Computational Continuum Mechanics
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Introduction to Tensors - 1
Lecture – 04
Tensor and Tensor Algebra - 2

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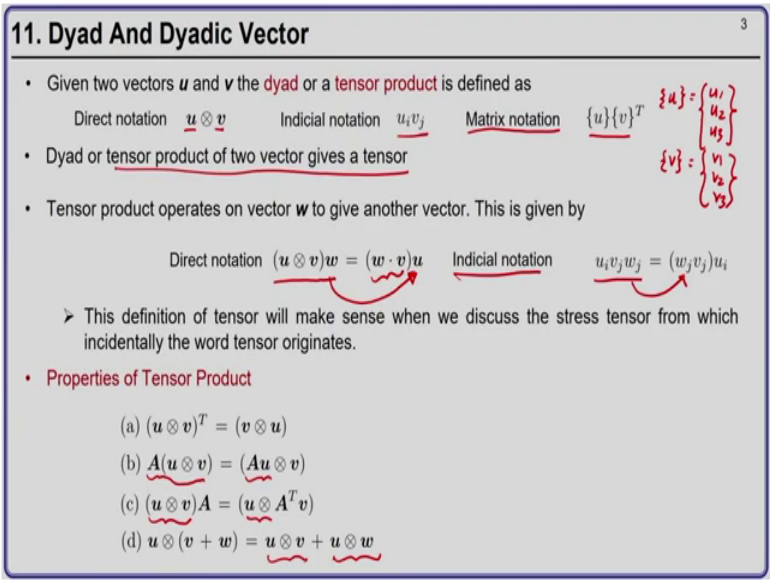

So, next we start with the concept of a dyad or a dyadic vector. So, given two vectors u and v, the dyad or the tensor product is defined as in direct notation; u tensor product v, ok. So, the symbol that you see here across with a circle is called tensor product, ok. So, the way to read this is u tensor product v ok; please do not read this as u multiplied by v or u multiplication v, ok. So, this is specifically read as u tensor product v. So, the in direct notation you write u tensor product v, in indicial notation it can be written as u i v j, ok.

So, the matrix notation, although I am writing it here for the sake of completeness; how we have come to this will be discussed later ok, in maybe a few slides later. So, it is written as vector u multiplied by transpose of vector v, ok. So, where vector u is u 1, u 2, u 3 and vector v is v 1, v 2, v 3. So, you can recognize from your concepts of linear algebra that, this matrix product is basically the outer product of two vectors ok; just like you had the inner product of two vectors which results in a scalar. So, the outer product of two vectors leads to a matrix, ok. And we will see that, you can actually write a second order tensor as a matrix ok, that for later, ok.

Now as I already mentioned that, the tensor product of two vectors gives you a tensor, ok. So, you can see from the indicial notation; you have two free indices i and j, ok. So, there are two indices i and j. So, there are total two indices, that is why it is a tensor or specifically a second order tensor.

Now, how does tensor product acts? So, if you take a arbitrary vector w and write u tensor product w; then what do you get? So, as you know a tensor maps one vector to another vector, ok. So, if you take a vector w; so how does this tensor product u tensor product v acts on w? So, the way it acts is, it projects the vector w along the vector u and scales it by a factor of v dot w dot v, ok.

So, in indicial notation you can write this as u i v j w j, ok. So, v j w j you can take in the bracket and v j w j is nothing, but w dot u, ok. So, a dyad operates on a vector by projecting the vector along one out, one of its vector and by scaling it. So, this definition of the tensor will make more sense I mean when we will discuss about the stress tensor ok, from which the word tensor itself originates. There are some properties of tensor product which I am stating here; the first one is the transpose of a dyad ok, u tensor product v is nothing, but dyad of v tensor product u, ok.

Now, if A is a second order tensor, then the product of A with the dyad u tensor product v is nothing, but A u tensor product v. Again the products of the dyad u tensor product v with a second order tensor A is nothing, but u tensor product A transpose v, ok. Finally, u tensor product v plus w is nothing, but u tensor product v plus u tensor product w.

So, you can actually show all of these properties that they hold by taking an arbitrary vector and using the operational approach that we discussed in the previous lectures; you take a arbitrary vector and operate and try to see what is the resulting output that you get, ok. With this concept operation of approach operational approach, you will be able to prove all these properties, ok.

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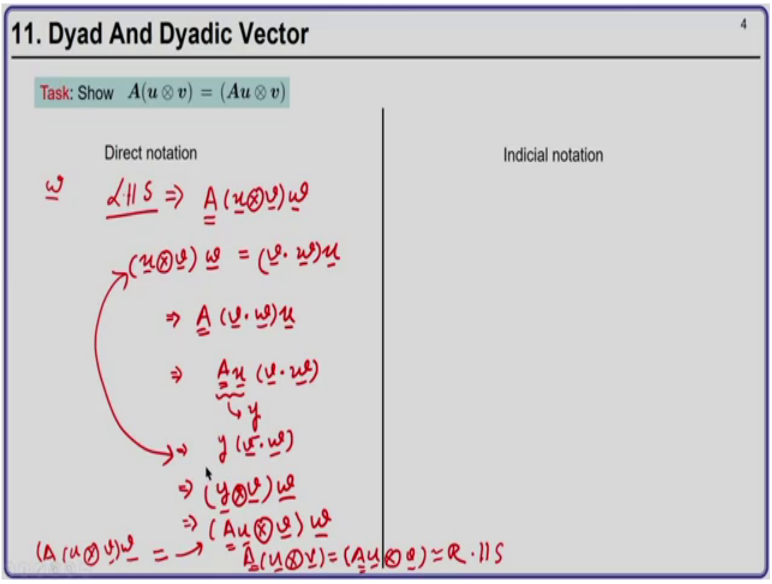

So, let us try to prove the second property that we had. So, now we want to show that A into u tensor product v is indeed equal to A u tensor product v, ok. So, we can either follow direct notation or we can follow indicial notation ok, and we use the operational approach. So, in operational approach, what we do? We consider or an arbitrary vector w and operate it on our given expression. So, we start with the left hand side, ok. I am doing the direct notation and I leave the indicial notation for you to complete, ok. If you have any problem in indicial notation, you can always write a email to me or you can get back to me and I will reply with necessary hints, ok.

So, let us start with the left hand side; so A u tensor product v operating on arbitrary vector w, ok. So, now, you know from the previous slide that, u tensor product v into w will be nothing, but v dot w into u. So, v dot w is a scalar, ok. So, the dot product of two vectors is a scalar. So, if we use this in our expression, we get A v dot w into u or I can write A u and v dot w

Now A u ok; so A is a second order tenser, u is a vector, so indeed A u will give you another vector say y, ok. So, now, what we have is v y v dot w, ok. Now again using the same property which is here, ok. So, you can relate to this property and you can write this expression as y dot v into w. Now y is nothing, but A sorry, this is tensor product A u tensor product v into w, ok.

So, since w is an arbitrary vector, we can say on the left hand side you had A u tensor product v into w, ok. Let me just write, which is equal to this quantity over here. Now w is an arbitrary vector; so you can see that A u tensor product v is nothing, but A u tensor product v, ok. So, that is how and this is nothing but your right hand side, ok.

So, similarly you can show the other properties using the similar concept ok; you take an arbitrary vector w and then operate it on the left hand side, ok. You start with the left hand side, it does not matter, you can also start with right hand side; but if you start from left hand side, you can follow the similar procedure and you can establish the right hand side by what we have applied to this particular case.

So, the indicial notation is left for you as an exercise and it will follow a similar procedure, ok.

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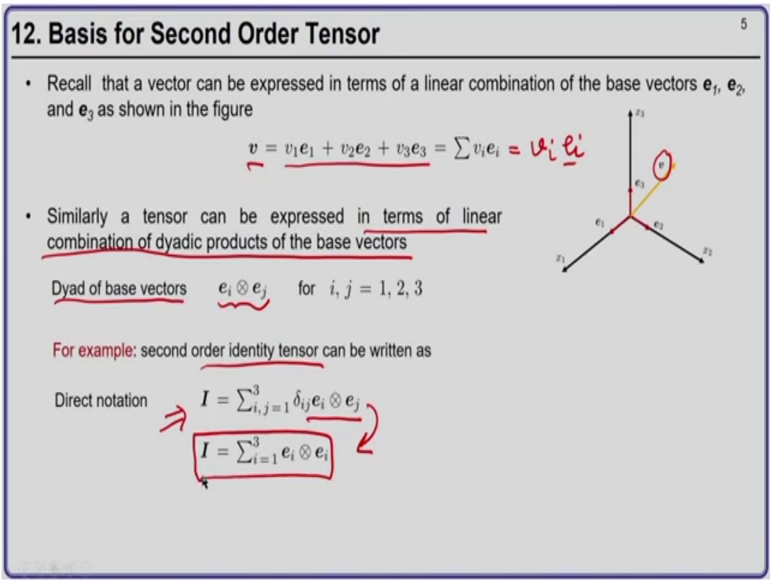

So, next we move to the basis of second order tensor, ok. So, you must, you may recall that a vector ok; as you see here you have a Cartesian coordinate system x 1, x 2, x 3 and e 1, e 2, e 3 are the base vector, ok. So, any vector v can be written in terms of the base vectors as v 1 e 1 plus v 2 e 2 plus v 3 e 3 or in short you can write summation v i e i; or if we follow our summation convention, so this is nothing, but simply v i e i with the implication that a summation over i is implied, ok.

Now, the question arises; can a second order tensor have a basis ok? And the answer is yes, a second order tensor can be expressed in terms of linear combination of dyadic product of the base vectors, ok. So, u tensor product v is also called the dyadic product. So, you should form a dyad of these base vectors which is written as e i tensor product e j, for i and j going from 1 to 3, you have total of 9 such dyads, ok.

You have e 1 tensor e 1, e 1 tensor e 2 all the way up to e 3 tensor e 3; you have 9 such dyads, ok. And any second order tensor can be written as scalar multiple of these base vectors ok, dyad of these base vector. For example, let us consider a second order identity tensor, ok. So, in direct notation it is written as i equal to delta i j e i tensor e j; or using the substitution property of chronicle delta, you can write i as summation over i e i tensor product e i, ok.

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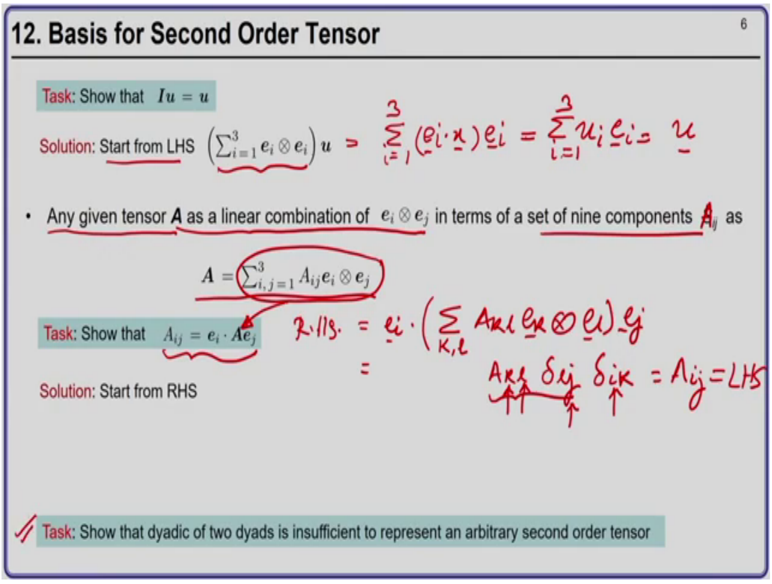

So, can you show that I u is equal to u. So, I is a identity tensor; so it will map the vector to itself, ok. So, let us start from the left hand side and we will substitute on the left hand side the expression for I in terms of its bases which is nothing, but summation over i e i tensor e i into u. Now using the property of dyad product, you can write this as i equal to 1 to 3 e i dot u into e i, ok. So, what is e i dot u? It is nothing but u i; it is nothing but the i th component of vector along the e ith base vector. So, you have summation i equal to 1 to 3 u i e i.

And what is this? This is nothing, but vector u itself. So, you can show that a second order identity tensor maps the vector to its itself, ok. So, generalizing we can write that any given tensor A can be written as a linear combination of the dyads of the base vectors and it has 9 components ok; sorry this should be A i j as A equal to a summation over i and j A i j A i tensor e j, ok. Now, how do you get each component of the tensor that is A i j? For this the formula is A i j is equal to e i dot A operating on e j, ok.

So, can you show that; yes, we can show. So, we can start from the right hand side now ok; I can start from the right hand side and what I will do is, I can substitute this expression over here ok, here on the right hand side ok; and instead of i and j, I will use k and l ok, k and l A k, l e k tensor product e l e j,. So, now, I can write this as e i dot summation over k, l A k l e l dot e j into e k, ok. So, what is e l dot e j? E l dot e j is nothing, but ok; let me write here e l dot e j is nothing but delta l j, ok.

And now I can take this e i dot inside ok; because both A k l and delta l j are nothing, but scalars, ok. So, I can take e i dot e k. And what is e i dot e k? It is nothing, but delta i k, ok. So, I can write delta i k and this summation sign also goes away. So, what I am left with is, A k l delta l j delta i k. So, using the substitution property, this k over here will be replaced by i and this l over here will replace by this j. So, finally, what I will get is A i j, ok. So, this is nothing but your left hand side, ok. So, to get the i jth component of a second order tensor; what you have to do is you have to take the operation which is e i dot A operated on e j.

Now, a task for you now is, to show that a dyadic of two dyads is insufficient to represent an arbitrary second order tensor. Till now we have not shown that u tensor product v is a second order tensor; but considering that this is indeed the fact that, u tensor product v is a second order tensor.

Then, can we say that all tensors can be written as u tensor product v; or this u tensor product v is insufficient to represent any given second order tensor, ok?

So, you can show that dyad of two vectors that is u tensor product v is insufficient for representing any second order tensor. For this what you have to do is, take an arbitrary vector w and operate u on and take the dyad u tensor product v and operate on w.

So, what you get? You will get a vector, ok. So, that vector will always be along the uth direction, ok. So, whatever may be your vector w; u tensor product v will always project that vector w in the direction of u, which is not general. So, this proof I will leave it to you, and now again if you have any doubt you can always come back to me; you can contact me and I will be able to help you out.

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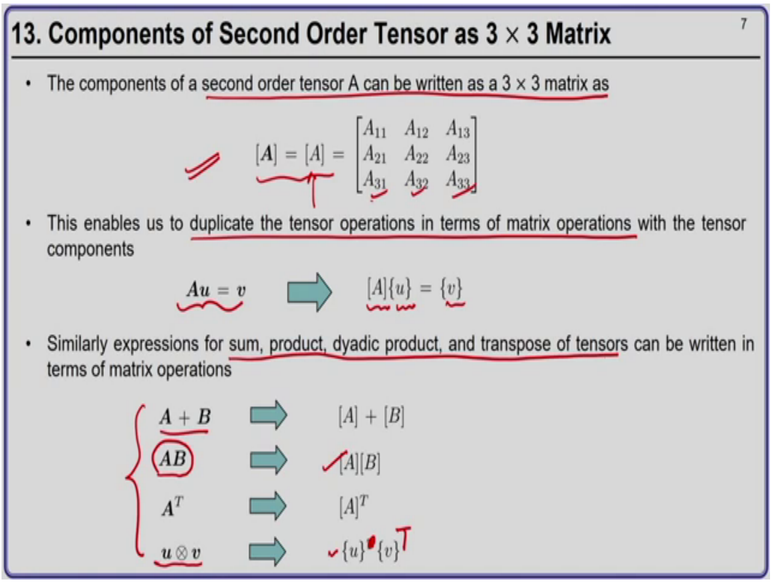

Now, in finest element context, it is always essential that we should be able to write the components of a second order tensor in a matrix form and that is how you can write, ok. So, the components of a second order tensor A can be written as a 3 by 3 matrix, where a symbol, following symbol is used; either you can put A as a bold or in when you are writing by hand, you can just put this square brackets, inside that you can put A and then A 1 1, A 1 2, A 1 3, A 2 1, A 2 2, A 2 3, A 3 1, A 3 2, A 3 3.

So, this forms here 3 by 3 matrix. So, all 9 components of a second order tensor can be nicely put into a 3 cross 3 matrix. With the help of this you can duplicate the tensor operations in terms of matrix operation. So, for example, tensor A operating on vector u gives another vector v; so in matrix form this can be written as matrix A multiplied by vector u gives you vector v, ok. Similarly you can write the expression for sum, product, dyadic product, and transpose of the tensors in the matrix operation ok, which are shown here, ok.

So, for example, the sum of two tensors A and B is nothing, but the sum of the matrix components of the two tensors; the product of two tensors A and B is nothing but the product of the matrix form of the two tensors A and B, ok. So, u tensor product v is nothing, but vector u and v transpose, u v transpose, ok. So, u transpose v will actually make it a dot product. So, u v transpose will make the matrix equivalent of the tensor expression u tensor product v, ok.

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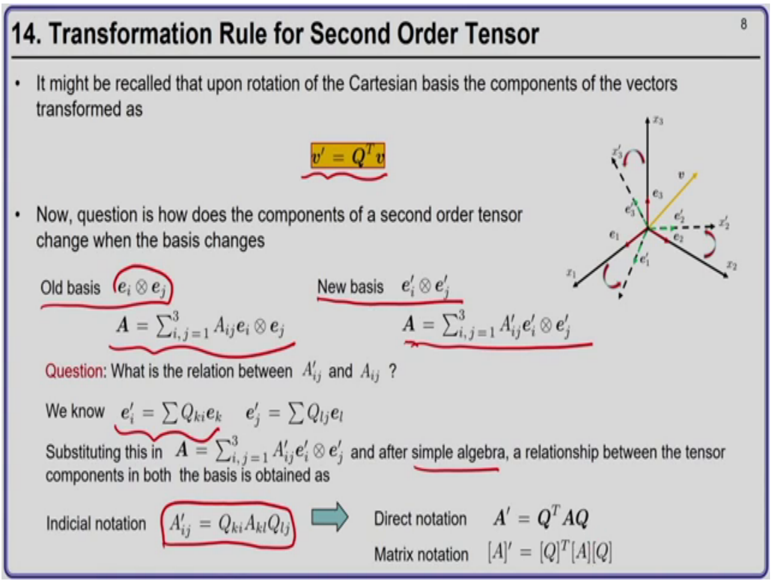

Now, you might recall that we had discussed how the components of a vector transform when the Cartesian basis are rotated, ok. So, this was the expression we dash equal to Q transpose v, where Q was the matrix of transformation; or it was T transformation vector, now transformation tensor.

Now again if the Cartesian basis are rotated, how does the components of the tensor A i j ok; for example, how do they transform, ok? So, now, if you rotate your basis; then in the old basis e i tensor product e j, the tensor A is given by following expression, ok. And just like in the case of vector, when you rotate the Cartesian system ok, your vector is not rotating; the vector is there itself, so the vector stays, ok. Now in the new basis e i dash e j dash, the same tensor can be written as A dash i j e i dash tensor e j.

Now, the question is, what is the relation between A dash i j and A i j? The component of the tensor in the new basis, and the component of the tensor in the old basis, ok. So, can we find this relation? So, the answer is yes, we can start by our expression; the relation between the basis vectors in the two coordinates system ok, in the two Cartesian coordinate systems. So, e i dash is nothing, but Q k i e k, and e j dash is nothing, but Q l j e l. So, these two we can substitute here ok, we can substitute it here. And then, what we can do? We can do some simple linear algebra and finally, we will end up getting the following relation.

So, this is the indicial notation first you will get this that, A dash i j is nothing, but Q k i A k l Q l j or in direct notation this is nothing, but A trance A dash is Q transpose A Q. In matrix notation this is matrix A dash equal to matrix Q transpose matrix A into matrix Q, ok. Now any tensor which satisfies this relation will be called a second order tensor.

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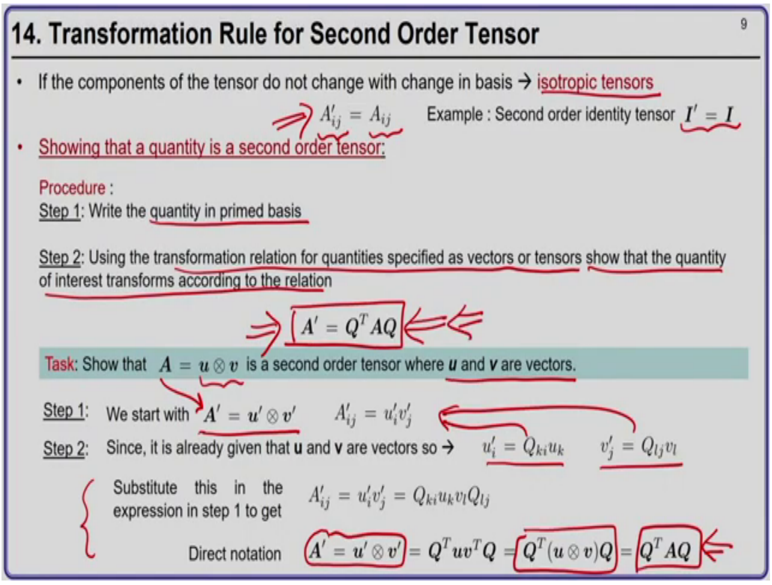

Now, question comes, are there certain tensors whose component will not change with the change in the coordinate system? The answer is yes, there are tensors, and these tensors are called isotropic tensors.

So, for isotropic tensors, the component of the tensor in the new basis is same as the component of the tensor in the old basis; and one of the examples of this is a identity tensor, ok. So, in identity tensor I dash is same as I, ok. So, you can actually show that, a quantity is a second order tensor ok; suppose you are given a quantity and you want to show that that quantity is a second order tensor.

So, how do you show that, this is a two step procedure; in the first step whatever is the quantity that you have been given, write that quantity in a prime system, that dash system, ok. And then using the transformation relation for quantities which are specified as vectors or tensors; you can show that the quantity of interest transforms according to this relation, A dash is Q transpose A Q, ok.

So, whatever quantity that you have been given, write that quantity first in the prime system; and then already you would have been given that certain quantities are vectors, tensors. So, which means those quantities will be satisfying certain transformation relation; use those transformation relation in the expression written in step one and then you can show that, if that quantity transforms according to this particular relation over here, then that quantity is a second order tensor. So, suppose you are asked to show that dyad u tensor v is a second order tensor. Now if u tensor v is a second order tensor, it should transform according to this relation over here, ok.

So, how do you begin? So, first note that, you have been given that u and v are vectors. So, if u and v are vectors; then in the first step what we do, we write A in the prime system which is A dash is u dash v dash that is the first step, ok. So, A dash is u dash tensor product v dash; in the second step we use the fact that u and v are vectors, so u dash is Q transpose u and v dash is Q transpose v, ok.

And now these two relations, you can substitute in your expression in the first step and you can solve, ok. So, this solution I will leave it to you to work it out, and finally you can show that A dash equal to u dash tensor product v dash is nothing, but Q transpose u tensor product v Q or Q transpose A Q. Hence, because the dyad transform, components of the dyad transform according to this relation; therefore dyad is a second order tensor.

So, remember is something has to be a second order tensor, the components of that quantity have to transform according to this relation A dash is Q transpose A Q, ok.

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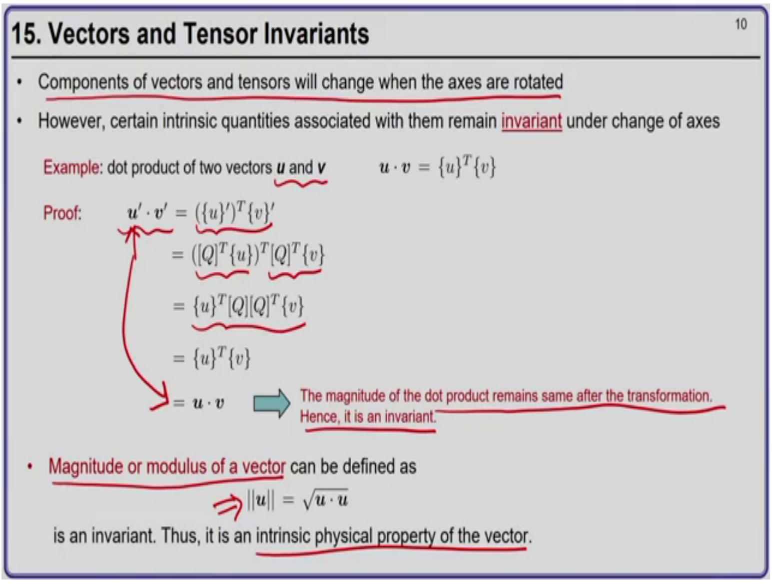

Now, coming to the next topic, which is vectors and tensor in variance. Now as you rotate the coordinate system what would happen? The components of the tensors or the vectors will change, ok. Now you might recall that tensors represent physical quantities.

So, the coordinate system is something that you have chosen, and these tensors represent something which is physical quantity. Now what you have chosen can be different for somebody else; somebody else can use some other coordinate system, but the physics of the problem should not change with the coordinate system, ok.

So, what happen? The physics of the problem should be described in terms of what is called the invariants of the quantity or the tensor ok; so quantities which do not change with the coordinate systems or when the axes are rotated, ok.

In general, you note that the components of vectors and tensors will change as the axes are rotated. However, there are certain intrinsic quantities which are called invariants which remain unchanged under the change of axes, ok. For example, consider two vectors u and v and if you take the dot product u dot v; then this u dot v is a invariant quantity. Once you rotate the coordinate system, u dot v, the value of u dot v will not change and this you can prove by starting in the prime system, ok.

U dash dot v dash; let us see what happens to u dash dot v dash, where u dash v dash are the new vectors which are in the prime system. Now writing in matrix notation, you can write u dash transpose v dash. Now u dash is Q transpose u and v dash is Q transpose v; that you can substitute here ok, that we have substituted here, ok. And then opening up the first bracket, we can write u transpose Q Q transpose v. Now Q is an orthogonal tensor or orthogonal matrix; so Q Q transpose is an identity, ok. So, finally, you get u transpose v which is nothing but u dot v, ok.

So, on the left hand side you have the dot product in the prime system which is the new coordinate, I mean new basis, and right hand side you have the dot product of the vector in the original basis. Now you see that u dash dot v dash is same as u dot v; which means that, the dot product of two vectors has not changed as the coordinate system has rotated, ok.

So, the magnitude or the dot product remains same after the transformation, hence it is called an invariant, ok. So, if you put u equal to v or v equal to u, you can define what is called the magnitude or modulus of a vector which is given by following expression; norm of u is root over u dot u and you can also show that this is also invariant quantity.

So, this modulus or the magnitude of the vector is an intrinsic physical property of the vector. Now coming to the invariants associated with the tensors, ok.

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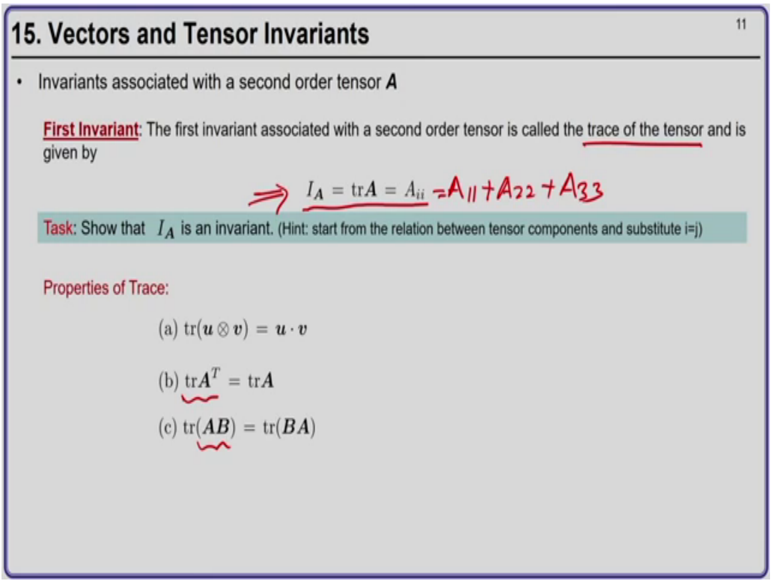

So, there are three invariants associated with the tensor; the first invariant is called the trace of the tensor and this is given by the symbol I subscript A, A is here just to denote that this is a first invariant associated with the tensor A, ok. And this I subscript a is nothing, but trace of A; and if you write A in matrix form, then trace of A matrix is nothing, but the summation of the diagonal component which is A i i. And if you expand it, this is A 1 1 plus A 2 2 plus A 3 3.

So, trace of A vector or trace of a tensor A i i is the first invariant of a second order tensor. So, you can easily show that I A is a invariant; you can start from the transformation relation A dash is Q transpose A Q, write in indicial notation and just substitute i equal to j; and then using the property that the transformation tensor is a orthogonal tensor ok, Q is Q Q transpose is the identity you can show that A dash i i is same as A i i, ok.

Again I leave it to you, you can always come back to me. So, there are some properties of trace; the first is trace of a dyad u tensor product v is nothing, but u dot v. Trace of transpose of A tensor is same as trace of a tensor itself; the trace of product of two tensors A and B is same as the trace of product of tensor B and A, ok

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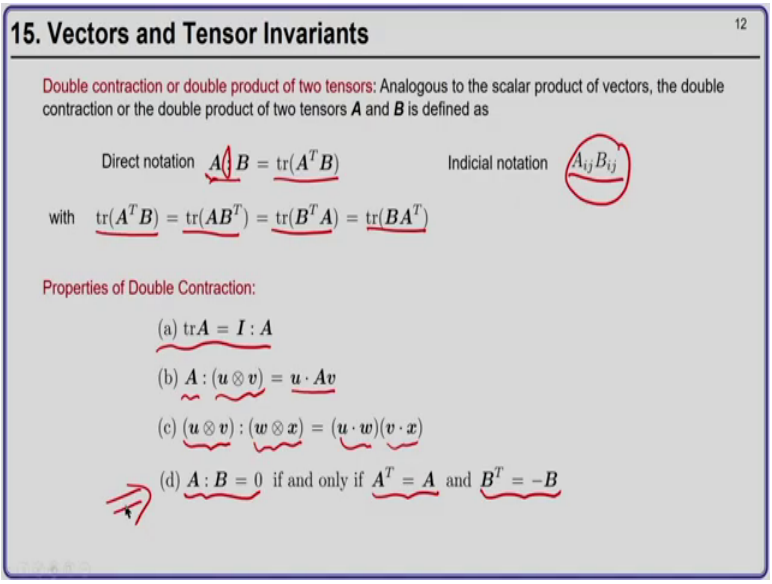

Now, next before discussing the next invariant; first we have to discuss what is meant by the double contraction or double product of two tensors, ok. So, just like you had a dot product of two vectors; similarly for tensors you can define dot product by what is called the double contraction, ok.

So, if you have two tensors A and B, then the dot product or the double contraction is defined as A and you see this symbol two dot line, ok. So, that is why it is called double contraction A double contraction with B is nothing, but trace of A transpose B. In indicial notation this is A i j B i j, ok.

Now trace of A transpose B you can show is same as trace of A B transpose, same as trace of B transpose A is same as trace of B A transpose, ok. So, some of the properties of double contraction are trace of A; a second order tensor is nothing but the double contraction of A with the identity tensor I, I double contraction with A. A contracted with dyad u tensor v is nothing, but u dot A v, ok.

And then double contraction of two dyads u tensor product v contract double contracted with v tensor product x is nothing but u dot w and v dot x, ok. So, notice that double contraction just like the dot product of two vectors leads to a scalar. You see this indicial expression over here, there is no free index here, there are only repeated index; say eventually you do not have any component, you just have the scalar magnitude, ok.

Now, the fourth property is A contracted with B ok, where A and B are two tensor is 0; if and only if A is a symmetric tensor and B is a antisymmetric tensor, ok. So, the double contraction of a symmetric tensor with an antisymmetric tensor leads to 0, ok. So, this plays a very important property with which you will come across again when we are discussing hyper elasticity and in kinectics.

So, you just remember this property that, the double contraction of a symmetric tensor and an antisymmetric tensor is equal to 0.

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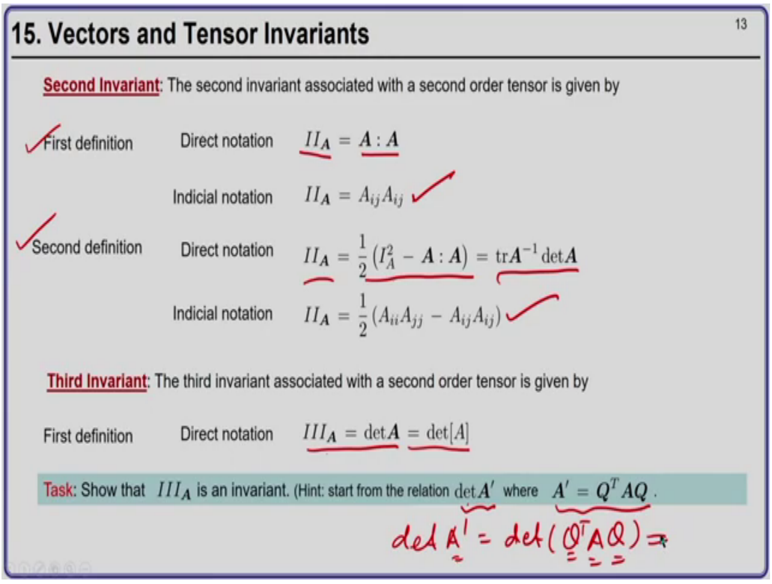

So, with this we can now define the second invariant of a second order tensor; the second invariant of a second order tensor is given by the symbol two subscript A is a double contraction with A itself.

There is an another definition which is very common in some literature, which is 1 by 2 I A square minus A contracted with A, which is nothing, but trace of A inverse determinant of A, ok. And these are the indicial representation of the second invariant of the second order tensor ok.

So, these are two definitions which are commonly used ok; depending on which resource you are using, you might find one of the other definition, ok. So, it is we have to be very careful when we are reading a text and a something like a second invariant of a tensor is used; we have to go back and see what definition of second invariant of a tensor they are using, either they are using first one or the second one.

Now, coming to the third invariant; the third invariant associated with a second order tensor is nothing, but the determinant of a second order tensor, ok. It is denoted by symbol three subscript A equal to determinant of A which is nothing but determinant of the matrix, 3 by 3 matrix of the components of the second order tensor, ok. So, you can easily show that this determinant is also an invariant, ok.

To show that, you have to start with; first you have to write the determinant in the primed basis that is the first step. So, you have to start with determinant of A dash, and A dash is nothing, but Q transpose A Q. If you take determinant of both the sides; determinant of A dash will be equal to determinant of Q transpose A Q, ok.

Now, determinant of Q transpose A Q will be determinant of Q transpose into determinant of A into determinant of Q, ok. Now determinant of Q is nothing, but 1; it is a proper orthogonal tensor. So, determinant of Q is 1; therefore determinant of A dash will be same as determinant of A, ok. So, these are three invariants associated with the second order tensor, which you need to remember.