Computational Continuum Mechanics
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Discretization
Lecture – 27 – 28
Discretization

So, today we are going to start the topic of Discretization.

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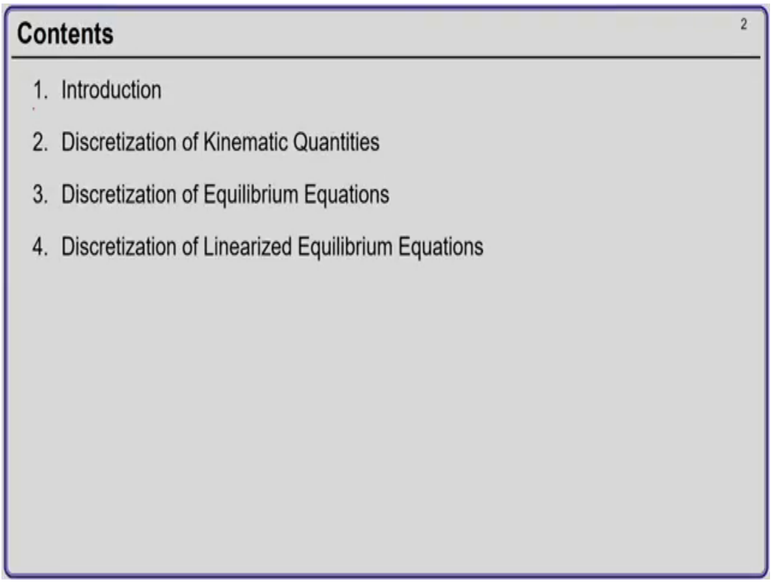

So, in this topic we are going to cover after a brief introduction the discretization of various kinematic quantities ok. This is very essential for getting the final finite element form ok; then we will get the discretized version of the continuum equilibrium equations in finite element form. And, finally, we discretize the linear linearized equilibrium equations that we derived in the previous lectures ok. So, each of these topics is very involved and I suggest you work it out on using pen and paper.

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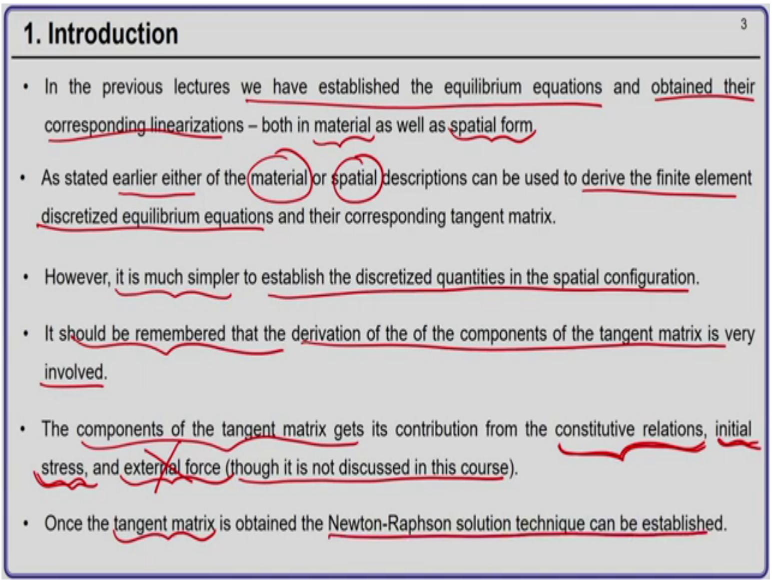

So, in the previous lectures we have established the equilibrium equations and also we had obtained their corresponding linearizations – both in the material or the Lagrangian configuration and the spatial or the Eulerian configuration. And, as stated earlier in the previous lectures either of the material or the spatial descriptions can be used to derive the finite element discretized equilibrium equations and their corresponding tangent matrices ok.

However, it was also mentioned that it is much simpler to establish the discretized quantities in the spatial configuration ok. Also, we like to point out at this point of time that the derivation of the components of the tangent matrix is very involved process and this will take a lot of time to understand and get used to.

So, the components of the tangent matrix get their contribution from the constitutive relations, from the initial stresses and also from the external forces, ok; external forces because if you have pressure dependent loading of follower loads in that case you will have a contribution to the tangent matrix which comes from these external forces.

But, we have mentioned in our previous lectures that we will not going to consider these kind of forces in this course ok. So, in this course we are not going to discuss those kind of forces. So, we will not have contribution to the tangent matrix because of external forces. We will only discuss the contribution of the tangent matrix because of the constitutive relations and the initial stresses.

And, once we have derived the tangent matrix we can set up the Newton – Raphson solution technique to get the desired solution.

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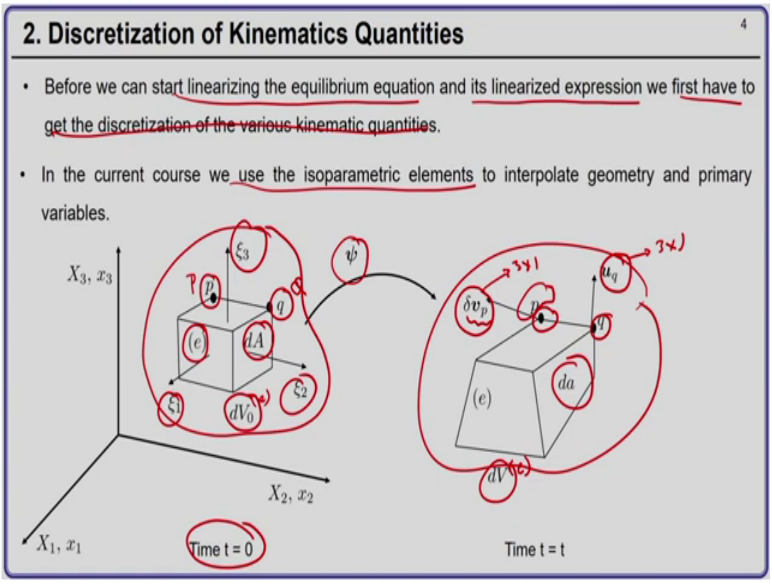

Now, let us come to discretization of kinematic quantities. Before we can start linearizing our equilibrium equation and also discretizing their linearized expressions, we first have to discretize the various kinematic quantities.

Now, in the present course we use what is called isoparametric finite elements, which means that the shape functions used to describe the geometry and also the shape function used to describe the primary variable in our case which is displacement are one and the same. So, we use the same shape function to characterize the geometry as well as the primary variable.

So, let us consider a typical finite element at finite element e at time 0 and let the volume of this element be dV 0 ok, not to confuse with the volume of the body if you are confused just put a subscript e ok. Then after deformation let us say this element deforms and gets this final shape and this volume is dV you can put dV e if you wish I will not use that and let us say d capital A is the area on which the external forces act ok.

So, it might be pressure loading it might be any kind of other kind of surface loads, but this is the area on which the surface forces act and this area deforms and you get the small a, ok. Remember this is one finite element of the entire finite element mesh of the body and xi 1, xi 2, xi 3 are the natural coordinates. Now, let us consider two nodes p and q and if you are very consistent we can put capital P and capital Q and after deformation these nodes go to this location p and q.

Let del v p subscript p means it is a virtual velocity of node p ok. So, p and q are the nodes of these elements and del v p is the virtual velocity associated with node p. And, let u subscript q be the displacement corresponding to node q. In 3-dimensional usually these are 3 cross 1 arrays ok , they have three components in finite element they will have we will write them as a vector of 3 cross 1.

So, with this setup in hand the first thing you will want to do is you want to discretize our various kinematic quantities that we discuss in the course till now ok.

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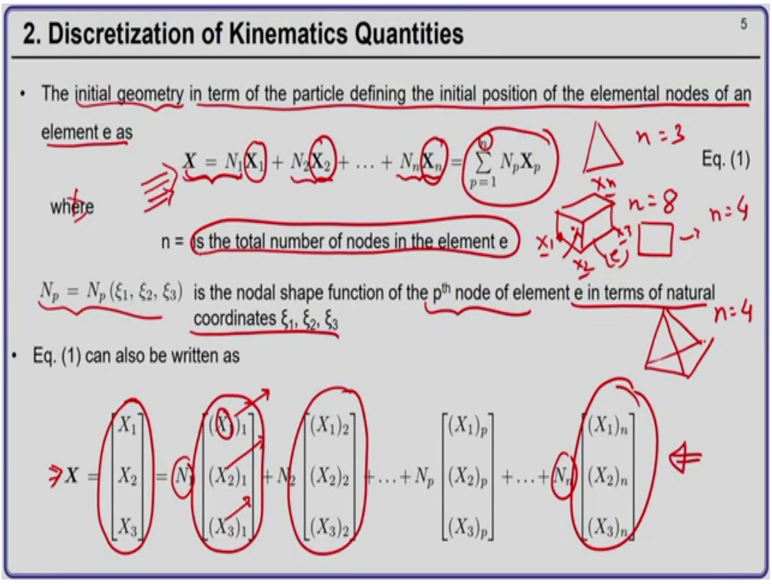

So, first we take the initial geometry that is the geometry of the body at time t equal to 0. So, the initial geometry in terms of the particle defining the initial position of the element elemental nodes of an element e is given by X equal to N 1 X 1 plus N 2 X 2 all the way up to N n X n and in terms of summation sign I can write this as p equal to 1 to n N p X p where N 1, N 2, N 3 all the way up to N n are the shape function.

So, what it means is inside the element e the material position of any point can be interpolated in terms of capital X 1 capital X 2 all the way up to capital X n where capital X 1, X 2 are the nodal material coordinates. So, this small n denotes the number of nodes the element has. So, n denotes the total number of nodes belonging to the element e ok.

So, in case if you have a brick element which is like a cube so, your n will be equal to 8. In 2D if you have a triangular element n will be equal to 3; in 2D if you have a square element or quadrilateral element your n will be equal to 4 ok. If you have a 3D and a tetrahedron in that case you have you have n equal to 4 like this n can vary.

So, the material point inside an element ok. So, this is the element ok; so, this is a finite element e and any material point inside this its position can be interpolated in terms of the nodal positions X 1, X 2, X 3 all the way up to X n where N 1, N 2, N 3 are the shape functions associated with node 1, node 2, node 3 and these shape functions are for the p-th node of the element e and they are written in terms of the natural coordinate xi 1, xi 2, xi 3 ok.

So, for better description I can write equation 1 as so, this is X. So, this is the material location inside the element and that can be written in terms of N 1 times the material location of node 1 plus N 2 times material location of node 2 all the way up to N m into material location of node n ok. So, here X 1 1 means the X 1 coordinate of node 1, X 2 1 means X 2 coordinate of node 1, X 3 1 means X 3 coordinate of node 1 ok.

So, equation 1 explicitly can be written like this. So, this is the only time I am writing like this, in coming slide I will not write ok. So, you can understand what actually equation 1 means ok. If we remember we have equation which is analogous to equation 1, we can understand that this is what we mean.

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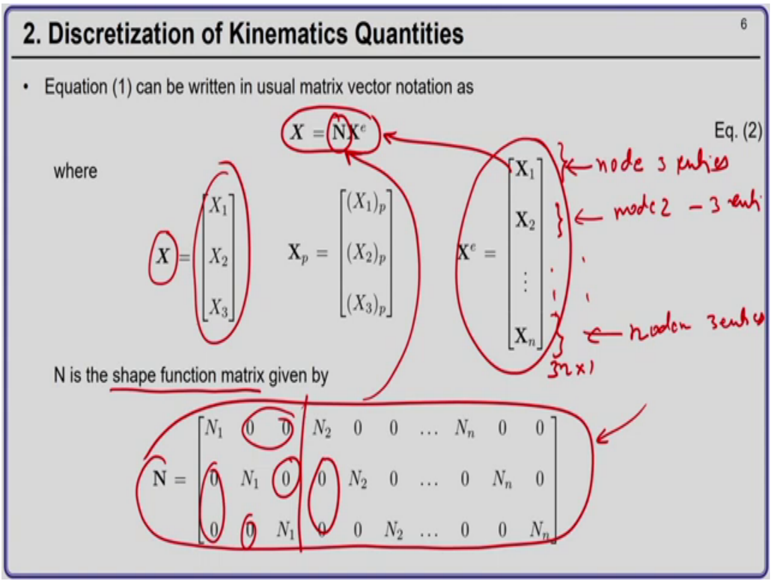

So, in traditional finite element equation 1 can also be written in terms of matrix vector multiplication given as X equal to matrix and multiplied by a vector X e ok, where X remains as it is. It is the material location of a point inside the element and X e is nothing but the vector or formed by the material location of all the n node. So, this is X 1 is node 1 and these are 3 entries. So, X 2 is for node 2 and also these are 3 entries like this you have node n and these are 3 entries. So, all in all this is 3 n cross 1 ok.

So, the matrix N can be written in this particular form you can do for yourself you can substitute it here and you can substitute this here and you will see that you can get equation 1 back. So, N is called the shape function matrix because it is composed of entirely of the shape function and you can also notice that there are lot of 0s here.

There are lot of 0s present. So, in fact, out of nine entries six entries are 0. So, there are two third 0 out of total entry two thirds are 0. So, if you use equation 2 you will in computational setting you will be having a lot of zero multiplications and you understand that multiplying any quantity by 0 gives you 0. So, equation 2 basically leads to a much more computationally expensive process though it does not change the result.

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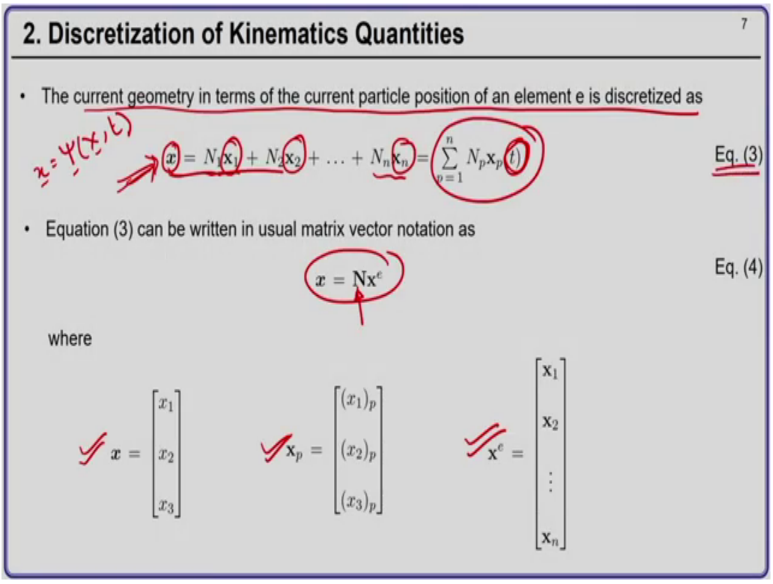

Now, the current geometry in terms of the current particle position of an element e can be discretized as given by this equation ok. So, small x is the current spatial location of material particle x. So, x is psi of X comma t ok. So, this is your t here and then the current spatial location can be interpolated in terms of the current spatial location of the nodes of element e ok.

So, it is X equal to N 1 x 1 plus N 2 x 2 all the way up to N n x n or in short N p x p; x p which is a function of time t ok. So, as usual I can write this in matrix vector notation as x equal to N x c where the form for various arrays is given here ok. Again, n has the same form is a shape function matrix which are described in the previous lectures or the previous slide.

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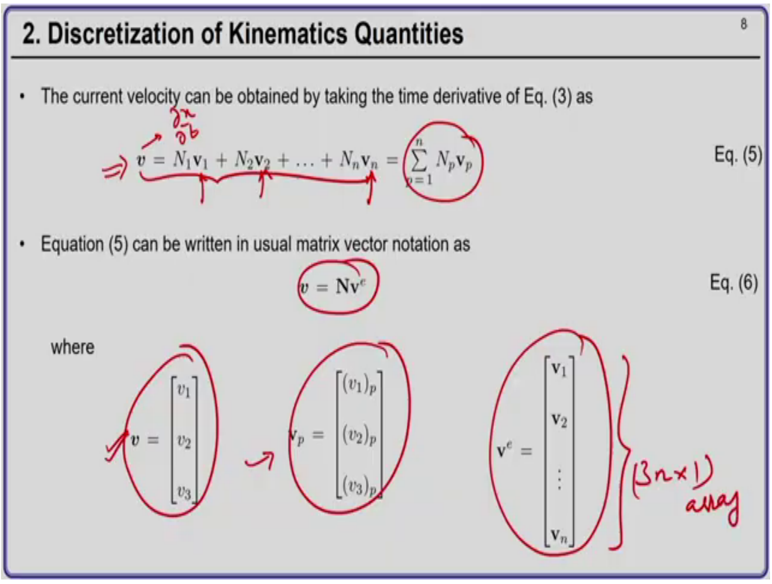

Now, we can differentiate equation 3 with respect to time and we will get what is called the velocity. So, velocity is del x by del t and we can get the expression for a velocity at a point inside the element in terms of the velocity of the nodes v 1, v 2 all the way up to v n are the velocity of node 1, node 2 all the way up to node n ok. And, in short it is N p v p.

And, this is the usual matrix vector notation expression and these are various ok. So, this is a component of the velocity of the point and these are the component of the velocity of node p and v e is nothing but the velocity vector composed of velocity of all the n nodes; as usual this is 3 n cross 1 array.

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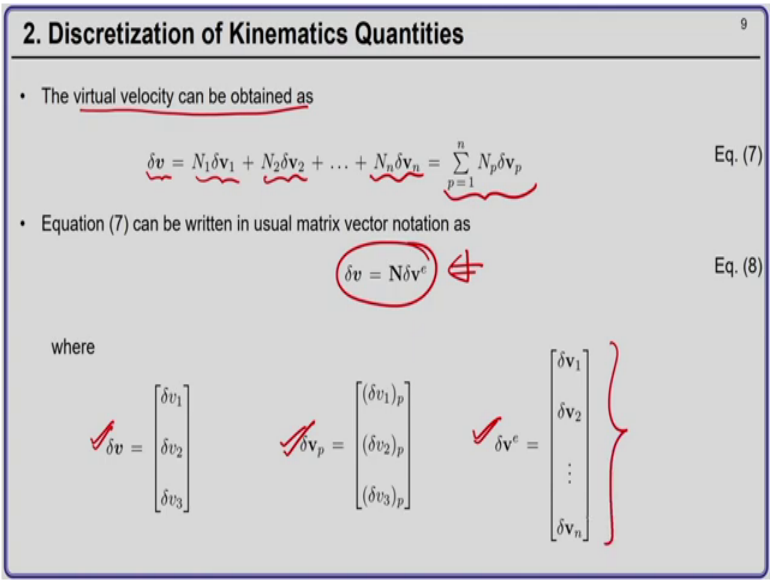

Now, the virtual velocity can be obtained by taking the virtual of equation number 5 ok. So, virtual variation you will take, you will get the virtual velocity of a point in terms of the virtual velocity of the nodes and this is nothing, but N p del v p and as usually the matrix vector notation helps us to write equation 7 more concisely as given in equation number 8.

And, these are the virtual velocities of a spatial point inside the element; this is the virtual velocity of node p and del v e is the virtual velocity composed of all the nodes of the element e.

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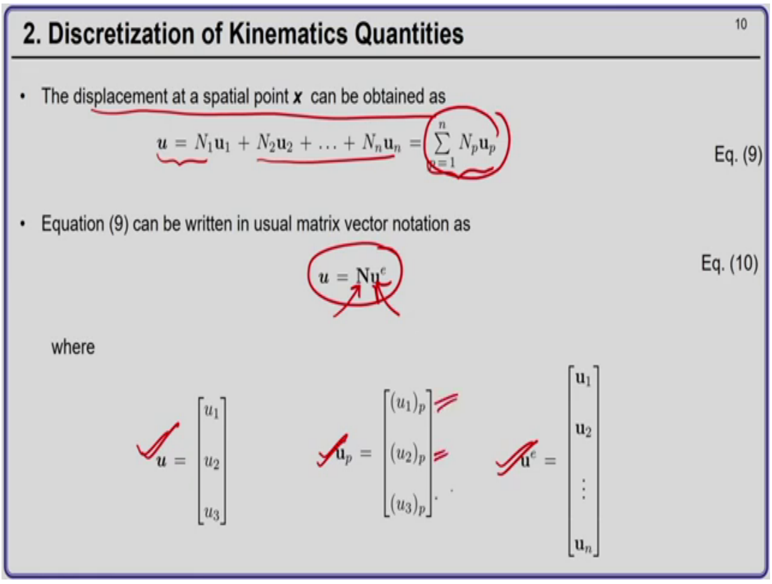

Next the displacement at a spatial point can be obtained as u equal to N 1 u 1 plus N 2 u 2 all the way up to N m u n or in short summation over the nodes N p u p ok. In matrix vector notation the displacement at a point can be written in terms of the shape function matrix and the elemental displacement vector ok.

So, this is the element displacement vector; u is the displacement of a spatial point inside the element and u p is the displacement of the node p ok. So, each node will have three displacements in three directions ok.

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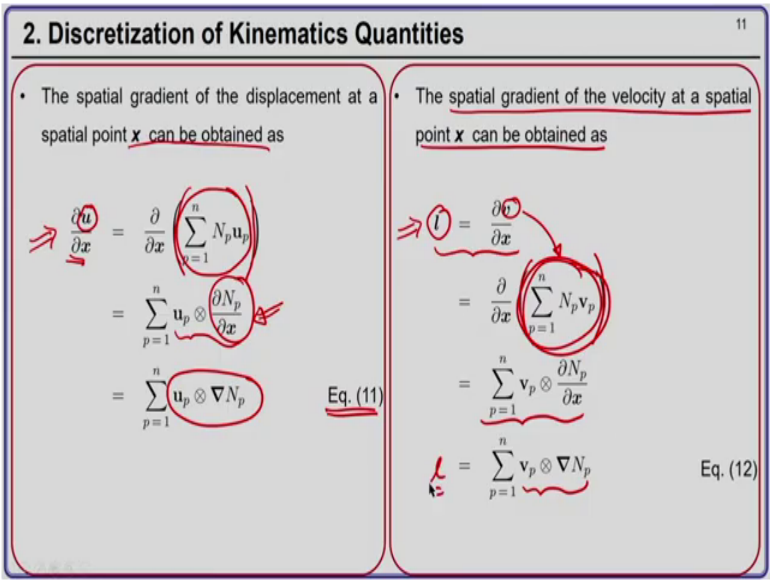

Next we come to the spatial gradient of the displacement at a spatial point x. So, the spatial gradient of a displacement at a point inside the element can be obtained as del u by del x and we know that u is summation over all the nodes N p u p and therefore, the gradient will be del by del x of this summation over all the nodes ok. So, if I bring that inside I can write this as up tensor product del N p by del x ok.

And, then I can use the Nabla symbol and I can write in short this as u p tensor product del N p and we will see how to compute the gradient spatial gradient of the shape function. Remember N p are given in terms of xi 1, xi 2, xi 3 natural coordinates to carry out the spatial gradient of these shape functions with respect to natural coordinates with which are a function of natural coordinates we have to do a special procedure which will come later ok. So, that is equation 11 which gives you the spatial gradient of the displacement at a spatial point ok.

Now, the spatial gradient of the velocity vector at a spatial position x which is nothing, but the velocity gradient tensor l is given by l equal to del v by del x. Now, I can substitute v as the interpolation of the velocity vectors of the nodes which is shown here ok. So, then the velocity gradient will be del by del x of this quantity in the bracket. So, again if I open up the bracket I will get v p tensor product del N p by del x or in short I can write v p tensor product gradient of N p ok. So, that is my velocity gradient tensor.

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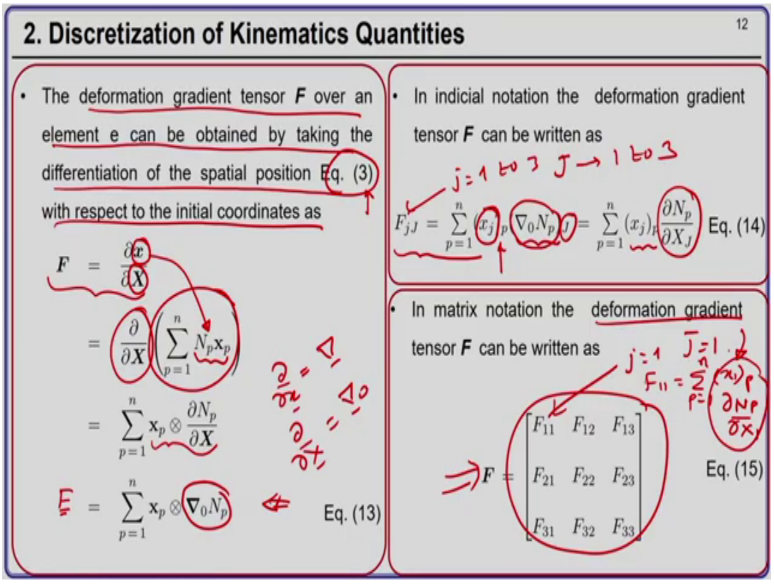

Next comes the deformation gradient tensor. So, how can I write the discretized version of the deformation gradient tensor? So, the deformation gradient tensor over an element e can be obtained by taking the differentiation of the spatial position that is equation 3 with respect to the initial or the material coordinate ok. So, that is what we know F is nothing but del x by del capital X the differentiation of the spatial position with respect to material position ok.

So, x is nothing but N p x p and then I take the material derivative of the summation over all the nodes N p x p. So, if I open up the bracket it becomes x p tensor product del N p by del X ok. Now, in short I can write this as del 0 N p ok. Remember del by del x can be written as delta and del by del capital X can be written as del 0.

So, the deformation the discretized version of the deformation gradient tensor is given by equation number 13 ok. So, in indicial notation the equation 13 can be written as F j J equal to the j-th component the j-th coordinate of the node p times t capital J-th value of the gradient of the shape function or more explicitly it is x j p del N p by del X J and in matrix notation the deformation gradient tensor ok.

So, here j goes from 1 to 3 and capital J goes from 1 to 3. So, therefore, deformation gradient tensor as you know is a 3 by 3 can be written as a 3 by 3 matrix and that is how you will get. For example, F 11 is j equal to 1 and capital J equal to 1 which will mean from equation number 1 F 11 is summation over p equal to 1 to n the x 1 of node p into del N p by del capital X 1.

Now, this has to be done over all the nodes ok. So, in case of brick element that is cuboid if your n will be equal to 8 so, this will be sum of 8 terms.

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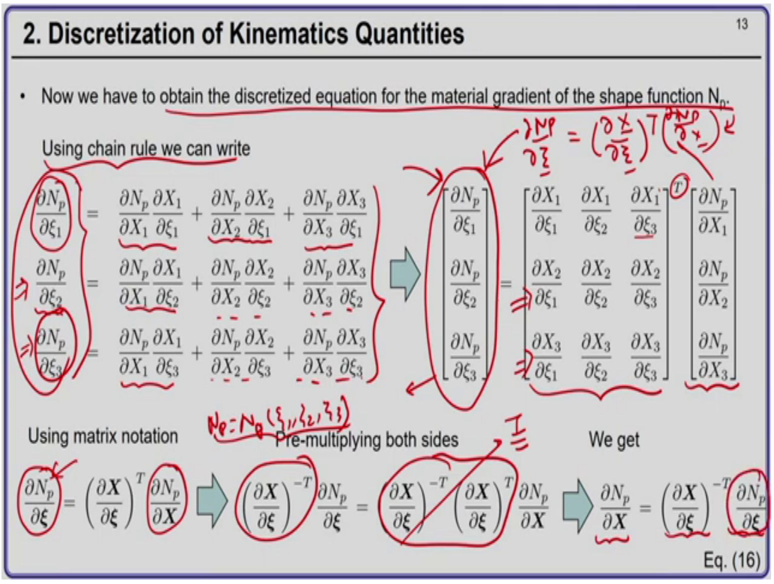

Now, we have to compute the material gradient of the shape function matrix N p shape functions N p ok. So, you have it here we have to compute del N p by del capital X that is the gradient with the material gradient of shape function N p. So, how to do that?

So, using chain rule you can write del N p by del xi 1 as del N p by del X 1 into del X 1 by del xi 1, del N p by del X 2 del del X 2 by del xi 1 plus del N p by del X 3 del X 3 by del xi 1. Similarly, you can write del N p by del xi 2 del N p by del X 1 del X 1 by del xi 2 like this the other two terms; del N p by del xi 3 you can write del N p by del X 1 into del X 1 by del xi 3 like and then the other two terms ok.

So, you can write the gradient of the shape function of in terms of the natural coordinates xi 1, xi 2, xi 3 as following three equation ok. So, now, you can put this in matrix form ok. So, the vector you can put these in this vector. So, this is a vector of the gradient of shape function with respect to the natural coordinates and you can put this as a matrix into a vector of the gradient of shape function with respect to material coordinate ok.

And, this is a matrix transpose del X 1 by del xi 1 del X 1 by del xi 2 del X 1 by del xi 3 transpose ok. So, there are other two rows ok. So, in short I can write this as this is nothing, but del N p by del xi ok; xi is a now xi is a vector xi 1, xi 2, xi 3 ok. So, the first array here is nothing, but del N p by del xi equal to this matrix I can write that as del X by del xi transpose and this I can write this as del N p by del capital X.

So, that is our spatial gradient that we require and we know N p in terms of xi 1, xi 2, xi 3. So, N p just after equation 1 we have written N p is function of xi 1, xi 2, xi 3. Once you have chosen a finite element you will know these function and this will not change during the course of simulation ok. So, this will be always constant during the simulation ok. So, you will always have del N p by del xi ok.

Now, I need this. So, what I do? I pre multiply both sides by del X by del xi inverse transpose and that is what we are going to do, pre multiply both side by del X by del xi inverse transpose and this term over here. This is nothing, but a into a inverse ok. So, this is nothing, but second order identity and then this becomes identity therefore, I get my material gradient of the shape function of node p that is del N p by del X as del X by del xi inverse transpose del N p by del xi. Now, as I said del N p by del xi is known to us because N p is given in terms of xi 1, xi 2, xi 3.

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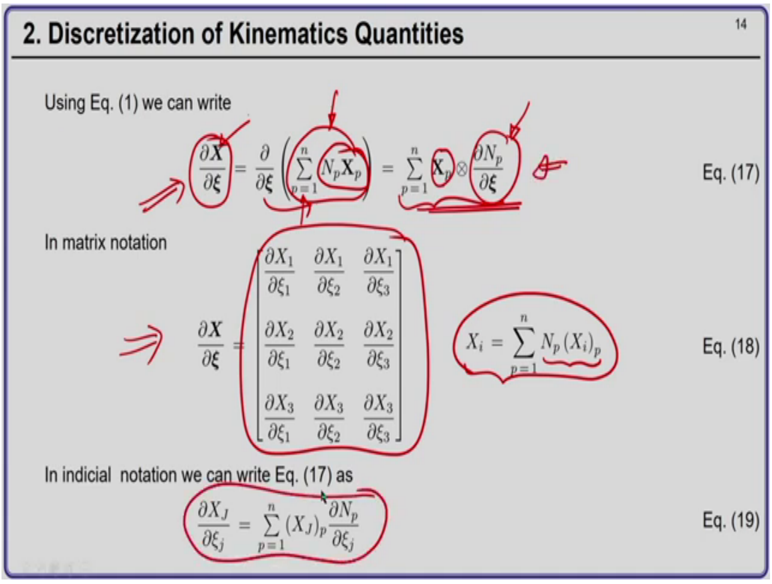

And, now I have to compute del X by del xi, how do I calculate that? So, del X for computing del X by del xi I just substitute X from equation number 1 what was that it is a X is a material coordinate of any point inside the element that can be written in terms of the linear interpolation or the interpolation with respect to the nodal position.

So, N p X p summation over all the nodes ok. So, I can take xi inside and this becomes X p tensor product del N p by del xi ok. So, I already know del N p by del xi, I know the nodal position and if I carry out these operation I will be able to find out my del X by del xi.

Once I have del X by del xi I can substitute that here I can take the inverse and I can take the transpose I can get this matrix and then what I can do? I can multiply by del N p by del xi to get my vector del N p by del X that is how we will obtain ok. Once you have this you can find out the expression for the deformation gradient tensor.

So, this del X by del xi in matrix notation is nothing, but given here. So, where X i can be interpolated in terms of the X i value of all the nodes p ok. So, in indicial rotation equation number 17 can be written as this.

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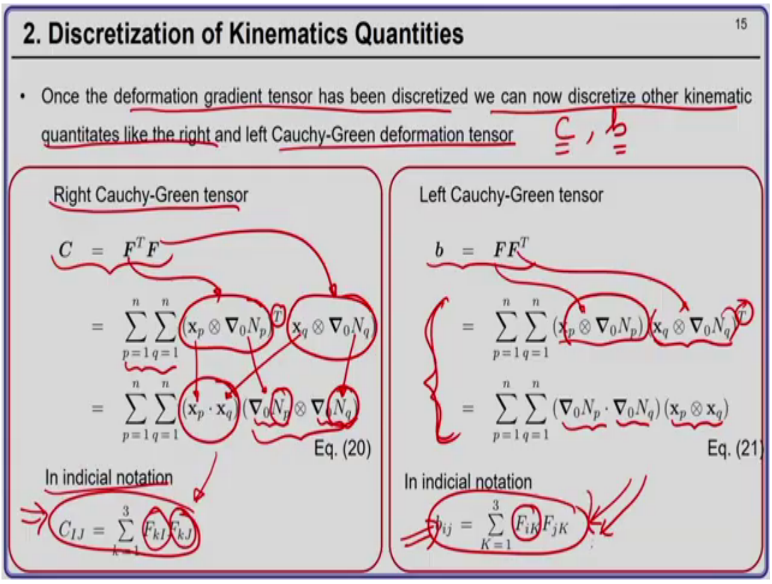

So, once the deformation gradient tensor has been discretized we can now discretize the other kinematic quantities like the right and left Cauchy – Green deformation tensor. We can have b and C discretized ok.

So, how do we discretize C which is the right Cauchy – Green tensor? C is F transpose F and I know F is x p tensor product del 0 N p and then the other. So, this is my F and this is a transpose and is the other F ok. So, I put that as X cube tensor product del 0 N q and summation over p and q ok.

So, if I simplify further I can write this as ok. So, this I can write as x p dot x q del N 0 p tensor product del 0 N q ok. Now, x p dot x q these are the nodal position of node p and node q belonging to element e. So, and we already have calculated the material gradient of shape functions N p and N q ok. So, this can be calculated very easily.

And, then in indicial notation equation 20 is nothing, but given like this ok. So, in actual setting you will calculate F and then using that F you can use this formula to directly compute C IJ all the components of right Cauchy – Green deformation tensor.

For the left Cauchy – Green tensor b is F F transpose ok. So, F is nothing, but x p tensor product del 0 N p and this F is x q tensor product del 0 N q and then you have to put a transpose and then when you simplify you get del 0 N p dot del 0 N q and x p tensor product x q ok. In indicial notation this is how you get ok.

So, natural computational setting you will find out the deformation gradient tensor and using that deformation gradient tensor you can actually find out all the values of components of b that is the left Cauchy – Green deformation tensor. This is just for computational this is just for understanding, but actual computational setting you will do this.

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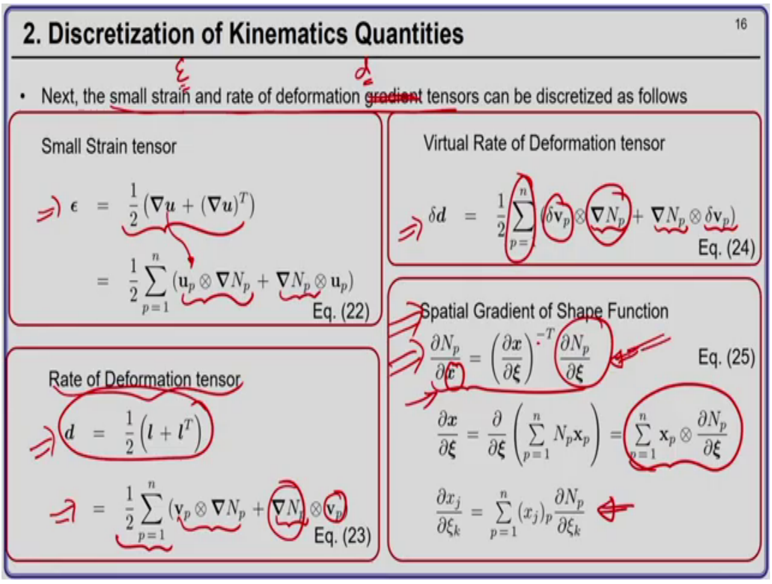

Now, next the small strain that is epsilon and the rate of deformation gradient tensor rate of deformation tensor d can be obtained as follows. The small strain tensor you know that epsilon is 1 by 2 del u plus del u transpose. Now, del u we have already seen is nothing, but u p tensor product spatial gradient of N p plus del u transpose is nothing but spatial gradient of N p tensor product up.

And, the rate of deformation tensor d is the symmetric part of the velocity gradient tensor and this is given by 1 by 2 summation over all the nodes v p tensor product spatial gradient of shape function N p plus spatial gradient of shape function N p tensor product velocity of node p.

So, the virtual rate of deformation tensor del d is nothing but the virtual velocity of node p tensor product the spatial gradient of node p shape function of node p plus the spatial gradient of shape function of node p tensor product the virtual velocity associated with node p summation over all the nodes ok.

Now, the spatial gradient so, we have a lot of spatial gradient of the shape function and the spatial gradient of the shape function can be derived like this ok. So, remember this material gradient of the shape function was of the similar form is just the same where the material coordinates x has been replaced by spatial coordinate small x ok.

So, del N p by del x is del x by del xi inverse transpose into del N p by del xi and del x by del xi is nothing, but x p tensor product del N p by del xi and this is the indicial notation ok.

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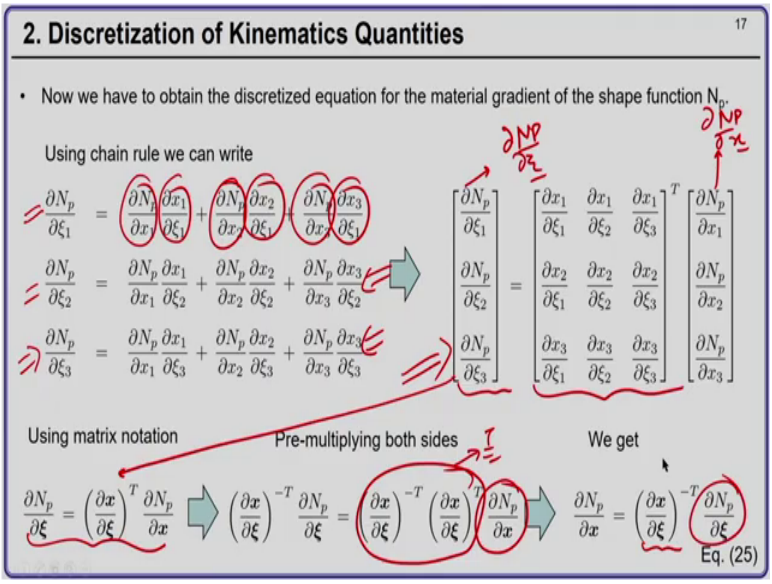

So, how we have derived this is shown here and this follows the similar steps as done for the material derivative of the shape functions ok.

So, here we have del N p by del xi 1, del N p by del xi 2, del N p by del xi 3 is del N p by del x 1 del x 1 by del xi 1 del N p by del x 2 del x 2 by del xi 1 del N p by del x 3 del x 3 by del xi 1, like this we can do for the other two equations and we can write this in matrix notation matrix vector notation and this one over here is nothing, but del N p by del xi and this is nothing, but del N p by del x you can see that we are following the similar procedure as we did previously for the material gradient of the shape function ok.

So, in matrix notation this can be written as this equation and now pre multiplying both side by del x by del xi inverse transpose and then recognizing that this 1 is equal to identity I eventually get the spatial gradient of the shape function as del x by del xi inverse transpose del N p by del xi ok, that is the derivation for you.

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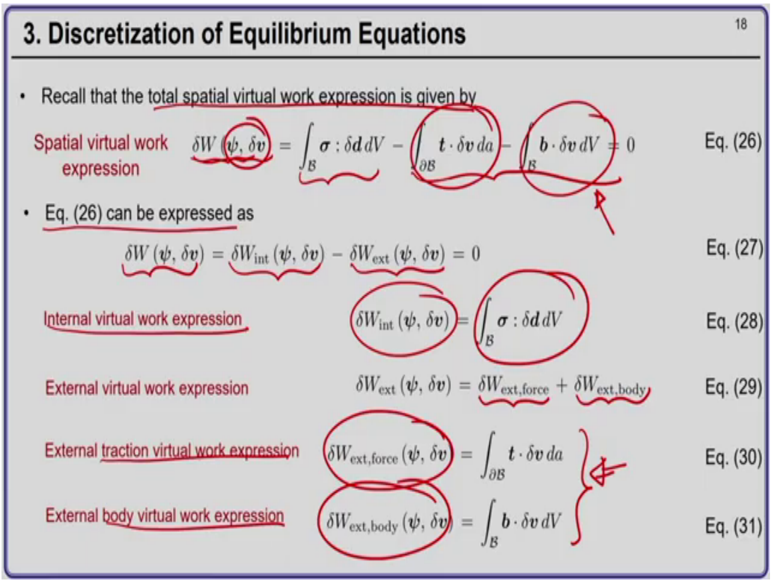

Now, we come to the discretization of the equilibrium equation ok. So, right now we have the equilibrium equation in the continuum setting, we like to get the equilibrium equation in the discretize setting that is in the finite element setting ok. So, like divergence of sigma plus b equal to 0 that is the equilibrium equation in the continuum setting. Now, what is the counterpart in the finite element setting is what we have going to look next.

So, recall that the total spatial virtual work expression is given by this equation. So, this is the total spatial virtual work which is a function of deformation mapping in the virtual velocities and the first term correspond to the internal virtual work; the second two terms correspond to the virtual work because of the external tractions and the body forces ok. So, this is the external virtual work external traction virtual work and external body virtual work ok. So, these are two ok. So, these are these two expressions and this is our internal virtual work expression.

Now, equation number 26 can be expressed as the total virtual work ok. So, the total virtual work can be split into the internal virtual work minus the external virtual work and the internal virtual work is the virtual work of the internal stresses and the external virtual work is the sum of the external virtual work because of the body forces and because of the surface tractions and equations 30 and 31 give you those expression ok.

Now, what we do is we use the discretize kinematic quantities that we had discussed in the previous slides and we can substitute those in equation number 26 and we can get the corresponding finite element discretize our equilibrium equation ok.

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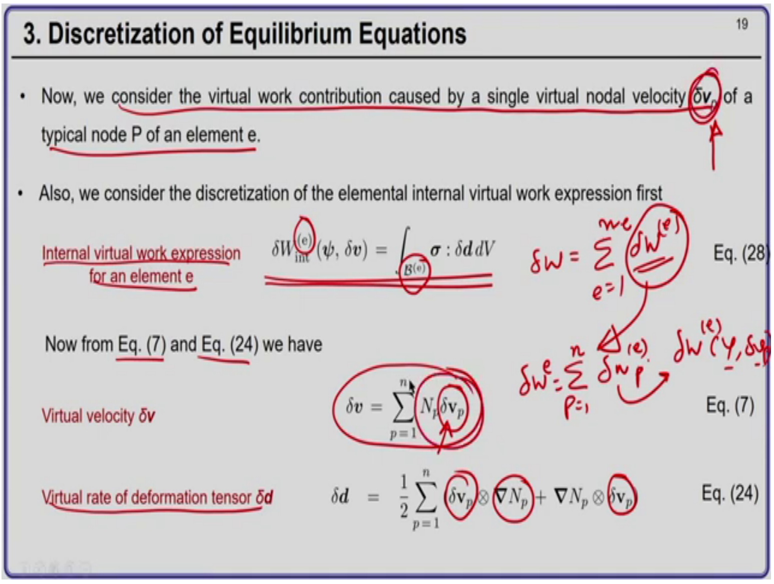

So, we then consider the virtual work contribution caused by a single virtual nodal velocities del p of a typical node P of an element e ok. So, what we do we take a node P of element e at time t and we consider what is the contribution to the virtual work because of this node.

So, the virtual velocity associated with that node is del p and then we take equation 26 and we first consider only the internal virtual work expression for an element e and inside that element we use equation number 7 and equation number 24. So, equation 7 corresponds to the interpolation of the virtual velocities in terms of the nodal velocities virtual velocities and equation 24 corresponds to the virtual rate of deformation tensor. And, that is also in terms of the nodal virtual velocities and the spatial gradient of the shape function ok.

So, what we do we write equation number 28 which is the virtual work coming from element e. Remember, see this the integration is also volume integral is performed over the volume of the element. So, the total virtual work will be nothing, but the sum of the virtual work let us say n e or the virtual work of all the elements.

And, now we choose one element and in that element we choose one node p and we try to see what is the contribution to the total virtual work because of this node p ok. So, now I substitute ok. So, this del e again is p equal to 1 ok. So, this del w e is p equal to 1 to n del w e of p and this is nothing, but del w e of psi comma del v p ok. This del v p and this del v p is nothing, but N p into the nodal virtual velocity ok.

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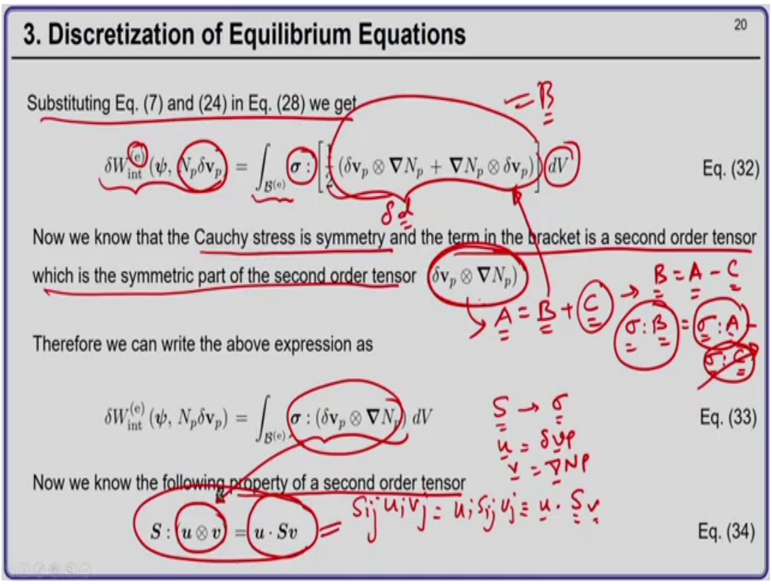

So, once we do that if we substitute equation 7 and 24 the internal virtual work contribution of node p belonging to element e is nothing, but integration over the element volume Cauchy stress double contracted with the virtual rate of deformation tensor and that integrated over the current volume of element e.

Now, we know that the Cauchy stress is symmetric and then also we recognize that this term inside the bracket is a second order tensor which is nothing, but the symmetric part of another second order tensor del v p tensor product del N p. Now, we know that this is our tensor let us say this is tensor A and this is B plus C, where B is the symmetric part ok. So, this is B and there is a anti-symmetric part. So, B is A minus C remember this is our B.

So, sigma double contracted with B A sigma double contracted with A minus sigma double contracted with C. So, this term sigma double contracted with C, C is the anti-symmetric tensor will be equal to 0 because C sigma is a symmetric tensor. So, sigma contracted with B is same as sigma contracted with A. So, this is our A. So, I can write equation 32 now as sigma double contracted with del v p tensor product del N p.

Now, I can use the following identity ok. Let S be a second order tensor and u and v are two vectors. So, S double contracted with u tensor product v is nothing, but u dot Sv in indicial notation I can write S ij u i v j which is nothing, but u i S ij v j and direct notation this is u dot S v ok.

So, if I see equation 33 and 34 I can recognize S is same as sigma my u is same as del v p and my v is spatial gradient of shape function. So, this is here.

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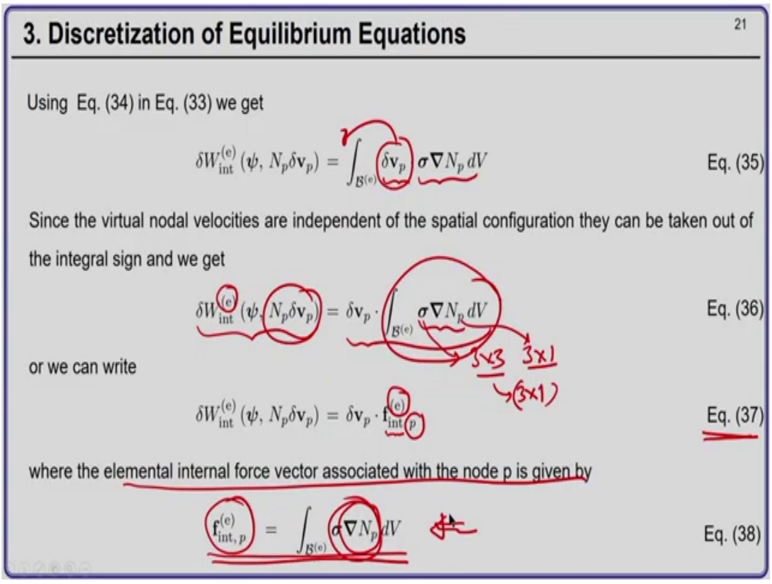

So, then using equation 34 equation 33 can be written as del v p dot sigma del N p dV. Now, the virtual velocities are independent of the configuration this we said at the start of the previous module. So, I can take virtual velocities outside the integral sign and I can write the internal virtual work contribution coming from note p of an element e as del v p dot sigma del N p integrated over the current volume of element e.

In short, I can replace this integral by this vector ok; remember, sigma is the 3 cross 3 matrix and this is a 3 cross 1 vector. So, eventually what I am going to get is a 3 cross 1 vector and this is that vector I denote this as F subscript int comma p superscript e ok. So, what this means is this is the internal force vector corresponding to node p of element e ok. So, equation 36 in short can be written as equation number 37.

So, the elemental internal force vector associated with node p of an element e is given by following expression. Now, remember the internal force vector can be computed using equation 38 and here sigma in general will have nine components and not necessarily all of them will be zero they may be all of them may be nonzero similarly del N p all the terms are nonzero. So, eventually what will happen? This multiplication over here is between all the nonzero quantities.

We will come to later we will again revisit this equation later and we will see how this equation number 38 is much more efficient in the computational setting.

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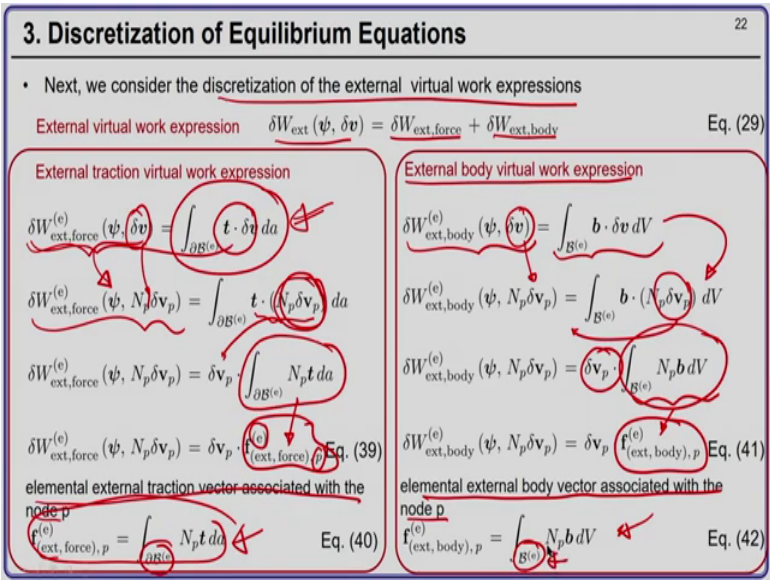

Now, next we consider the discretization of the external virtual work expressions ok. So, the external virtual work is the virtual work of the tractions and the virtual work of the body forces. So, similarly we can write the elemental external force of the traction as the integral over the elemental volume t dot del v and remember t is the surface traction externally applied surface traction ok.

And, now some of the finite element maybe in the bulk or the inside the body, therefore, they may not have any surface traction acting over them in that case they will you will have no such contribution. So, this integral only takes place over the those elements which are on the surface of the body ok.

And, now, the elemental traction virtual work expression corresponding to node p of element e is given by following expression. So, this del v is replaced by N p del v p and therefore, del v replaced by NP del v p and then I can write t dot del v p as del v p dot t. And, then because virtual velocities are independent of the configuration I can take them outside and then I am left with this particular integral and this I will write as this expression ok. This is the external force vector because of the external tractions of node p belonging to element e ok.

So, this is the elemental external traction vector associated with node p N p t da. Now, the second part of the external virtual work is because of the body forces and we know the elemental external body force body virtual work expression is given by following integral and then for a node p of element e del v is N p del v p ok. So, this expression becomes this.

And, then we can take del v p outside and this I can this integral I can denote by the following vector which is the elemental external body force vector associated with node p ok. So, corresponding to the external virtual work we are getting two force vectors one is because of traction and one which is because of the body forces ok. Remember, these integrals are carried out over the elemental area and volume of the element e.

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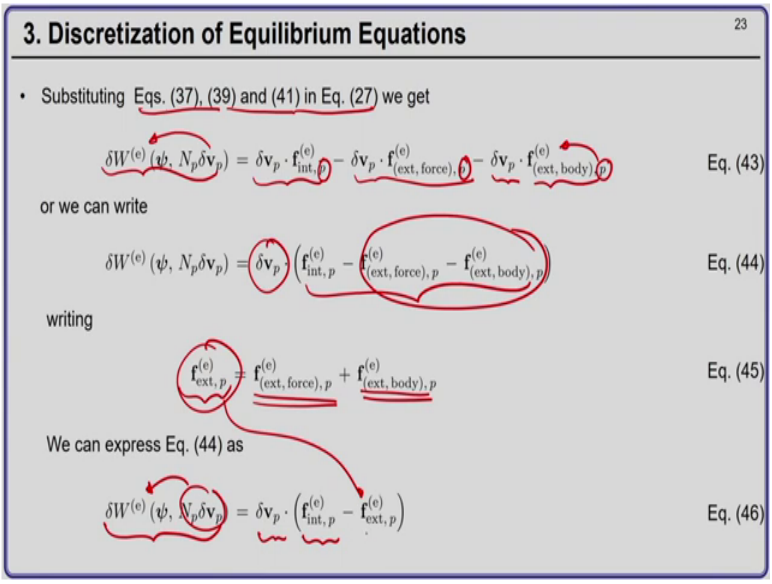

Now, I can substitute equation 37, 39 and 41 in equation number 27 ok. Once I do this the virtual work of node p belonging to element e is del v p dot internal force vector of node p belonging to element e minus del v p dot the external force vector because of traction of node p belonging to element e minus del v p dot the external force vector because of the body forces at node p belonging to element e.

So, I can take del v p outside the bracket and I am left with following quantity inside the bracket ok. So, these two I can combine together and I can write them as the 1 force vector that is the external force vector which is nothing, but the sum of the tractions component and the body components body force component ok.

Once I do this I can get the virtual work contribution from node p belonging to element e as del v p dot the internal force vector because of node p belonging to element e minus the external force vector of node p belonging to element e ok.

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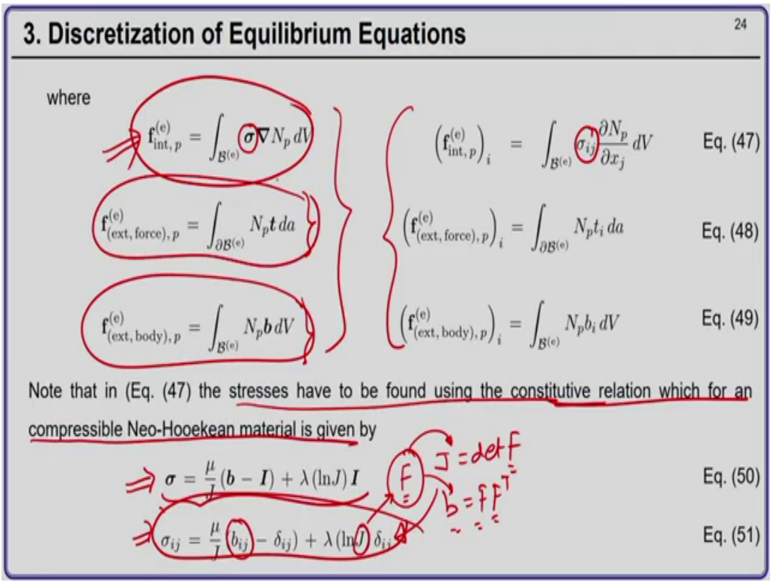

And, then these are the explicit expressions in direct notation and these are the indicial notations ok. And, we note that the stresses in the internal force vector ok. So, the stresses in the internal force vector that is here can be found using the constitutive relation.

So, in our case we are using and compressible Neo-Hooekean material for which we have already derived the Cauchy stress as following relation given by equation 50 or in indicial notation given by equation number 50. So, what will happen? To compute the internal force vector you have to first get the stresses ok. To get these stresses what we need to first do is we need to get the b and J. b and J both you can obtain if you have got the deformation gradient tensor F ok. Then J is determinant of F and b is F F transpose ok.

So, inside a finite element setting the first thing you will get is the deformation gradient tensor then you will obtain J, then you will check for the value of J. And, you will do check whether you have a valid deformation and then you will find out b and using b you will find out sigma. Once you have found out sigma you can find out the internal stresses. That is F internal and if that element has some body forces you will find the external body force vector. And, if that element belongs to the surface of the body then you will find the external force vector because of the external tractions.

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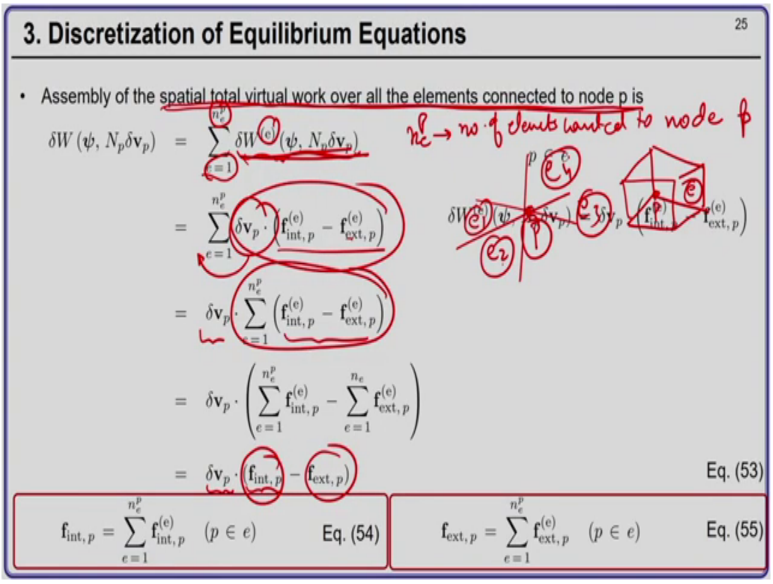

Then the assembly of the spatial total work over all the elements which are connected to node p ok. In a finite element setting if you have a node p then a lot of elements e 1, e 2, e 3, e 4 like this a lot of elements will connect to node p.

It may also happen that your node may belong to a corner element in that case p will have only one element or in general p may have different elements connecting to it. And, then if we take the contribution to the spatial virtual work because of all the elements which are connected to node p, then we know add the virtual work contribution over all the elements from e equal to N p e, where N p e is the number of elements connected to node p ok.

So, the number of n elements connected to node p ok. So, this information has to be there and then I can substitute the expression here as del v p dot F internal minus F external ok. Now, virtual velocities I can take outside and this is the integral over all the elements connected to node p of the internal minus the external force vector.

So, this finally, I can write this as del v p dot F internal total internal force at node p minus the total external force at node p.

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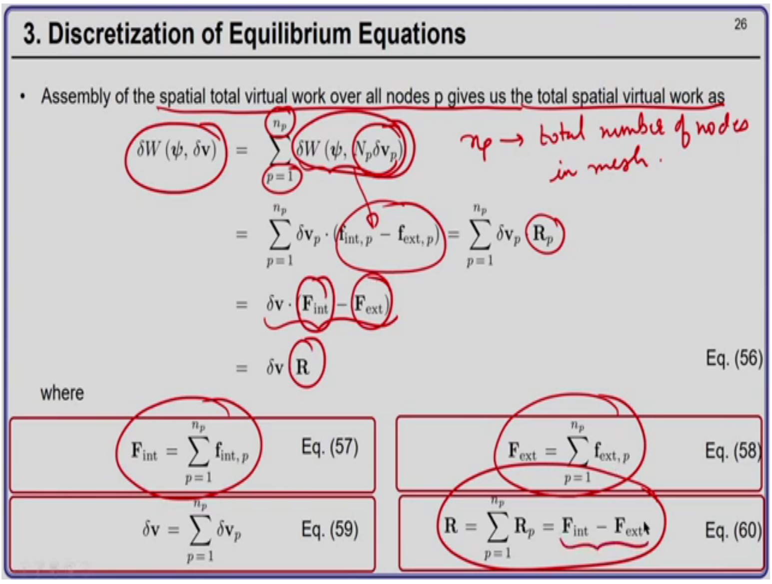

Next I can do the ok. So, next I can assemble the spatial total virtual work over all the nodes to give us the total spatial virtual work. Remember, this is a total spatial virtual work and this can be obtained as the total virtual work of all the nodes ok. So, this is the contribution to the total virtual work for node p and this when integrated over all the nodes ok.

Here n p is the total number of nodes in the mesh you may have thousands of nodes. So, this n p maybe thousand and when you substitute this expression from the previous slide I can write in short this as R p, where R p is the residual at node p ok.

And, assembly over all the nodes give me the global internal force vector minus the global external force vector. And, the difference of the internal and the external force vector is nothing but the residual R where the internal force vector as you can see here is the assembly over all the nodes the external force vector is assembly over all external forces of all the nodes ok.

And, then the residual is nothing, but the difference of the internal and the external forces and now my objective is to reduce this residual. To bring this residual equal to 0, so that when residual goes to 0 the internal forces will balance out the external forces and that is when you achieve your equilibrium.

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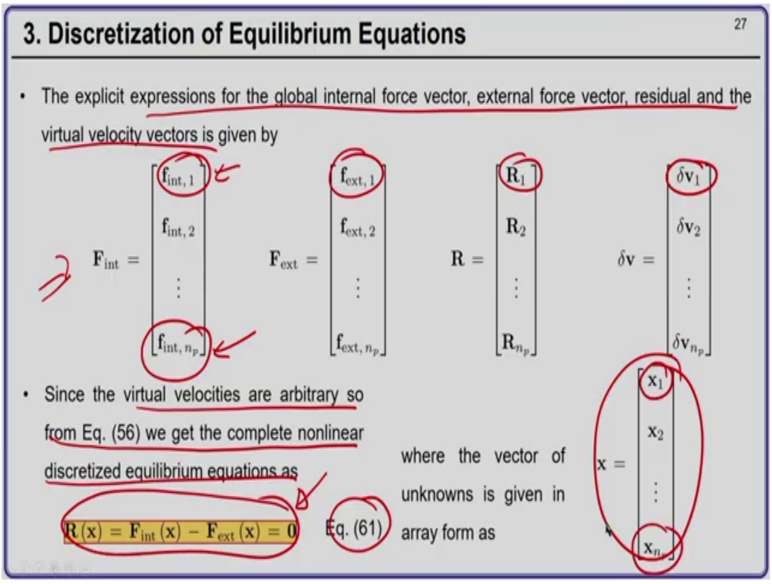

So, this is the explicit expression of the global internal forces, global external force, residual and virtual velocity vector in the previous slides where each of these here correspond to one particular node. So, this correspond to node 1 and this correspond to the final node in the mesh that you have.

Since the virtual velocities are arbitrary, so, equations 56 gives us the complete non-linear discretized equilibrium equation as R x equal to F internal minus F x external ok. So, the internal and external forces are the non-linear functions of x and then equation number 61 gives you the finite element discretized continuum equilibrium equation that we had which is divergence of sigma plus b equal to 0.

And, this is the corresponding finite element discretized part and x is the vector of unknowns that we need to find out ok. So, this is the unknown at node 1 and this is the unknown at final load N p.

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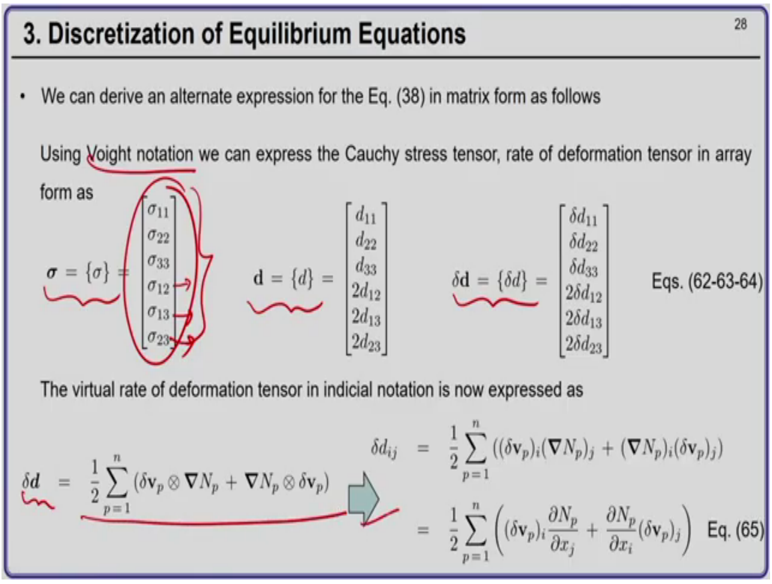

So, we can derive an alternate expression for equation number 38 in matrix form I can write using Voight notation the Cauchy stress tensor in the vector form like this. So, instead of writing nine components, I can express them as six components because the other three are symmetric because sigma 12 is same as sigma 21, 13 is sigma 31s and sigma 32 is same as sigma 23.

And, this is the rate of deformation tensor and this is a virtual rate of deformation tensor ok. So, this virtual rate of deformation tensor which is given by the following expression and this is the indicial notation.

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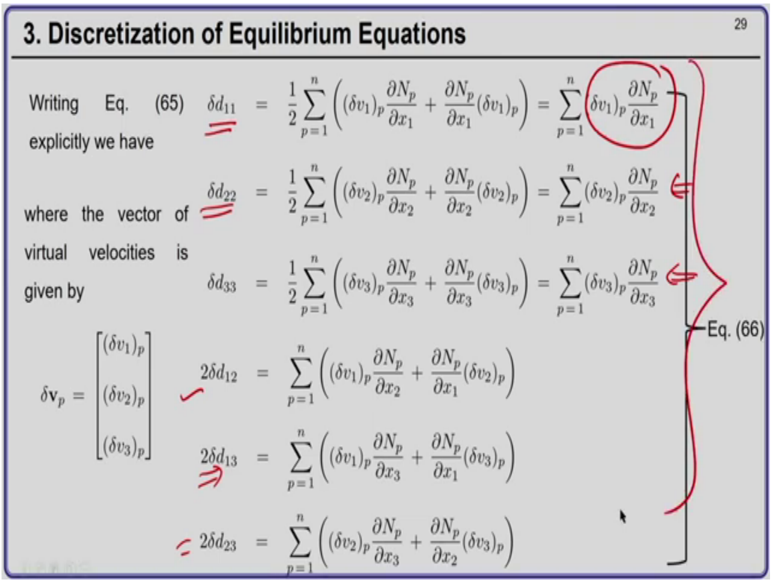

I can write them explicitly as del d 11 as del v 1 p del N p by del x 1 ok; this you can do from the indicial notation. So, del dd 2 del d 22 is given by this expression; del d 33 is given by this expression; this is 2 del d 12 this is 2 del d 13 and this is 2 del d 23.

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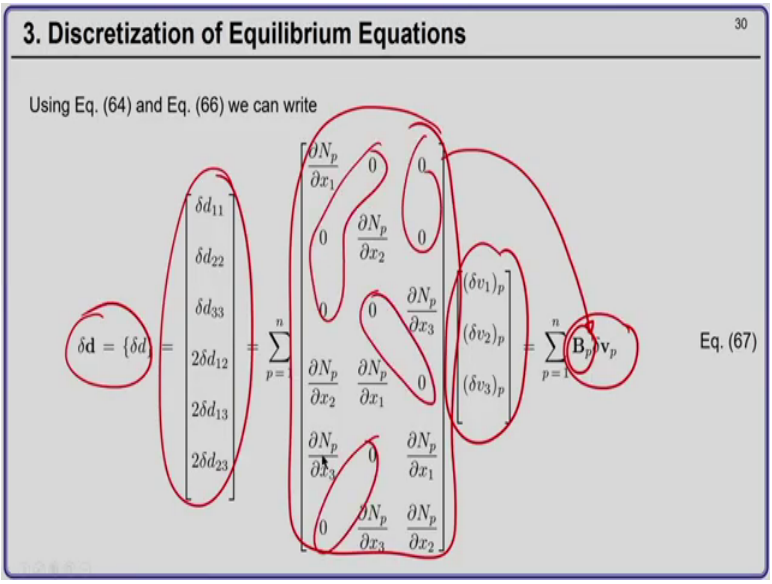

If I write all these six equations in matrix form I can express my rate of deformation tensor in vector form as following expression. So, this is what I define. This is a matrix and this is a virtual velocities. So, this matrix I can write define a matrix called B p ok. So, this is B p; the matrix B associated with node p into virtual velocity del v p.

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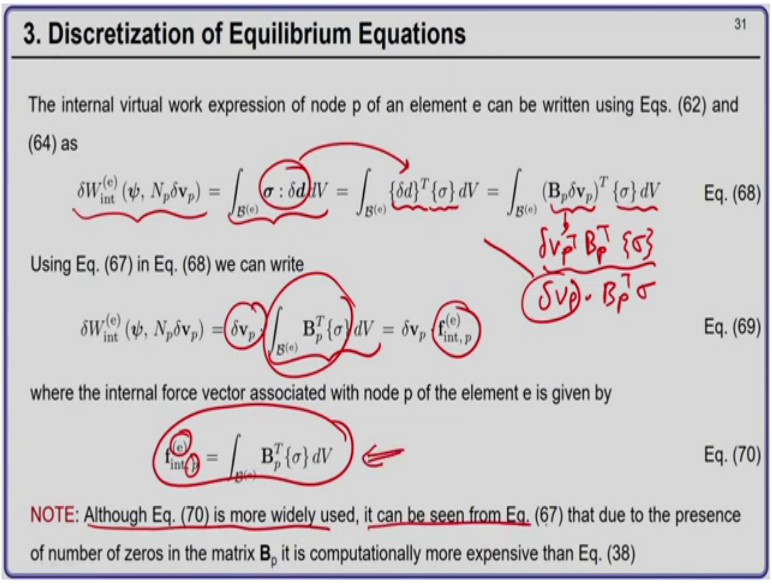

Now, the internal virtual work expression which has given like this the same can be written as del d transpose sigma and del d from my previous slide is B p del v del v p. So, this is v B p del v p transpose this is sigma. And, now del v p transpose this becomes del v p transpose B p transpose sigma and then I can write this as del v p dot B p transpose sigma. And, virtual velocities I can take outside the bracket and this is my relation B transpose sigma dV.

And, this is nothing, but this should be same as the internal force vector corresponding to node p of element e. So, you see here we get an alternative expression for the internal force vector of node p belonging to element e as integral over the current volume v p transpose sigma dV.

You note that although this equation is what you find mostly in the books. It can be seen from equation 67 which is here that there are lot of zeros present over here there are lot of zeros you see here. So, that due to the presence of a number of zeros in the matrix B equation 70 is computationally more expensive than equation number 38 ok. So, equation 38 in the actual computational setting is what we use 70 is what is mostly present in the finite element text ok.

Thank you.