So, a second order tensor say a dyad u tensor product v and C is the fourth order tensor. So, the double contraction, the way its defined is C double contracted with u tensor product v will be equal to Cv into u. So, in indicial notation the way I will write is Cijkl; four indices because it is a fourth order tensor and because there is a double contraction, the last two indices k and l will be same for the next second order tensor which will be u k v l ok.

So, this is nothing but C ijkl vl u k, this is indirect notation Cv. Notice, I have put 4 under bars u sorry u 1 under bar ok. So, that is how you can show. So, example of this relation ok, this property is this relation over here where as the stress tensor, Cauchy stress tensor can we obtain as double contraction of the fourth order material constitutive tensor C with the strain tensor ok, that is where this double contraction of a fourth order tensor with a second order tensor comes ok.

Now, there are certain properties of double contraction associated with fourth order tensor and this, we have listed here ok. So, you can again use the indicial notation, you can use operational approach to verify each and every one of them and I strongly suggest that you try these yourself. It will give you a lot of practice and if you have any problem, please do not hesitate to contact me.

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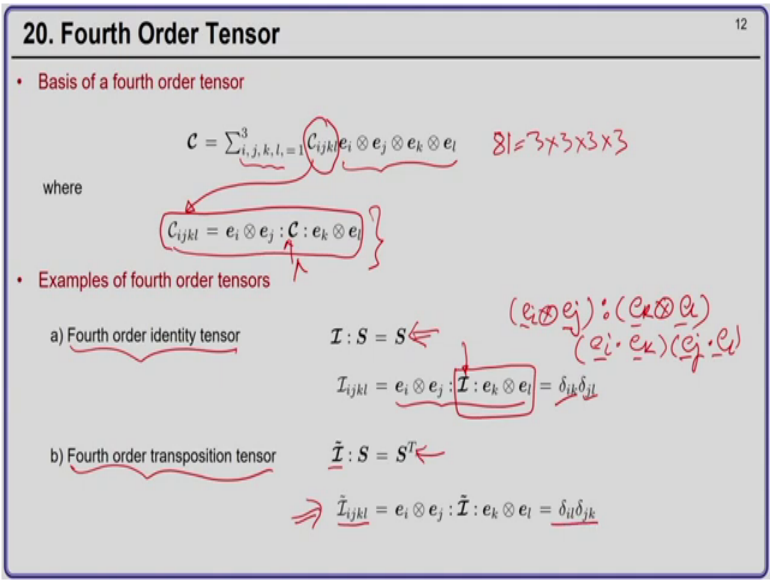

So, now as was with the case with second order tensors and third order tensors, there is indeed a basis for fourth order tensor and this is obtained by taking the tensor product of the basis vectors e i, e j, e k. e l ok; where, i j k l will be going from 1 to 3 and corresponding to each base basis, you will have a component which is C ijkl. So, in total there will be 81 such components ok. Because at each place, each indices can have 3 options; so, 3 into 3 into 3 into 3, total that is 81. So, there are 81 components of a general fourth order tensor and you can get the component of a fourth order tensor by using this relation over here ok.

C ijkl is e i tensor product e j double contracted with the fourth order tensors C double contracted with e k tensor product e l ok. Now, there are two specific examples of fourth order tensor which are of much use ok, they are fourth order identity tensor. What is a fourth order identity tensor? It is denoted by this calligraphic bold I. This fourth order identity tensor when contracted with double contracted with a second order tensor S, will give you the second order tensor S itself ok, that is the property of fourth order identity tensor ok.

Once you have this property and you know this relation were here ok, this relation, you can get the components of this fourth order identity tensor. So, this is the relation instead of C here, you just substitute the fourth order identity tensor and then you can use this property that I contracted with e k tensor product e l will give you nothing but e k tensor product e l and then, you have e i tensor product e j double contracted with e k tensor product e l ok.

And what is that? That is e i dot e k e j dot el ok. e i dot e k is nothing but delta i k and e j dot e l will be delta j l ok. So, the components of fourth order identity tensor is delta i k delta j l. Similarly, there is fourth order transposition tensor which is I tilde here and when it double contracts with S, it gives you the transpose of the tensor and the components of this transposition tensor here I tilde i jkl is nothing but delta i l delta j k ok.

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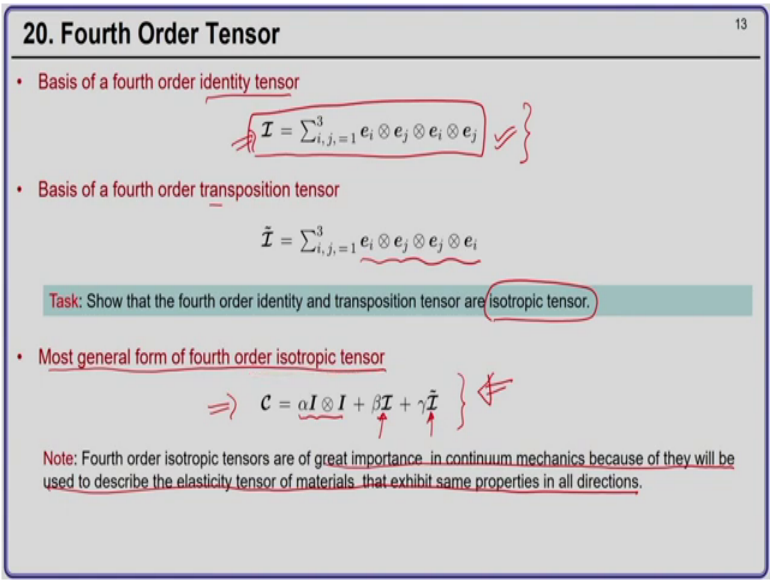

So, once you have these components ok, if we go to our previous slides. So, this is ij kl, the component delta ik delta jl. Now, this if you substitute ok; if you substitute here in this expression, then the fourth order identity tensor can be written as summation over i j from 1 to 3 e i tensor product e j tensor product e i tensor product j ok. This is very simple to show.

Similarly, the basis for a fourth order transposition tensor will be ei tensor ej tensor product e j tensor product e i ok. Now, you can show that both of these fourth order identity tensor which is the identity tensor and the transposition tensor are isotropic tensor. So, as you would remember an isotropic tensor is one, whose component do not change with the change in the basis ok.

Once you rotate the basis or your coordinate system, then the component of the tensors do not change ok. So, what you have to do is write these relations over here in the prime bases and then because e i, e j, e k, e l are given to be base vector. You know the relation between e i dash and e i ok, use this in expressions over here and then, you can show that the component of say identity tensor in the prime bases and the un prime bases are both same ok.

So, the most general form of a fourth order isotropic tensor can be obtained as scalar multiple of the tensor product of second order identity tensor plus fourth order identity tensor and fourth order transposition tensor ok. So, C will be alpha I tensor product I plus beta I plus gamma I tilde. So, what is this?

Why this tensor is important? It is important because fourth order identity tensors are of great importance in continuum mechanics. Because they will be used to describe the elasticity tensor of materials, that exhibit same property in all the directions ok. So, if you want to represent properties of material which has same property in all the direction, then the most general form for that fourth order tensor will be this isotropic tensor gain by expression over here.

Thank you.